A Sequence of Cauchy Sequences Which Converge to the Imaginary Parts of the Zeros of the Riemann Zeta Function

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Abstract

An iteration function which has fixed-points at the zeros of the Hardy Z function is constructed and it is shown that it is impossible for this function converge to a non-real number when started with a real number. If there were any zeros of $\zeta(t)$ with $\mathrm{Re}(t) \neq \frac{1}{2}$ they would correspond to zeros of $Z(t)$ with $\mathrm{Im}(t) \neq 0$ and thus the constructed interation function must be able to converge for at least one real-valued starting point to a number with non-zero imaginary part, but this is impossible because the iteration function is real-valued when its argument is real. Thus, the Riemann hypothesis is shown to be true.

1 Preliminary Outline

1.1 Definitions

Let $\zeta(t)$ be the Riemann zeta function

$$\begin{aligned} \zeta(t) &= \sum_{n=1}^{\infty} n^{-s} && \forall \mathrm{Re}(s) > 1 \\ &= (1 - 2^{1-s}) \sum_{n=1}^{\infty} n^{-s} (-1)^{n-1} && \forall \mathrm{Re}(s) > 0 \end{aligned} \tag{1}$$

and $\vartheta(t)$ be Riemann-Siegel vartheta function

$$\vartheta(t) = -\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4} + \frac{i t}{2}\right) - \ln\Gamma\left(\frac{1}{4} - \frac{i t}{2}\right)\right) - \frac{\ln(\pi)\, t}{2} \tag{2}$$

so that the Hardy Z function[2] can be defined by

$$Z(t) = e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + i t\right) \tag{3}$$

which is real-valued when t is real and satisfies the identity

$$\zeta(t) = e^{-i\vartheta\left(\frac{i}{2} - i t\right)} Z\left(\frac{i}{2} - i t\right) \tag{4}$$

where $\ln\Gamma(z)$ is the principal branch of the logarithm of the Γ function defined by

$$\ln\Gamma(z) = \ln(\Gamma(z)) = (z-1)! = \prod_{k=1}^{z-1} k \forall z \in \mathbb{R} > 0 \tag{5}$$

which is analytically continued from the positive real axis when $z \in \mathbb{C}$ is complex. Each of the points $z \in \mathbb{Z} = \{0, -1, -2, \ldots\}$ is a singularity and a branch point so that the union of the branch cuts is the negative real axis. On the branch cuts, the values of $\ln\Gamma(z)$ are determined by continuity from above. Let $S(t)$ denote the normalized argument of $\zeta(t)$ on the critical line

$$\begin{aligned} S(t) &= \pi^{-1} \arg\left(\zeta\left(\frac{1}{2} + it\right)\right) \\ &= -\frac{i}{2\pi}\left(\ln\zeta\left(\frac{1}{2} + it\right) - \ln\zeta\left(\frac{1}{2} - it\right)\right) \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0} \mathrm{Im}\left(\ln\zeta\left(\frac{1}{2} + it + \varepsilon\right)\right) \end{aligned} \tag{6}$$

1.2 Transcendental Equations Satisifed By The Nontrivial Riemann Zeros

Definition 1. *The* `critical line` *is the line in the complex plane defined by* $\mathrm{Re}(t) = \frac{1}{2}$.

Definition 2. *The* `critical strip` *is the strip in the complex plane defined by* $0 < \mathrm{Re}(t) < 1$.

Definition 3. *The* `exact equation` *for the n-th zero of the Hardy Z function y_n is given by [1, Equation 20]*

$$\vartheta(y_n) + S(y_n) = \left(n - \frac{3}{2}\right)\pi \tag{7}$$

where y_n enumerate the zeros of Z on the real line and the zeros of ζ on the critical line

$$\mathrm{Z}(y_n) = 0 \,\text{and}\, \zeta\left(\frac{1}{2} + iy_n\right) \forall n \in \mathbb{Z}^+ \tag{8}$$

where $\mathbb{Z}^+$ denotes the positive integers. [1, Equation 14]

By replacing the $\ln\Gamma$ function in (2) with Stirling's asymptotic expansion as in [1, Equation 13] we get

$$\tilde{\vartheta}(t) = \frac{t}{2}\ln\left(\frac{t}{2\pi e}\right) - \frac{\pi}{8} + O(t^{-1}) \tag{9}$$

and substitute $\vartheta(t)$ with $\tilde{\vartheta}(t)$ in Equation 7 which leads to

Definition 4. *The* `asymptotic equation` *for the n-th zero of the Hardy Z function*

$$\frac{t_n}{2\pi}\ln\left(\frac{t_n}{2\pi t}\right) + S(t_n) = n - \frac{11}{8} \tag{10}$$

[1, Equation 20]

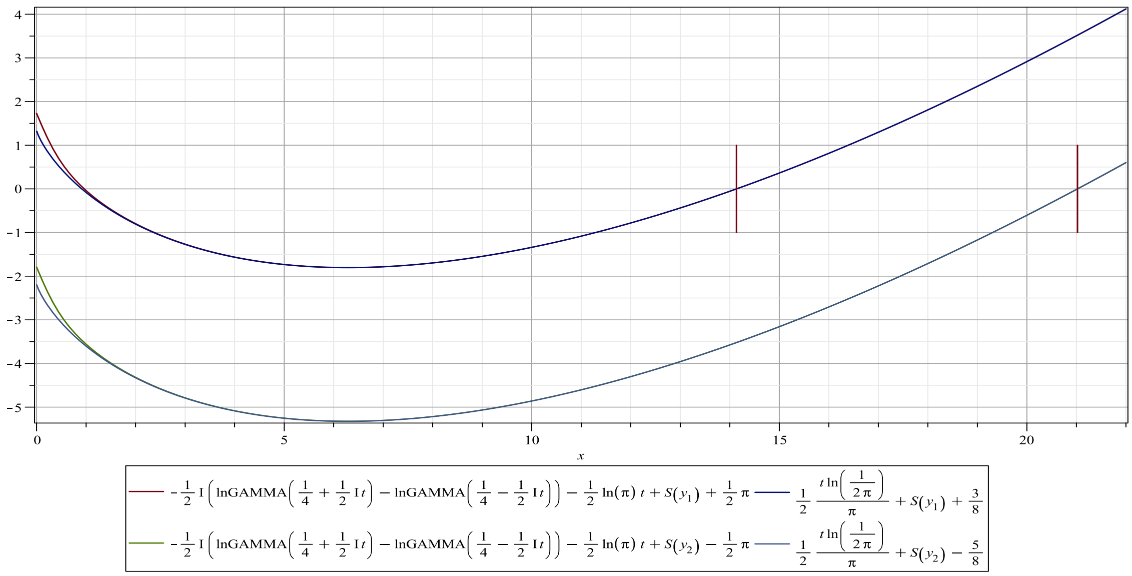

Figure 1. The functions $\vartheta(y_n) + S(y_n) - \left(n - \frac{3}{2}\right)\pi$ and $\tilde{\vartheta}(y_n) + S(y_n) - \left(n - \frac{3}{2}\right)\pi$ for $n = 1, 2$ with the zeros at y_1 and y_2 marked with vertical lines.

Remark 5. The fact that the exact and asymptotic equations have two solutions when $n = 1$ can be understood by noting that Equations (7) and (10) are derived from the equation

$$n = \tilde{\vartheta}(t) + \frac{\pi}{8} - \frac{5}{8} + S(t) \tag{11}$$

which has a minimum in the interval $(-2, -1)$ and thus $n \geqslant -1$ so that, in order to follow the convention that the zeros are enumerated by the positive integers, the substituion $n \rightarrow n - 2$ is made in Equation (11) so that

$$n - 2 = \tilde{\vartheta}(t) + \frac{\pi}{8} - \frac{5}{8} + S(t) \tag{12}$$

[1, Equation 12]

Theorem 6. *If the limit*

$$\lim_{\delta \rightarrow 0^+} \arg\left(\zeta\left(\frac{1}{2} + \delta + i t\right)\right) \tag{13}$$

is exists and is well-defined $\forall t$ *then the left-hand side of Equation (10) is well-defined* $\forall t$*, and due to monotonicity, there must be a unique solution for every* $n \in \mathbb{Z}^+$*. [1, II.A]*

Corollary 7. *The number of solutions of Equation (10) over the interval* $[0, t]$ *is given by*

$$N_0(t) = \frac{t}{2\pi} \ln\left(\frac{t}{2\pi e}\right) + \frac{7}{8} + S(t) + O(t^{-1}) \tag{14}$$

which counts the number of zeros `on the critical line`.

Conjecture 8. *(The Riemann hypothesis) All solutions* t *of the equation*

$$\zeta(t) = 0 \tag{15}$$

besides the trivial solutions $t = -2n$ with $n \in \mathbb{Z}^+$ have real-part $\frac{1}{2}$, that is, $\mathrm{Re}(t) = \frac{1}{2}$ *when $\zeta(t) = 0$ and $t \neq -2n$.*

Definition 9. *The Riemann-von-Mangoldt formula makes use of Cauchy's argument principle to count the number of zeros* `inside the critical strip` $0 < \mathrm{Im}(\rho_n) < t$ *where $\zeta(\sigma + i\rho_n)$ with $0 < \sigma < 1$*

$$N(t) = \frac{t}{2\pi}\ln\left(\frac{t}{2\pi e}\right) + \frac{7}{8} + S(t) + O(t^{-1}) \tag{16}$$

and this definition does not depend on the Riemann hypothesis(Conjecture 8). This equation has exactly the same form as the asymptotic Equation 10. [1, Equation 15]

Lemma 10. *If the exact Equation (7) has a unique solution for each $n \in \mathbb{Z}^+$ then Conjecture 8, the Riemann hypothesis, follows.*

Proof. If the exact equation has a unique solution for each n, then the zeros obtained from its solutions on the `critical line` can be counted since they are enumerated by the integer n, leading to the counting function $N_0(t)$ in Equation (14). The number of solutions obtained on the `critical line` would saturate counting function of the number of solutions on the `critical strip` so that $N(t) = N_0(t)$ and thus all of the non-trivial zeros of ζ would be enumerated in this manner. If there are zeros off of the critical line, or zeros with multiplicity $m \geqslant 2$, then the exact Equation (7) would fail to capture all the zeros on the critical strip which would mean $N_0(t) < N(t)$. [1, IX] □

Corollary 11. *The Riemann hypothesis(RH) is not necesarily false if the exact Equation (7) does not have a unique solution for every n, since the solutions could still be on the critical line but not necessarily simple, that is, a root on the critical line could have multiplicity $m \geqslant 2$ and the RH would still be true.*

Corollary 12. *The Riemann hypothesis is true and all of the zeros on the critical line are simple if the exact Equation (7) has a unique solution for each $n \in \mathbb{Z}^+$. [1, IX]*

1.3 Iterated Function Systems

Definition 13. *A* `fixed-point` *α of a function $f(x)$ is a value α such that*

$$f(\alpha) = \alpha \tag{17}$$

[6, 3.]

Definition 14. *The* `multiplier` *of a fixed point α of a map $f(x)$ is equal to the absolute value of the derivative of the map evaluated at the point α.*

$$\lambda_f(\alpha) = |\dot{f}(\alpha)| \tag{18}$$

If $\lambda_f(\alpha) < 1$ then α is a said to be an ***attractive fixed-point*** *of the map $f(x)$. If $\lambda_f(\alpha) = 1$ then α is an* ***indifferent fixed point****, and if $\lambda_f(\alpha) > 1$ then α is a* ***repelling fixed-point****. [6, 3.]*

Definition 15. *Let*

$$Y_{n,m}(t) = \begin{cases} t & m = 0 \\ t + h_{n,m}\cos(\pi n)\tanh\left(\frac{Z(Y_{n,m-1}(t))}{|\Omega(t)|\prod_{k=1}^{n-1}\tanh(Y_{n,m-1}(t) - y_k)}\right) & m \geqslant 1 \end{cases}$$

denote the m-th iterate of the n-th iteration function corresponding to the n-th zero of the Hardy Z function where

$$\Omega(t) = \begin{cases} 1 & t = e \\ e^{\frac{3}{4}\sqrt{\frac{\log(t)}{\log(\log(t))}}} & t \neq e \end{cases} \tag{19}$$

is a lower bound for the running maximum of $|Z(s)|$

$$\max_{0 \leqslant s \leqslant t} |Z(s)| > \Omega(t) \forall t \geqslant 45.590... \tag{20}$$

ensuring that

$$\frac{|Z(t)|}{\Omega(t)} > 0 \forall t \geqslant 45.590... \tag{21}$$

which normalizes the range of $Z(t)$ *which is known to grow in both maximum and average value as* $t \to \infty$ *and* $h_{n,m}$ *is factor which influences the rate of convergence*

$$h_{n,m} = \begin{cases} 1 & m \leqslant 2 \\ h_{n,m-1} & \operatorname{sign}(\Delta Y_{n,m-2}(t)) = \operatorname{sign}(\Delta Y_{n,m-1}(t)) \\ \frac{h_{n,m-1}}{2} & \operatorname{sign}(\Delta Y_{n,m-2}(t)) \neq \operatorname{sign}(\Delta Y_{n,m-1}(t)) \end{cases} \tag{22}$$

where

$$\Delta Y_{n,m}(t) = Y_{n,m}(t) - Y_{n,m-1}(t) \tag{23}$$

is the 1*-st difference of the m-th iterate for the n-th zero. [5, Theorem 3.2.3]*

Lemma 16. *The roots of* $Z(t)$ *are fixed-points of* $Y_n(t) \forall n \in \mathbb{Z}^+$.

Proof. If $Z(t) = 0$ then $\tanh\left(\frac{Z(t)}{|\Omega(t)| \prod_{k=1}^{n-1} \tanh(t - y_k)}\right) = \tanh\left(\frac{0}{|\Omega(t)| \prod_{k=1}^{n-1} \tanh(t - y_k)}\right) = \tanh(0) = 0$ so that $Y_n(t) = t + \cos(\pi n) 0 = t + 0 = t$ when $Z(t) = 0$. □

Theorem 17. $Y_{n,m}(t)$ *has indifferent fixed-points at each point* y_k *where* $k = 1...n-1$

Proof. The product in the denominator $\prod_{k=1}^{n-1} \tanh(t - y_k) \to 0$ smoothly as t approaches any $y_k \in \bigcup_{k=1}^{n-1} y_k$ since $\tanh(0) = 0$ and tanh is a smooth function. When any element of the product is zero the value of the product is zero regardless of the values of any other elements of the product. Since $\frac{1}{s} \to \infty$ as $s \to 0$ and $\tanh(|x|) \to 1$ as $|x| \to \infty$ we have $\tanh(\infty) = 1$ and $\tanh(-\infty) = -1$ so that $Y_n(t) = t + \cos(\pi n) \forall t \in \bigcup_{k=1}^{n-1} y_k$. Since $Y_n(t) = t \pm 1 \forall t \in \bigcup_{k=1}^{n-1} y_k$ when n is an integer, we see that $\frac{\mathrm{d}}{\mathrm{d}t} Y_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}(t \pm 1) = 1$ so that the multiplier $\lambda_{Y_n(t)} = \left|\frac{\mathrm{d}}{\mathrm{d}t} Y_n(t)\right| = 1 \forall t \in \bigcup_{k=1}^{n-1} y_k$. □

Proposition 18. *When n is an odd number,* $Y_n(t)$ *has attractive fixed-points at the odd-numbered roots* $y_{2k-1} \forall 2k - 1 \geqslant n$ *and repulsive fixed-points at the even-numbered roots* $y_{2k} \forall 2k \geqslant n$.

Proposition 19. *When n is an even number,* $Y_n(t)$ *has attractive fixed-points at the even-numbered roots* $y_{2k} \forall 2k \geqslant n$ *and repulsive fixed-points at the odd-numbered roots* $y_{2k-1} \forall 2k - 1 \geqslant n$.

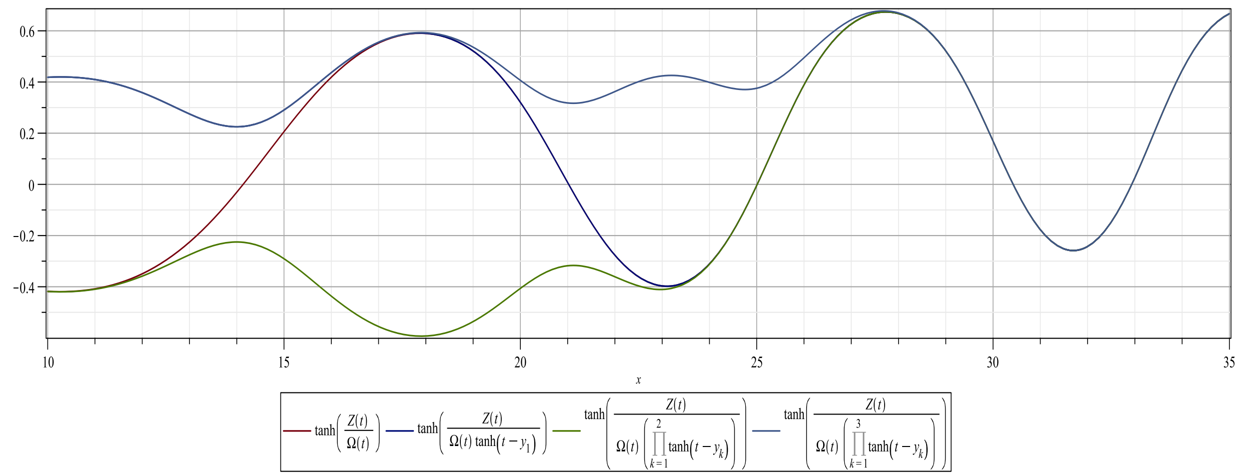

Figure 2. The functions which are subtracted or added to t to get $Y_1(t), Y_2(t), Y_3(t), Y_4(t)$. When n is odd $\cos(\pi n) = -1$ so that the the value is subtracted from t, when n is even $\cos(2\pi) = 1$ so it is added. It is plain to see that the curves $\tanh\left(\frac{Z(t)}{\Omega(t)\prod_{k=1}^{n-1}\tanh(t-y_k)}\right)$ do not cross the zero axis for any $t < y_n$

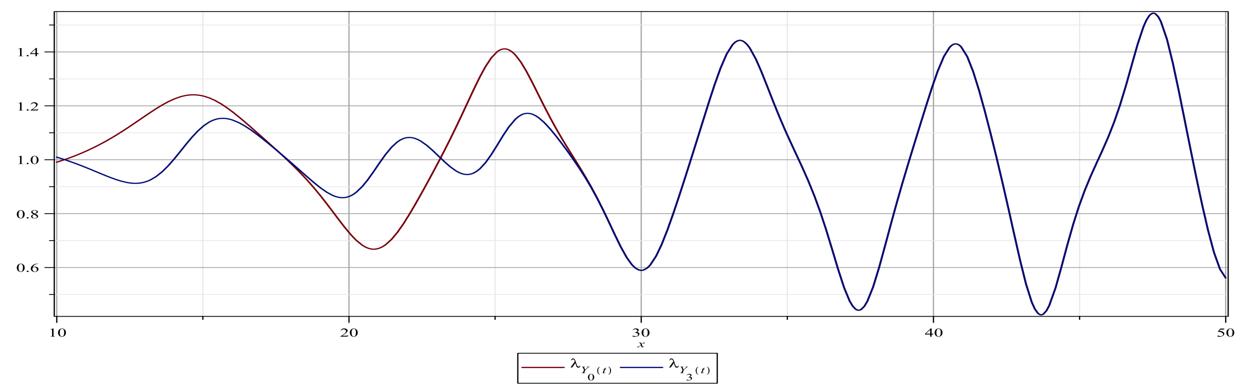

Figure 3. Multipler of the maps $Y_1(t)$ and $Y_3(t)$

Remark 20. The function $h_{n,m}$ is defined to be 1 when $1 \leqslant m \leqslant 2$. If $\text{sign}(\Delta Y_{n,m-1}(t)) \neq \text{sign}(\Delta Y_{n,m}(t))$ then $h_{n,m+1} = \frac{h_{n,m}}{2}$ so that the convergence rate is halved when the sign of the difference between successive iterates changes, indicating that it jumped across the root. This prevents the sequence generated by the iteration from getting stuck in an artifical 2-cycle and jumping back and forth across the root with equal magnitude indefinately when implementing this method with finite-precision arithmetic on a digital computer. Without this successive relaxation, the iterates still converge in theory however the number of iterations required could be several million or higher, while still having the difficulty of possibly getting stuck in a 2-cycle in computer implementations.

Theorem 21. *The Lipschitz constant M of the map $Y_{n,m}(t) < 1 \forall t > e$ therefore $Y_{n,m}(t)$ is a contraction mapping*

$$|Y_{n,m}(t) - Y_{n,m}(s)| \leqslant M|t-s| \tag{24}$$

Proof. The Lipschitz constant of a continuous differentiable function $f(x)$ is equal to the maximum absolute value of its derivative

$$M = \sup_x \left|\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right| \tag{25}$$

The derivative of $t - \tanh(t)$ is $\tanh(t)^2$. Since the maximum absolute value of $\tanh(t)$ is 1 then the maximum value of its square is also 1. Since $\Omega(t) > 1$ and $h_{n,m} \leqslant 1$ the derivative $\frac{\mathrm{d}}{\mathrm{d}t} Y_{n,m}(t)$ can never have an absolute value $\geqslant 1$ since that would require $\left|\tanh\left(\frac{Z(t)}{|\Omega(t)| \prod_{k=1}^{n-1} \tanh(t-y_k)}\right)\right| = 1$ which is only possible if $Z(t) = \pm\infty$ which is only the case when $t = \pm\frac{i}{2}$ which corresponds to the pole at $\zeta(1)$. Since $Z(t) \in \mathbb{R}$ when $t \in \mathbb{R}$ it can never be the case that $Z(t) = \infty$ so that $\left|\frac{\mathrm{d}}{\mathrm{d}t} Y_{n,m}(t)\right| \neq 1 \forall t \in \mathbb{R}$ and the Lipschitz constant M is strictly less than 1. □

Proposition 22. *The limit*

$$y_n = \lim_{m \to \infty} Y_{n,m}(s_n) \tag{26}$$

where

$$s_n = \begin{cases} 14 & n = 1 \\ 21 & n = 2 \\ \frac{y_{n-1} + y_{n-2}}{2} & n \geqslant 3 \end{cases} \tag{27}$$

exists and is equal to the n-th zero of the Hardy Z function for all integer $n \subset \mathbb{Z}^+$. That is, $Y_{n,m}(z_n)$ forms a Cauchy sequence, due to the contraction mapping property proved in Theorem 21 whose elements are indexed by m converging to the n-th root y_n where the n-th starting point is defined to be half-way between the $(n-2)$-th and the $(n-1)$-th root y_n when $n > 2$ and equal to a point close to the first known zero at 14.134.... when $n = 1$ and a point close to the 2nd zero at 21.022... when $n = 2$

Remark 23. The mid-way point between the nearest neighbors to the left of y_n is used as the starting point for the iteration since any point less than y_n and greater than e is within the immediate basin of attraction of y_n. The precise location of any roots y_p where $p < n$ cannot be used as a starting point since the map $Y_{p,m}(t)$ is a non-expansive mapping with Lipschitz constant precisely equal to 1 when $t \in \bigcup_{k=1}^{p-1} y_k$ so that the hyperbolic tangent has an argument of infinity resulting in a value of 1. Trajectories are neither attracted or repelled to any point $\bigcup_{k=1}^{n-1} y_k$ under the action of the map $Y_{n,m}(t)$ however, trajectories started precisely on any point $t \in \bigcup_{k=1}^{n-1} y_k$ will never attain a value other than t since any y_k is a fixed-point of $Y_{n,m}(t)$.

Note 24. The truth of Propositon 22 has been verified computationally up to $n = 628,737$ with a computer program which implements the methods described here using the arbitrary precision complex ball arithmiticlibrary arblib[3] and compares the results against the tables published by Andrew Odlyzko[4].

Theorem 25. *The Cauchy sequence $\lim_{m \to \infty} Y_{n,m}(s_n)$ will never converge to any y_k where $k < n$.*

Proof. All y_k are indifferent fixed-points of $Y_{n,m}(t)$ and the trajectories generated by $Y_{n,m}(s_n)$ are never started from a point y_k since $s_n \notin \bigcup_{k=1}^{n-1} y_k$ and the only way $Y_{n,m}(t)$ would ″convege″ to an indifferent fixed-point is if it was started precisely on one, and s_n is by definition equal to the mid-point between successive y_n. □

Theorem 26. *The Cauchy sequence $Y_{n,m}(s_n)$ will never converge to any $y_{n+2k-1} \forall k \in \mathbb{Z}^+$ if Proposition 18 is true.*

Proof. If Propositions 18 is true then y_{n+2k-1} are repelling fixed-points for $Y_{n,m}(t)$. □

Note 27. If Propositions 18 and 19 are true then $Y_{n,m}(s_n)$ will never converge to y_q with q odd and n even nor to y_r with r even and n odd. It suffices to prove that $Y_{n,m}(s_n) < y_{n+1} \forall n, m \in \mathbb{Z}^+$ which would mean that $Y_{n,m}(s_n)$ can never jump across the repelling fixed-point at y_{n+1} to land on any of the attractive fixed-points in $\bigcup_{k=1}^{\infty} y_{n+2k}$

Lemma 28. *The truth of Proposition 18 is true due to Theorem 26 which implies the truth of Proposition 22 which implies the existence of a soluion for each* n *of the exact Equation (7) since*

$$\begin{aligned} S(y_n) &= \left(n - \frac{3}{2}\right)\pi - \vartheta(y_n) \\ &= \left(n - \frac{3}{2}\right)\pi - \vartheta\left(\lim_{m\to\infty} Y_{n,m}(s_n)\right) \end{aligned} \tag{28}$$

would be well-defined$\forall n \in \mathbb{Z}^+$ *and therefore Conjecture 8, the Riemann hypothesis, would be true due to Lemma 10.*

Lemma 29. $Z(t)$ *is real-valued for real values of* t.

Proof. The functional equation for ζ is

$$\zeta(s) = \chi(s)\zeta(1-s) \tag{29}$$

where

$$\chi(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}(1-s)\right)}{\Gamma\left(\frac{s}{2}\right)} \tag{30}$$

satisfies $\chi(s)\chi(1-s) = \chi\left(\frac{1}{2}+it\right)\chi\left(\frac{1}{2}-it\right) = 1$ so that $\left|\chi\left(\frac{1}{2}+it\right)\right| = 1$. Therefore,

$$Z(t) \in \mathbb{R} \forall t \neq 0 \in \mathbb{R} \tag{31}$$

since

$$\chi\left(\frac{1}{2}+it\right)^{-\frac{1}{2}}\zeta\left(\frac{1}{2}+it\right) = \chi\left(\frac{1}{2}-it\right)\zeta\left(\frac{1}{2}-it\right) \tag{32}$$

[7] □

Theorem 30. *All of the roots of* $Z(t)$ *in the critical strip are real, that is to say,* $\Big\{t \in \mathbb{R}\colon Z(t) = 0, t \neq \bigcup_{n=1}^{\infty} \frac{i}{2}(4n+1)\Big\}$ *where* $\bigcup_{n=1}^{\infty} \frac{i}{2}(4n+1)$ *are the trivial zeros.*

Proof. If there did exist a root $Z(u)$ with $\mathrm{Im}(u \neq 0)$ then there must be at least one starting point t on the real line and some integer n such that $\lim_{m\to\infty} Y_{n,m}(t) = u \exists n \in \mathbb{Z}^+, t \in \mathbb{R}^+$. This is because, every root of $Z(t)$ except $t \in \bigcup_{k=1}^{n-1} y_k$ either repels or attracts trajectories generated by $Y_{n,m}(t_0)$ whether or not they are on the critical line and whether or not the Riemann hypothesis is true. If there were a trajectory $Y_{n,m}(t_0)$ which converged to some $u = \lim_{m\to\infty} Y_{n,m}(t)$ with $\mathrm{Im}(u \neq 0)$ then there must be at least one m such that $\mathrm{Im}(Y_{n,m}(t)) \neq 0$ however that is an impossibility since $\mathrm{Im}(\tanh(x)) = 0 \forall x \in \mathbb{R}$ and all of the arguments of the hyperbolic tangent function in $Y_{n,m}(t)$ are real-valued, $h_{n,m}, Z(t), |\Omega(t)|, (\prod_{k=1}^{n-1} \tanh(Y_{n,m-1}(t) - y_k)) \forall t \in \mathbb{R}$. The $\cos(\pi n)$ factor which is multiplied by the hyperbolic tangent is also real-valued for real-arguments, therefore $\mathrm{Im}(Y_{n,m}(t)) = 0 \forall t \in \mathbb{R}$. Since it is already known that there are no roots of $Z(t)$ for $0 < t < e$, this completes the proof. □

Corollary 31. *The Riemann hypothesis, Lemma 8, is true.*

Proof. *Since all of the non-trivial zeros* $Z(t) = 0$ *have* $\mathrm{Im}(t) = 0$ *then all of the non-trivial zeros of* $\zeta(v)$ *have* $\mathrm{Re}\left(\frac{1}{2}\right)$ *since the inverse of* $\frac{i}{2} - iy = t$ *is* $y = -\frac{i}{2}(i-2t)$ *we have* $\mathrm{Re}\left(-\frac{i}{2}(i-2t)\right) = \frac{1}{2} \forall t \in \mathbb{R}$. □

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