A Sequence of Cauchy Sequences Convergent to Almost All the Riemann Zeta Zeros

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Abstract

A sequence of Cauchy sequences which converge to (almost all) the Riemann zeros is constructed.

1 Preliminary Outline

Let $\zeta(t)$ be the Riemann zeta function

$$\begin{aligned} \zeta(t) &= \sum_{n=1}^{\infty} n^{-s} & \forall \mathrm{Re}(s) > 1 \\ &= (1-2^{1-s}) \sum_{n=1}^{\infty} n^{-s} (-1)^{n-1} & \forall \mathrm{Re}(s) > 0 \end{aligned} \tag{1}$$

and $\vartheta(t)$ be Riemann-Siegel vartheta function $\vartheta(t)$

$$\vartheta(t) = -\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4}+\frac{it}{2}\right) - \ln\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\right) - \frac{\ln(\pi)\, t}{2} \tag{2}$$

The exact equation for the n-th Riemann zero t_n is [1, Equation 20]

$$\vartheta(t_n) + S(t_n) = \left(n - \frac{3}{2}\right)\pi \tag{3}$$

where $S(t)$ is the the normalized argument of $\zeta(t)$ on the critical line

$$\begin{aligned} S(t) &= \pi^{-1} \arg\left(\zeta\left(\frac{1}{2}+it\right)\right) \\ &= -\frac{i}{2\pi}\left(\ln\zeta\left(\frac{1}{2}+it\right) - \ln\zeta\left(\frac{1}{2}-it\right)\right) \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0} \mathrm{Im}\left(\ln\zeta\left(\frac{1}{2}+it+\varepsilon\right)\right) \end{aligned} \tag{4}$$

The approximate equation for the n-th zero can be obtained by dropping the argument term $S(t_n)$ and replacing ϑ with its Stirling expansion

$$\tilde{\vartheta}(t) = \frac{t}{2}\ln\left(\frac{t}{2\pi e}\right) - \frac{\pi}{8} \tag{5}$$

to get

$$\tilde{\vartheta}(\tilde{y}_n) = \left(n - \frac{3}{2}\right)\pi \tag{6}$$

whose exact solution is given by

$$\tilde{y}_n = \frac{2\pi\left(n-\frac{11}{8}\right)}{W\left(\frac{n-\frac{11}{8}}{e}\right)} \tag{7}$$

where $W(x)$ is the Lambert W function which is the solution to the equation

$$x = W(x)e^{W(x)} \tag{8}$$

Let $Z(t)$ denote the Hardy Z function[2] defined by

$$Z(t) \quad = e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2}+it\right) \tag{9}$$

which is real-valued when t is real and satisfies the identity

$$\zeta(t) = e^{-i\vartheta\left(\frac{i}{2}-it\right)} Z\left(\frac{i}{2}-it\right) \tag{10}$$

Let $Z(t,s)$ be the 1-reduced Hardy Z function,

$$Z(t,s) = \frac{Z(t)}{t-s} \tag{11}$$

The function $Z(t,s)$ is called 1-reduced because if $Z(s)=0$ then $\lim_{t\to s} Z(t,s)\neq 0$ since the denominator $t-s$ approaches 0 faster than the numerator $Z(t)$ as $t\to s$. In that way, the roots of $Z(t)$ and $Z(t,s)$ are identical except for s which cannot be a root of $Z(t,s)$. This nomenclature and technique is inspired by the method of deflating or reducing polynomials described in [5].

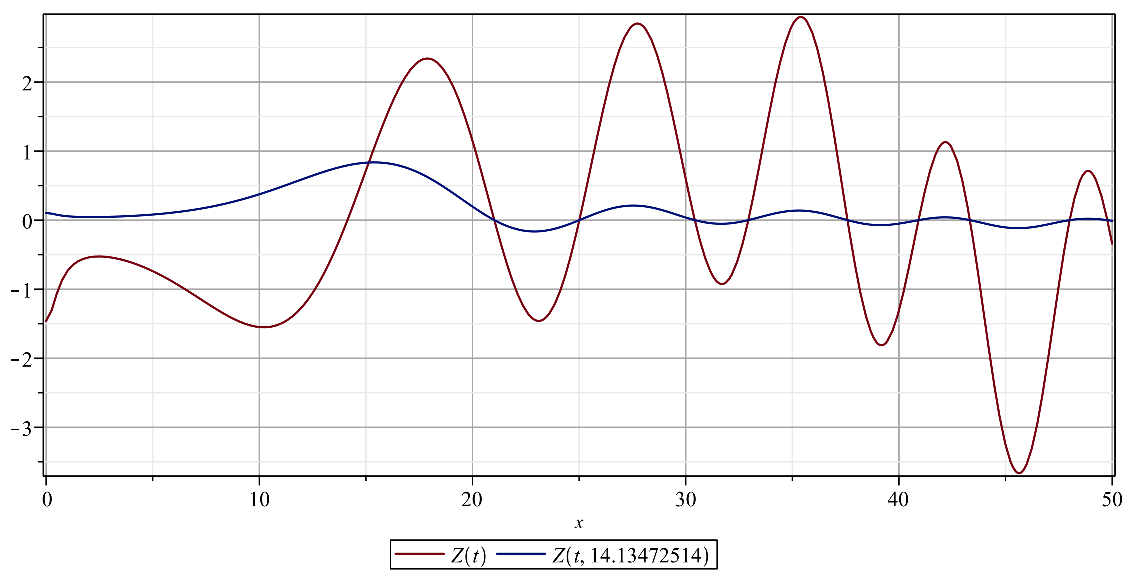

Figure 1. $Z(t)$ and $Z(t)$ with the first root removed

Let

$$\Omega(t) = e^{\frac{3}{4}\sqrt{\frac{\log(t)}{\log(\log(t))}}} \tag{12}$$

then it is known that

$$\max_{0\leqslant s\leqslant t}|Z(s)|>\Omega(t)\forall t>e \tag{13}$$

which ensures that

$$\frac{|Z(t)|}{\Omega(t)}>0\forall t>e \tag{14}$$

See [4, Theorem 3.2.3] Now define the iteration function for the n-th zero

$$C_{n,m}(t)=\begin{cases} t & m=0\\ t-h_{n,m}\frac{\tanh(Z(C_{n,m-1}(t)))}{\Omega(t)} & n=1\\ t-h_{n,m}\frac{\tanh(Z(C_{n,m-1}(t),y_{n-1}))}{\Omega(t)} & n>1 \text{ is odd and } m>0\\ t+h_{n,m}\frac{\tanh(Z(C_{n,m-1}(t),y_{n-1}))}{\Omega(t)} & n>1 \text{ is even and } m>0 \end{cases} \tag{15}$$

where the relaxation factor $h_{n,m}$ is defined by

$$h_{n,m}=\begin{cases} 1 & m\leqslant 1\\ h_{n,m-1} & \operatorname{sign}(\Delta C_{n,m-2}(t))=\operatorname{sign}(\Delta C_{n,m-1}(t))\\ \frac{h_{n,m-1}}{2} & \operatorname{sign}(\Delta C_{n,m-2}(t))\neq\operatorname{sign}(\Delta C_{n,m-1}(t)) \end{cases} \tag{16}$$

where

$$\Delta C_{n,m}(t)=C_{n,m}(t)-C_{n,m-1}(t) \tag{17}$$

is the 1-st difference of the m-th iterate for the n-th zero.

Remark 1. The function $h_{n,m}$ is defined to be 1 when $m=0$. If $\operatorname{sign}(\Delta C_{n,m-1}(t))\neq\operatorname{sign}(\Delta C_{n,m}(t))$ then $h_{n,m+1}=\frac{h_{n,m}}{2}$. That is, the convergence speed is cut in half when the sign of the difference between successive iterates changes, indicating that it jumped across the root. This prevents the iteration from getting stuck in an artifical 2-cycle and jumping back and forth across the root with equal magnitude indefinately when implementing this method with finite-precision arithmetic on a digital computer. Without this successive relaxation, the iterates still converge in theory however the number of iterations required could be several million or higher, while still having the difficulty of possibly getting stuck in a 2-cycle in computer implementations.

Theorem 2. *The Lipschitz constant of the map $C_{n,m}(t)$ is equal to 1 therefore $C_{n,m}(t)$ is a non-expansive mapping*

$$|C_{n,m}(t)-C_{n,m}(s)|\leqslant|t-s| \tag{18}$$

Proof. The Lipschitz constant of a continuous differentiable function $f(x)$ is equal to the maximum absolute value of its derivative

$$\sup_x\left|\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right| \tag{19}$$

The derivative of $t-\tanh(t)$ is $\tanh(t)^2$. Since the maximum absolute value of $\tanh(t)$ is 1 then the maximum value of its square is also 1. Since $\Omega(t)>1$ and $h_{n,m}\leqslant 1$ the derivative $\frac{\mathrm{d}}{\mathrm{d}t}C_{n,m}(t)$ can never have an absolute value greater than 1. □

Definition 3. *The multiplier of a fixed point α of a map $f(\alpha) = \alpha$ is equal to the absolute value of the derivative of the map evaluated at the point α.*

$$\lambda_f(\alpha) = |\dot{f}(\alpha)| \tag{20}$$

If $\lambda_f(\alpha) < 1$ then α is a said to be an attractive fixed-point of the map $f(x)$. If $\lambda_f(\alpha) = 1$ then α is an indifferent fixed point, and if $\lambda_f(\alpha) > 1$ then α is a repelling fixed-point.

Proposition 4. *When n is odd, the odd-numbered zeros are attractive fixed-points of the map $C_{n,m}(t)$ and the even numbered zeros are repelling fixed-points. The converse is also true, when n is even, the even-numbered zeros are attractive fixed-points of $C_{n,m}(t)$ and the odd numbered zeros are repelling fixed-points.*

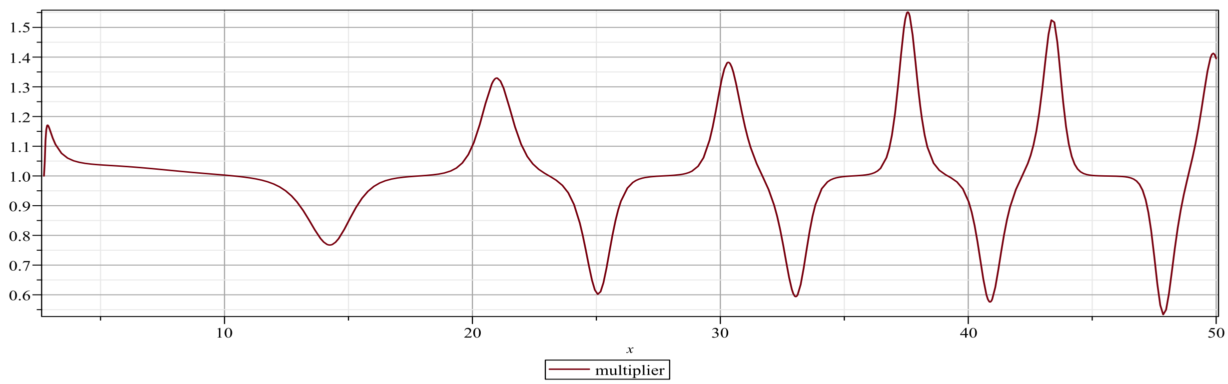

Figure 2. Multipler $\lambda_f(t) = |\dot{f}(t)|$ of the map $f(t) = t - \frac{\tanh(Z(t))}{\Omega(t)}$

Proposition 5. *The limit*

$$y_n = \lim_{m\to\infty} C_{n,m}(z_n) \tag{21}$$

exists and is equal to the n-th zero of the Hardy Z function for all integer $n \geqslant 1$. That is, $C_{n,m}(z_n)$ forms a Cauchy sequence whose elements are indexed by m converging to the n-th root y_n where the n-th starting point is defined to be half-way between the $(n-1)$-th zero y_{n-1} and the n-th approximation zero $\tilde{y}_n$

$$\begin{aligned} z_n &= \begin{cases} n=1 & \tilde{y}_1 \\ n>1 & \frac{y_{n-1}+\tilde{y}_n}{2} \end{cases} \\ &= \begin{cases} n=1 & \tilde{y}_1 \\ n>1 & \frac{1}{2}\left(\lim_{m\to\infty}\left(C_{n-1,m}(z_n) + \frac{2\pi\left(n-\frac{11}{8}\right)}{W\left(\frac{n-\frac{11}{8}}{e}\right)}\right)\right) \end{cases} \end{aligned} \tag{22}$$

Note 6. Proposition 5 is true only up to level $1 \leqslant n \leqslant 580,492$ because $\lim_{m\to\infty} C_{580493,m}(z_n) = y_{580491} \neq y_{580493}$ since the iteration converges at that point to an already-found fixed-point due to the varying magnitudes of the attractive fixed-points, the zero on the left attracts the trajectories more strongly than the zero corresponding to the starting point. However, if one considers the iteration function

$$Y_n(t) = t - \frac{\tanh(Z(t))}{\Omega(t)\prod_{k=1}^{n}\tanh(t-y_k)} \tag{23}$$

then it would be impossible for the iteration to converge to any root indexed by $m \leqslant n$ so that the starting point of the iteration could be sufficiently far to the left so that it converges to the first attractive fixed-point $m > n$. It should be noted that the majority of the terms of the product in the denominator of $Y_n(t) \sim 1$ since $\tanh(2.646652412\ldots) = 0.99$ indicating that already found zeros sufficiently far away from t can be excluded from the product.

Remark 7. The iterates do not always converge to the n-th zero if starting from $\tilde{y}_n$ instead of z_n because sometimes $\tilde{y}_n > y_{n+1}$ and the iterates cannot cross the barrier and hence converges to the wrong zero. The $(n+1)$-th zero cannot be removed because its precise location is not yet known when converging to y_n.

Note 8. The truth of Propositon 5 has been verified up to $n = 580, 492$ with a computer program which implements the methods described here with arbitrary precision complex ball arithmitic and compares the results against the tables published by Andrew Odlyzko[3].

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