Solving Coupled Riccati ODEs as Solution of Incompressible Non-Stationary 3D Navier-Stokes Equations

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Abstract

In a recent paper, Ershkov derived a system of two coupled Riccati ODEs as solution of nonstationary 3D Navier-Stokes equations. Now in this paper, we will solve these coupled Riccati ODEs using Maxima computer algebra package. The result seems to deserve further investigation in particular for finding nonstationary Navier-Stokes equations for real fluid.

Introduction

The Riccati equation, named after the Italian mathematician Jacopo Francesco Riccati, is a basic first-order nonlinear ordinary differential equation (ODE) that arises in different fields of mathematics and physics.[4]

In fluid mechanics, there is an essential deficiency of the analytical solutions of Navier–Stokes equations for 3D case of non-stationary flow. The Navier-Stokes system of equations for incompressible flow of Newtonian fluids should be presented in the Cartesian coordinates as below (under the proper initial conditions):[1]

$$\nabla . \vec{u} = 0, \tag{1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla p}{\rho} + \nu \cdot \nabla^2 \vec{u} + \vec{F}, \tag{2}$$

Where u is the flow velocity, a vector field; ρ is the fluid density, p is the pressure, v is the kinematic viscosity, and F represents external force (per unit mass of volume) acting on the fluid.[1]

In ref. [1], Ershkov explores the ansatz of derivation of non-stationary solution for the Navier–Stokes equations in the case of incompressible flow, which was suggested earlier. In general case, such a solution should be obtained from the mixed system of 2 coupled Riccati ordinary differential equations (in regard to the time-parameter t). But instead of solving the problem analytically, we will try to find a numerical solution.

The coupled Riccati ODEs read as follows:[1]

$$a' = \frac{w_y}{2} \cdot a^2 - (w_x \cdot b) \cdot a - \frac{w_y}{2}(b^2 - 1) + w_z \cdot b, \tag{3}$$

$$b' = -\frac{w_x}{2} \cdot b^2 - (w_y \cdot a) \cdot b - \frac{w_x}{2}(a^2 - 1) + w_z \cdot a \tag{4}$$

We are going to rewrite the above coupled equations in Maxima language.

Computer algebra solution

The above coupled Riccati ODEs (1) and (2) can be rewritten as follows:[3]

$$a(t)' = \frac{v}{2} \cdot a(t)^2 - (u \cdot b(t)) \cdot a(t) - \frac{v}{2}(b(t)^2 - 1) + w \cdot b(t), \tag{5}$$

$$b(t)' = -\frac{u}{2} \cdot b(t)^2 - (v \cdot a(t)) \cdot b(t) - \frac{u}{2}(a(t)^2 - 1) + w \cdot a(t) \tag{6}$$

Maxima expression of coupled Riccati ODEs (1) and (2) are as follows:[3]

‘diff(a(t),t)=v/2*a(t)^2-(u*b(t))*a(t)-v/2*(b(t)^2-1)+w*b(t), (7)

‘diff(b(t),t)=-u/2*b(t)^2-(v*a(t))*b(t)-u/2*(a(t)^2-1)+w*a(t). (8)

The Maxima results are as shown below:

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(%i3) 'diff(a(t),t)=v/2*a(t)^2-(u*b(t))*a(t)-v/2*(b(t)^2-1)+w*b(t);
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$$(\%o3)\quad \frac{d}{d\,t}\,a(t)=b(t)\,w-\frac{(b(t)^2-1)\,v}{2}+\frac{a(t)^2\,v}{2}-a(t)\,b(t)\,u$$

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(%i4) 'diff(b(t),t)=-u/2*b(t)^2-(v*a(t))*b(t)-u/2*(a(t)^2-1)+w*a(t);
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$$(\%o4)\quad \frac{d}{d\,t}\,b(t)=a(t)\,w-a(t)\,b(t)\,v-\frac{b(t)^2\,u}{2}-\frac{(a(t)^2-1)\,u}{2}$$

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(%i5) desolve([%o3,%o4],[a(t),b(t)]);

(%o5) [a(t)=ilt(-(

((laplace(b(t)^2,t,g34120)-laplace(a(t)^2,t,g34120))*v+2*laplace(a(t)*b(t),t,g34120)*u-2*a(0))*

g34120^2+(

(2*laplace(a(t)*b(t),t,g34120)*v+(laplace(b(t)^2,t,g34120)+laplace(a(t)^2,t,g34120))*u-2*b(0))*

w-v)*g34120-u*w)/(2*g34120^3-2*w^2*g34120),g34120,t),b(t)=ilt(-(

(2*laplace(a(t)*b(t),t,g34120)*v+(laplace(b(t)^2,t,g34120)+laplace(a(t)^2,t,g34120))*u-2*b(0))*

g34120^2+(

((laplace(b(t)^2,t,g34120)-laplace(a(t)^2,t,g34120))*v+2*laplace(a(t)*b(t),t,g34120)*u-2*a(0))*

w-u)*g34120-v*w)/(2*g34120^3-2*w^2*g34120),g34120,t)]
```

Concluding remarks

Using Maxima package we solve the two coupled Riccati ODEs as solution of non-stationary 3D Navier-Stokes equations.

However, we admit that the obtained computer solution is not easily plotted graphically using Maxima, therefore it is advisable to verify this result with other computer algebra packages, such as Maple or Mathematica.

References:

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