A Sequence of Cauchy Sequences Convergent to the Imaginary Parts of the Zeros of the Riemann Zeta Function

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Abstract

A sequence of Cauchy sequences which converge to the Riemann zeros is constructed and related to the LeClair-França criteria for the Riemann hypothesis.

1 Preliminary Outline

1.1 Definitions

Let $\zeta(t)$ be the Riemann zeta function

$$\begin{aligned}\zeta(t) &= \sum_{n=1}^{\infty} n^{-s} && \forall \mathrm{Re}(s) > 1 \\ &= (1-2^{1-s}) \sum_{n=1}^{\infty} n^{-s}(-1)^{n-1} && \forall \mathrm{Re}(s) > 0\end{aligned} \tag{1}$$

and $\vartheta(t)$ be Riemann-Siegel vartheta function

$$\vartheta(t) = -\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4}+\frac{it}{2}\right) - \ln\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\right) - \frac{\ln(\pi)\,t}{2} \tag{2}$$

where $\ln\Gamma(z)$ is the principal branch of the log-Gamma function defined by

$$\ln\Gamma(z) = \ln(\Gamma(z)) = (z-1)! = \prod_{k=1}^{z-1} k \forall z \in \mathbb{R} > 0 \tag{3}$$

and is defined by analytic continuation from the positive real axis when $z \in \mathbb{C}$ is complex. Each of the points $z \in \mathbb{Z} = \{0, -1, -2, \ldots\}$ is a singularity and a branch point, and the union of the branch cuts is the negative real axis. On the branch cuts, the values of $\ln\Gamma(z)$ are determined by continuity from above. Let $S(t)$ denote the normalized argument of $\zeta(t)$ on the critical line

$$\begin{aligned}S(t) &= \pi^{-1} \arg\left(\zeta\left(\frac{1}{2}+it\right)\right) \\ &= -\frac{i}{2\pi}\left(\ln\zeta\left(\frac{1}{2}+it\right) - \ln\zeta\left(\frac{1}{2}-it\right)\right) \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0} \mathrm{Im}\left(\ln\zeta\left(\frac{1}{2}+it+\varepsilon\right)\right)\end{aligned} \tag{4}$$

Let $Z(t)$ denote the Hardy Z function[2] defined by

$$Z(t) = e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2}+it\right) \tag{5}$$

which is real-valued when t is real and satisfies the identity

$$\zeta(t) = e^{-i\vartheta\left(\frac{i}{2}-it\right)} Z\left(\frac{i}{2} - it\right) \tag{6}$$

1.2 Transcendental Equations Satisifed By The Nontrivial Riemann Zeros

Definition 1. *The* `critical line` *is the line in the complex plane defined by* $\mathrm{Re}(t) = \frac{1}{2}$.

Definition 2. *The* `critical strip` *is the strip in the complex plane defined by* $0 < \mathrm{Re}(t) < 1$.

Definition 3. *The* `exact equation` *for the n-th zero of the Hardy Z function y_n is given by [1, Equation 20]*

$$\vartheta(y_n) + S(y_n) = \left(n - \frac{3}{2}\right)\pi \tag{7}$$

where y_n enumerate the zeros of Z on the real line and the zeros of ζ on the critical line

$$\mathrm{Z}(y_n) = 0 \,\mathrm{and}\, \zeta\left(\frac{1}{2} + i y_n\right) \forall n \in \mathbb{Z}^+ \tag{8}$$

where $\mathbb{Z}^+$ denotes the positive integers. [1, Equation 14]

By replacing the $\ln\Gamma$ function in (2) with Stirling's asymptotic expansion as in [1, Equation 13] we get

$$\tilde{\vartheta}(t) = \frac{t}{2}\ln\left(\frac{t}{2\pi e}\right) - \frac{\pi}{8} + O(t^{-1}) \tag{9}$$

and substitute $\vartheta(t)$ with $\tilde{\vartheta}(t)$ in Equation 7 which leads to

Definition 4. *The* `asymptotic equation` *for the n-th zero of the Hardy Z function*

$$\frac{t_n}{2\pi}\ln\left(\frac{t_n}{2\pi t}\right) + S(t_n) = n - \frac{11}{8} \tag{10}$$

[1, Equation 20]

Theorem 5. *If the limit*

$$\lim_{\delta \to 0^+} \arg\left(\zeta\left(\frac{1}{2} + \delta + it\right)\right) \tag{11}$$

is exists and is well-defined $\forall t$ then the left-hand side of Equation (10) is well-defined $\forall t$, and due to monotonicity, there must be a unique solution for every $n \in \mathbb{Z}^+$. [1, II.A]

Corollary 6. *If Propositon 5 is true then number of solutions of Equation (10) is given by*

$$N_0(t) = \frac{t}{2\pi}\ln\left(\frac{t}{2\pi e}\right) + \frac{7}{8} + S(t) + O(t^{-1}) \tag{12}$$

which counts the number of zeros `on the critical line`.

Conjecture 7. *The Riemann hypothesis) All non-trivial solutions of the equation*

$$\zeta(t)=0 \tag{13}$$

besides the trivial solutions $t=-2n$ *with* $n\in\mathbb{Z}^+$ *have real-part* $\frac{1}{2}$*, that is,* $\mathrm{Re}(t)=\frac{1}{2}$ *when* $\zeta(t)=0$ *and* $t\neq-2n$*.*

Definition 8. *The Riemann-von-Mangoldt formula makes use of Cauchy's argument principle which gives the number of zeros* `inside the critical strip` $0<\mathrm{Im}(\rho_n)<t$ *where* $\zeta(\sigma+i\rho_n)$ *with* $0<\sigma<1$

$$N(t)=\frac{t}{2\pi}\ln\left(\frac{t}{2\pi e}\right)+\frac{7}{8}+S(t)+O(t^{-1}) \tag{14}$$

and this definition does not depend on the Riemann hypothesis and it is noted that this equation has exactly the same form as the asymptotic Equation 10 [1, Equation 15]

Lemma 9. *If the exact Equation (7) has a unique solution for each* $n\in\mathbb{Z}^+$ *the Riemann hypothesis follows.*

Proof. If the exact equation has a unique solution for each n, then the zeros obtained from its solutions on the `critical line` can be counted since they are enumerated by the integer n, leading to the counting function $N_0(t)$ in Equation (12). The number of solutions obtained on the `critical line` would saturate counting function of the number of solutions on the `critical strip` so that $N(t)=N_0(t)$ and thus all of the non-trivial zeros of ζ would be enumerated in this manner. If there are zeros off of the critical line, or zeros with multiplicity $m\geqslant 2$, then the exact Equation (7) would fail to capture all the zeros on the critical strip which would mean $N_0(t)<N(t)$. [1, IX] □

Corollary 10. *The Riemann hypothesis(RH) is not necesarily false if the exact Equation (7) does not have a unique solution for every* n*, since the solutions could still be on the critical line but not necessarily simple, that is, a root on the critical line could have multiplicity* $m\geqslant 2$ *and the RH would still be true.*

Corollary 11. *The Riemann hypothesis is true and all of the zeros on the critical line are simple if the exact Equation (7) has a unique solution for each* $n\in\mathbb{Z}^+$*. [1, IX]*

1.3 Iterated Function Systems

Definition 12. *A* `fixed-point` α *of a function* $f(x)$ *is a value* α *such that*

$$f(\alpha)=\alpha \tag{15}$$

[6, 3.]

Definition 13. *The* `multiplier` *of a fixed point* α *of a map* $f(x)$ *is equal to the absolute value of the derivative of the map evaluated at the point* α*.*

$$\lambda_f(\alpha)=|\dot{f}(\alpha)| \tag{16}$$

If $\lambda_f(\alpha)<1$ *then* α *is a said to be an* ***attractive fixed-point*** *of the map* $f(x)$*. If* $\lambda_f(\alpha)=1$ *then* α *is an* ***indifferent fixed point****, and if* $\lambda_f(\alpha)>1$ *then* α *is a* ***repelling fixed-point****. [6, 3.]*

Definition 14. *Let*

$$Y_{n,m}(t)=\begin{cases} t & m=0 \\ t+h_{n,m}\cos(\pi n)\tanh\left(\frac{Z(Y_{n,m-1}(t))}{\Omega(t)\prod_{k=1}^{n-1}\tanh(Y_{n,m-1}(t)-y_k)}\right) & m\geqslant 1\end{cases}$$

denote the m-th iterate of the n-th iteration function corresponding to the n-th zero of the Hardy Z function where

$$\Omega(t)=e^{\frac{3}{4}\sqrt{\frac{\log(t)}{\log(\log(t))}}} \tag{17}$$

is a lower bound for the running maximum of $|Z(s)|$

$$\max_{0\leqslant s\leqslant t}|Z(s)|>\Omega(t)\forall t\geqslant 45.590... \tag{18}$$

ensuring that

$$\frac{|Z(t)|}{\Omega(t)}>0\forall t\geqslant 45.590... \tag{19}$$

which normalizes the range of $Z(t)$ *which is known to grow in both maximum and average value as* $t\rightarrow\infty$ *and* $h_{n,m}$ *is factor which controls the rate of convergence*

$$h_{n,m}=\begin{cases} 1 & m\leqslant 2 \\ h_{n,m-1} & \text{sign}(\Delta Y_{n,m-2}(t))=\text{sign}(\Delta Y_{n,m-1}(t)) \\ \frac{h_{n,m-1}}{2} & \text{sign}(\Delta Y_{n,m-2}(t))\neq\text{sign}(\Delta Y_{n,m-1}(t))\end{cases} \tag{20}$$

where

$$\Delta Y_{n,m}(t)=Y_{n,m}(t)-Y_{n,m-1}(t) \tag{21}$$

is the 1*-st difference of the m-th iterate for the n-th zero. [5, Theorem 3.2.3]*

Lemma 15. *The roots of* $Z(t)$ *are fixed-points of* $Y_n(t)\forall n\in\mathbb{Z}^+$.

Proof. If $Z(t)=0$ then $\tanh\left(\frac{Z(t)}{\Omega(t)\prod_{k=1}^{n-1}\tanh(t-y_k)}\right)=\tanh\left(\frac{0}{\Omega(t)\prod_{k=1}^{n-1}\tanh(t-y_k)}\right)=\tanh(0)=0$ so that $Y_n(t)=t+\cos(\pi n)0=t+0=t$ when $Z(t)=0$. □

Theorem 16. $Y_{n,m}(t)$ *has indifferent fixed-points at each point* y_k *where* $k=1...n-1$

Proof. The product in the denominator $\prod_{k=1}^{n-1}\tanh(t-y_k)\rightarrow 0$ smoothly as t approaches any $y_k\in\bigcup_{k=1}^{n-1}y_k$ since $\tanh(0)=0$ and tanh is a smooth function. When any element of the product is zero the value of the product is zero regardless of the values of any other elements of the product. Since $\frac{1}{s}\rightarrow\infty$ as $s\rightarrow 0$ and $\tanh(|x|)\rightarrow 1$ as $|x|\rightarrow\infty$ we have $\tanh(\infty)=1$ and $\tanh(-\infty)=-1$ so that $Y_n(t)=t+\cos(\pi n)\forall t\in\bigcup_{k=1}^{n-1}y_k$. Since $Y_n(t)=t\pm 1\forall t\in\bigcup_{k=1}^{n-1}y_k$ when n is an integer, we see that $\frac{\mathrm{d}}{\mathrm{d}t}Y_n(t)=\frac{\mathrm{d}}{\mathrm{d}t}(t\pm 1)=1$ so that the multiplier $\lambda_{Y_n(t)}=\left|\frac{\mathrm{d}}{\mathrm{d}t}Y_n(t)\right|=1\forall t\in\bigcup_{k=1}^{n-1}y_k$. □

Proposition 17. *When n is an odd number,* $Y_n(t)$ *has attractive fixed-points at the odd-numbered roots* $y_{2k-1}\forall 2k-1\geqslant n$ *and repulsive fixed-points at the even-numbered roots* $y_{2k}\forall 2k\geqslant n$

Proposition 18. *When n is an even number, $Y_n(t)$ has attractive fixed-points at the even-numbered roots $y_{2k} \forall 2k \geqslant n$ and repulsive fixed-points at the odd-numbered roots $y_{2k-1} \forall 2k-1 \geqslant n$.*

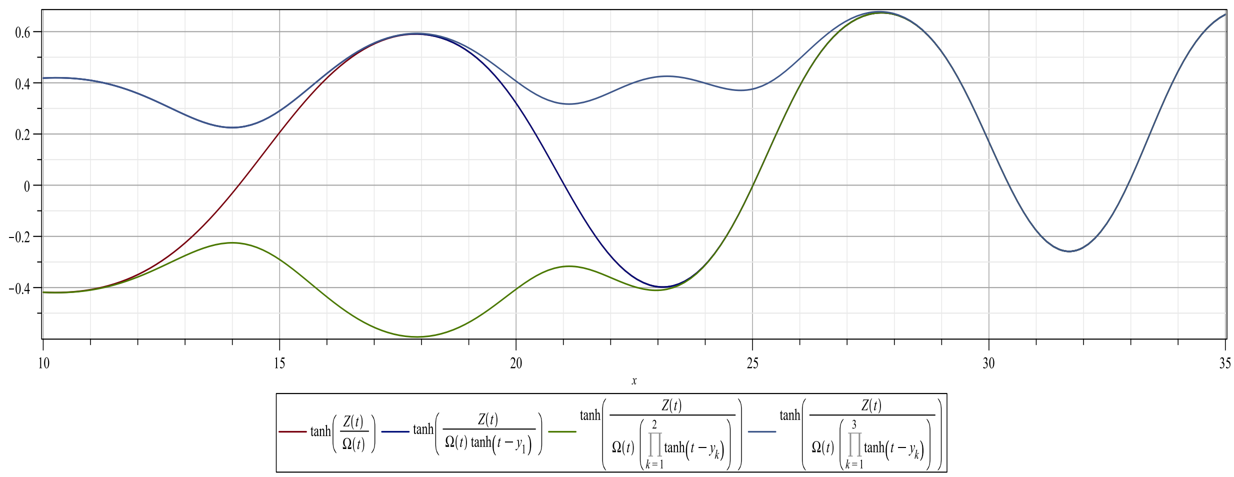

Figure 1. The functions which are subtracted or added to t to get $Y_1(t), Y_2(t), Y_3(t), Y_4(t)$. When n is odd $\cos(\pi n) = -1$ so that the the value is subtracted from t, when n is even $\cos(2\pi) = 1$ so it is added. It is plain to see that the curves $\tanh\left(\frac{Z(t)}{\Omega(t)\prod_{k=1}^{n-1}\tanh(t-y_k)}\right)$ do not cross the zero axis for any $t < y_n$

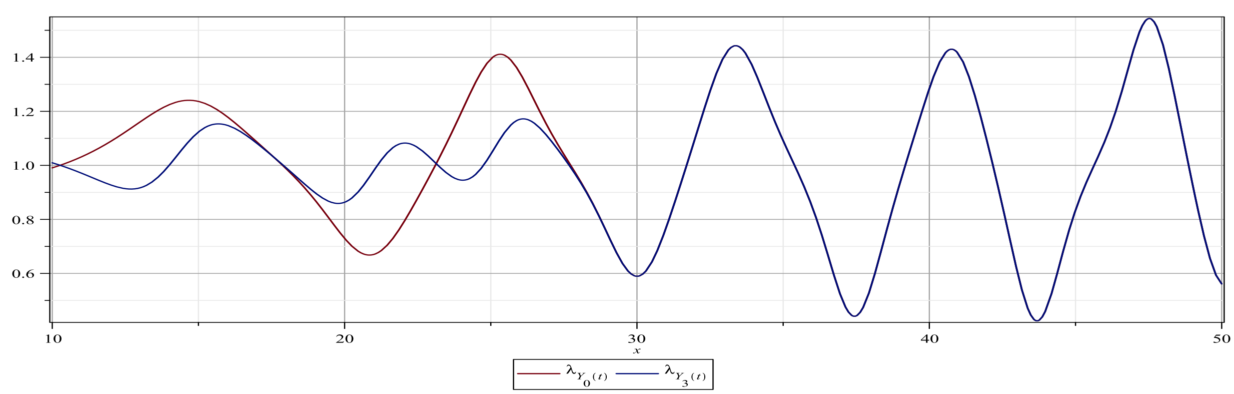

Figure 2. Multipler of the maps $Y_0(t)$ and $Y_3(t)$

Remark 19. The function $h_{n,m}$ is defined to be 1 when $1 \leqslant m \leqslant 2$. If $\operatorname{sign}(\Delta Y_{n,m-1}(t)) \neq \operatorname{sign}(\Delta Y_{n,m}(t))$ then $h_{n,m+1} = \frac{h_{n,m}}{2}$ so that the convergence rate is halved when the sign of the difference between successive iterates changes, indicating that it jumped across the root. This prevents the sequence generated by the iteration from getting stuck in an artifical 2-cycle and jumping back and forth across the root with equal magnitude indefinately when implementing this method with finite-precision arithmetic on a digital computer. Without this successive relaxation, the iterates still converge in theory however the number of iterations required could be several million or higher, while still having the difficulty of possibly getting stuck in a 2-cycle in computer implementations.

Theorem 20. *The Lipschitz constant M of the map $Y_{n,m}(t)$ is strictly less than 1 therefore $Y_{n,m}(t)$ is a contraction mapping*

$$|Y_{n,m}(t) - Y_{n,m}(s)| \leqslant M|t-s| \tag{22}$$

Proof. The Lipschitz constant of a continuous differentiable function $f(x)$ is equal to the maximum absolute value of its derivative

$$M = \sup_x \left| \frac{\mathrm{d}}{\mathrm{d}x} f(x) \right| \tag{23}$$

The derivative of $t - \tanh(t)$ is $\tanh(t)^2$. Since the maximum absolute value of $\tanh(t)$ is 1 then the maximum value of its square is also 1. Since $\Omega(t) > 1$ and $h_{n,m} \leqslant 1$ the derivative $\frac{\mathrm{d}}{\mathrm{d}t} Y_{n,m}(t)$ can never have an absolute value $\geqslant 1$ since that would require $\left| \tanh\left(\frac{Z(t)}{\Omega(t) \prod_{k=1}^{n-1} \tanh(t - y_k)} \right) \right| = 1$ which is only possible if $Z(t) = \pm\infty$ which is only the case when $t = \pm \frac{i}{2}$ which corresponds to the pole at $\zeta(1)$. Since $Z(t) \in \mathbb{R}$ when $t \in \mathbb{R}$ it can never be the case that $Z(t) = \infty$ so that $\left| \frac{\mathrm{d}}{\mathrm{d}t} Y_{n,m}(t) \right| \neq 1 \forall t \in \mathbb{R}$ and the Lipschitz constant M is strictly less than 1. □

Proposition 21. *The limit*

$$y_n = \lim_{m \to \infty} Y_{n,m}(s_n) \tag{24}$$

where

$$s_n = \begin{cases} 14 & n = 1 \\ 21 & n = 2 \\ \frac{y_{n-1} + y_{n-2}}{2} & n \geqslant 3 \end{cases} \tag{25}$$

exists and is equal to the n*-th zero of the Hardy Z function for all integer* $n \subset \mathbb{Z}^+$*. That is,* $Y_{n,m}(z_n)$ *forms a Cauchy sequence, due to the contraction mapping property proved in Theorem 20 whose elements are indexed by* m *converging to the* n*-th root* y_n *where the* n*-th starting point is defined to be half-way between the* $(n-2)$*-th and the* $(n-1)$*-th root* y_n *when* $n > 2$ *and equal to a point close to the first known zero at* 14.134.... *when* $n = 1$ *and a point close to the* 2nd *zero at* 21.022... *when* $n = 2$

Remark 22. The mid-way point between the nearest neighbors to the left of y_n is used as the starting point for the iteration since any point less than y_n and greater than e is within the immediate basin of attraction of y_n. The precise location of any roots y_p where $p < n$ cannot be used as a starting point since the map $Y_{p,m}(t)$ is a non-expansive mapping with Lipschitz constant precisely equal to 1 when $t \in \bigcup_{k=1}^{p-1} y_k$ so that the hyperbolic tangent has an argument of infinity resulting in a value of 1. Trajectories are neither attracted or repelled to any point $\bigcup_{k=1}^{n-1} y_k$ under the action of the map $Y_{n,m}(t)$ however, trajectories started precisely on any point $t \in \bigcup_{k=1}^{n-1} y_k$ will never attain a value other than t since any y_k is a fixed-point of $Y_{n,m}(t)$.

Note 23. The truth of Propositon 21 has been verified computationally up to $n = 628,737$ with a computer program which implements the methods described here using the arbitrary precision complex ball arithmiticlibrary arblib[3] and compares the results against the tables published by Andrew Odlyzko[4].

Theorem 24. *The limit of the Cauchy sequence* $\lim_{m\to\infty} Y_{n,m}(s_n)$ *will never converge to* y_{n+1} *since* y_n *is a repelling fixed-point of* $Y_{n,m}(t)$*.*

Proof. A sequence of iterated function compositions $f(x)$ will never converge to a repelling fixed-point of $f(x)$. □

Theorem 25. *The Cauchy sequence* $\lim_{m\to\infty} Y_{n,m}(s_n)$ *will never converge to any* y_k *where* $k < n$*.*

Proof. All y_k are indifferent fixed-points of $Y_{n,m}(t)$ and the trajectories generated by $Y_{n,m}(s_n)$ are never started from a point y_k since $s_n \notin \bigcup_{k=1}^{n-1} y_k$ and the only way $Y_{n,m}(t)$ would ″convege″ to an indifferent fixed-point is if it was started precisely on one, and s_n is by definition equal to the mid-point between successive y_n. □

Proposition 26. *The Cauchy sequence* $\lim_{m\to\infty} Y_{n,m}(s_n)$ *will never converge to any* y_k *where* $k > n$.

Note 27. We know that $Y_{n,m}(s_n)$ will never converge to y_q with q odd and n even nor to y_r with r even and n odd. It suffices to prove that $Y_{n,m}(s_n)$ will never cross the repulsive fixed-point at y_{n+1}.

Lemma 28. *The truth of Proposition 26 implies the existence of a soluion for each* n *to the exact Equation (7) since*

$$\begin{aligned} S(y_n) &= \left(n - \frac{3}{2}\right)\pi - \vartheta(y_n) \\ &= \left(n - \frac{3}{2}\right)\pi - \vartheta\left(\lim_{m\to\infty} Y_{n,m}(s_n)\right) \end{aligned} \tag{26}$$

therefore implies the truth of Proposition 21 and therefore Conjecture 7, the Riemann hypothesis.

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