Exact quantum mechanics of quadratic Liénard type oscillator equations with bound states energy spectrum

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Abstract

The quantization of second order dissipative dynamical systems is well known to be a complicated Sturm-Liouville problem. This work is devoted to the exact quantization of a given quadratic Liénard type oscillator equation. The bound state solutions of the resulting Schrödinger equation in terms of associated Laguerre polynomials and the possibility to recover the energy spectrum of the quantum harmonic oscillator are discussed following the specific values of system parameters, using the Nikiforov-Uvarov method.

Keywords: Quadratic Liénard equation, Schrödinger equation, bound state solutions, quantum mechanics, Laguerre polynomials.

1. Introduction

Many problems in physics and science were found to be adequately solved by considering the harmonic oscillator with position-dependent mass [1], so that the study of classical and quantum harmonic oscillator with a spatially varying mass has fast become an attrative research field of the mathematical physics. Numerous applications in various areas of engineering have been developped on the basis of harmonic oscillator with position dependent mass [2]. However, exact analysis is often hard to be carried out, and most research contributions are limited to the approximate and numerical investigations of differential equations governing the classical as well as quantum features of systems with position-dependent mass. The quadratic Liénard type differential equations constitute an important class of position-dependent mass oscillators, since it allows a more satisfactory description of nonlinear dissipative dynamical systems [3-6]. In this context, it appears reasonable to be interested to the problem of finding exact quantum mechanics of quadratic Liénard type oscillator equations. More again, a high interest should be accorded to exact quantum mechanics of classical quadratic Liénard type oscillator equations having exact harmonic periodic solutions, since such nonlinear dissipative oscillator studies are known to be rare in the mathematical physics literature. Analytical quantum mechanics of quadratic Liénard type equations leads to solve in general a complicated Schrödinger equation due to the quadratic term in velocity. In [7] the exact eigenfunctions are expressed in terms of associated Legendre functions and Gegenbauer polynomials. The position-dependent mass Schröndinger equation in [8-10] is analytically solved in terms of prolate spheroidal wave functions. In [11] the eigensolutions of the Schrödinger wave equation with position-dependent mass are exactly formulated as the prolate spheroidal wave functions. However, as before mentioned, it is not difficult to notice that few works about exact quantization of quadratic Liénard type equations are available in the literature. Recently in [12], it is shown for the first time the existence of a family of quadratic Liénard type nonlinear equations

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + \omega^2 x \exp(2\gamma\varphi(x)) = 0 \tag{1}$$

which admits an exact trigonometric periodic solution but with amplitude dependent frequency, where the dot over a symbol stands for differentiation with respect to time and the prime denotes differentiation with respect to x. The choice $\varphi(x) = \frac{1}{2}\ln(1+\mu x)$, yields the nonlinear differential equation

$$\ddot{x} - \frac{\gamma\mu}{2(1+\mu x)}\dot{x}^2 + \omega^2 x(1+\mu x)^{\gamma} = 0 \tag{2}$$

This equation may be regarded as a nonlinear oscillator equation with a quadratic dissipative term. The exact harmonic periodic solution may be written

$$x(t) = A_0 \sin\phi(t) \tag{3}$$

where

$$\phi(t) = \omega\tau + \theta_0, \qquad \frac{d\tau}{dt} = [1+\mu A_0 \sin\phi(t)]^{\frac{\gamma}{2}} \tag{4}$$

so that

$$\omega(t-t_0)=\int_{\phi_0}^{\phi}[1+\mu A_0\sin\phi(t)]^{-\frac{\gamma}{2}}d\phi \tag{5}$$

where t_0 is a constant of integration.

For $\gamma=0$, or $\mu=0$, (2) reduces to the linear harmonic oscillator equation with well known trigonometric solution, so that the parameter $\omega=\omega_o$, becomes the natural frequency in this situation. In such a context a problem to investigate may be the exact quantization of (2) for a fixed γ and arbitrary μ, or conversely, for a fixed μ and arbitrary γ in order to analyze the effects of nonlinearity on the discrete bound state eigensolutions and energy spectrum of the quantum harmonic oscillator. To be more precise, the question to be answered in this work may be addressed as follows: Can we perform the exact quantization of the equation (2) in terms of classical orthogonal polynomials with discrete energy spectrum? The present work postulates that the classical equation (2) may be exactly quantized in terms of discrete bound states in order to study the nonlinearity effects. This prediction is of a great interest since the energy spectrum of the quantum harmonic oscillator may be recovered as a limiting case of that of (2) and conditions for obtaining discrete bound states solutions with negative energy spectrum may be stated. The present formulation in terms of associated Laguerre polynomials of the eigensolutions of the Schrödinger wave equation is theoretically and practically interesting since the hypergeometric type polynomials are deeply and intensively studied from mathematical as well as physical and numerical standpoint [13]. The associated Laguerre polynomials are well known in quantum mechanics since they arise in the bound state solutions of the radial part of the Schrödinger equation for the hydrogen atom [13]. Exact solutions are physically important since they will enable to better understand and capture analytically interesting features of the quantum system under question and are also well convenient for engineering calculations. Several mathematical techniques like point-canonical transformations, supersymmetric quantum theory and Nikiforov-Uvarov approach, are used to analytically solve the position-dependent mass Schrödinger equations [14]. The Nikiforov-Uvarov theory [15] has however, the advantage to be a coherent solving process and suitably transform the second order linear Schrödinger equation into a hypergeometric type differential equation, so that it ensures the bound state eigensolutions to be expressed in terms of classical hypergeometric type polynomials and the necessary conditions to obtain the associated energy spectrum. That being so, to demonstrate the preceding prediction, it is suitable to first establish the appropriate Schrödinger equation with position-dependent mass associated to the equation (2) (section 2), and secondly perform the solution using the Nikiforov-Uvarov method [15] (section 3). Finally the predicted results are discussed (section 4) and a conclusion is given for the developed work.

2. Schrödinger equation

The one dimensional Schrödinger differential equation requires the knowledge of the Hamiltonian associated to (2). Usually the Hamiltonian operator is derived from the classical Hamiltonian. As regards the equation (2) the mass distribution function may be written

$$M(x) = m_0 e^{-\mu\gamma\int\frac{1}{1+\mu x}dx}$$

or

$$M(x) = m_0(1+\mu x)^{-\gamma} \tag{6}$$

where m_0 is the integration constant, so that the potential energy

$$V(x) = \omega^2\int xM(x)(1+\mu x)^{\gamma}\,dx$$

becomes

$$V(x) = \frac{1}{2}m_0\omega^2 x^2 \tag{7}$$

In this perspective the classical Hamiltonian

$$H = \frac{p^2}{2M(x)} + V(x)$$

with $p = M(x)\dot{x}$, reads

$$H = \frac{1}{2}p^2(1+\mu x)^{\gamma} + \frac{1}{2}m_0\omega^2 x^2 \tag{8}$$

which is not invariant Hamiltonian such that the associated Hamiltonian operator is not Hermitian for $\gamma \neq 0$. In such a case the momentum and position operators do no longer commute. To overcome this difficulty, one may use the von Roos quantum Hamiltonian formulation [16] to write the Schrödinger eigenvalue problem.

2.1. Schrödinger equation with mass M(x)

In the literature various forms of Hamiltonian related to the von Roos formulation [17]

$$H=-\frac{\hbar}{4}\left[M(x)^{a}\partial_{x}M(x)^{b}\partial_{x}M(x)^{c}+M(x)^{c}\partial_{x}M(x)^{b}\partial_{x}M(x)^{a}\right]+V(x) \qquad (9)$$

where the ambiguity parameters a, b and c, should satisfy $a+b+c=-1$, in order to render H Hermitian are used. Indeed, there is no law to fix the value of these parameters for a specific system of interest. So, a judicious choice of these parameters consists of a prerequisite for an adequate Schrödinger equation satisfying the expected performance objective.

The requirement that it is desired to express the Schrödinger equation solution in terms of hypergeometric type polynomials involves to adequately solving the ambiguity parameters problem. To successfully perform this task, the set of parameters is chosen such that the Schrödinger equation becomes [11]

$$\psi''(x)-\frac{M'(x)}{M(x)}\psi'(x)+\frac{2M(x)}{\hbar^{2}}\left[E-V(x)\right]\psi(x)=0$$

where E denotes the energy eigenvalue, $\psi(x)$ the wave function, and the prime means derivative with respect to x. Let us now precise the Schrödinger equation of interest.

2.2. Schrödinger equation under study

As the mass function $M(x)=\frac{m_0}{(1+\mu x)^{\gamma}}$ and the potential energy $V(x)=\frac{1}{2}m_0\omega^2x^2$, for the equation (2), the preceding Schrödinger equation reduces, for $m_0=\hbar=1$, to

$$\psi''(x)+\frac{\mu\gamma}{1+\mu x}\psi'(x)+\left[2E-\omega^2x^2\right](1+\mu x)^{-\gamma}\psi(x)=0 \qquad (10)$$

The equation (10) constitutes the Schrödinger wave equation with variable coefficients related to the classical quadratic Liénard type oscillator equation (2). This equation consists of a Sturm-Liouville eigenvalue problem which may be exactly solved using the Nikiforov-Uvarov (NU) method. The solution of (10) clearly depends on the value of parameter γ. In this contribution the solution of equation (10) will be investigated under $\gamma=2$, in order to attain the fixed objective. To do this, the mathematical problem to solve should be clearly stated. Let us consider the equation (10) on the semi-infinite interval $[0,+\infty[$ with $\omega \geq 0$. The problem of interest can then be formulated as follows. Find the bound state solutions $\psi_n(x)$ and associated energy eigenvalues E_n for the Schrödinger wave equation

$$\frac{d^2\psi(x)}{dx^2}+\frac{2\mu}{1+\mu x}\frac{d\psi(x)}{dx}+\left[\frac{2E-\omega^2x^2}{(1+\mu x)^2}\right]\psi(x)=0 \tag{11}$$

, which remains finite for $x=0$ and $x\to+\infty$, that is $\psi(x)\to 0$ for $x\to 0$ and $x\to+\infty$.

3. Exact bound state solutions to Schrödinger equation

The exact solution of Schrödinger equation (11) with $\gamma=2$, is exhibited in this section using as before mentioned the Nikiforov-Uvarov approach.

3.1. Discrete energy spectrum

The exact solution of (11) under the boundary conditions previously mentioned may suitably computed using the Nikiforov-Uvarov method. By application of the Nikiforov-Uvarov approach [15] the requirement is that the Schrödinger wave equation (11) should be written as

$$\psi''(x)+\frac{\tilde{\tau}(x)}{\sigma(x)}\psi'(x)+\frac{\tilde{\sigma}(x)}{\sigma(x)^2}\psi(x)=0 \tag{12}$$

with

$$\psi\ (x)=\phi(x)\,y_n(x) \tag{13}$$

where $y_n(x)$ becomes the solution of the hypergeometric type differential equation

$$\sigma(x)y_n''(x)+\tau(x)y_n'(x)+\lambda y_n(x)=0$$

and

$$\frac{\phi'(x)}{\phi(x)}=\frac{\pi(x)}{\sigma(x)} \tag{14}$$

so that $\sigma(x)$ and $\tilde{\sigma}(x)$ are polynomials at most of second order degree, $\tau(x)$ and $\tilde{\tau}(x)$ are polynomials at most of first degree, λ is a constant, and

$$\pi(x) = (\frac{\sigma'(x) - \tilde{\tau}(x)}{2}) \pm \sqrt{\left(\frac{\sigma'(x) - \tilde{\tau}(x)}{2}\right)^2 - \tilde{\sigma}(x) + k\sigma(x)} \qquad (15)$$

The function $\pi(x)$ is a polynomial of degree at most one such that

$$\tau(x) = \tilde{\tau}(x) + 2\pi(x) \qquad (16)$$

$$k = \lambda - \pi'(x) \qquad (17)$$

and

$$\lambda = \lambda_n = -n\tau'(x) - \frac{n(n-1)}{2}\sigma''(x)\ ,\ n = 0,1,2,3,... \qquad (18)$$

The hypergeometric-type function $y_n(x)$ defined as a polynomial of degree n is given by the Rodrigues formula

$$y_n(x) = \frac{A_n}{\rho(x)} \frac{d^n}{dx^n}\left[\sigma(x)^n \rho(x)\right] \qquad (19)$$

such that the weight function $\rho(x)$ obeys

$$\frac{d}{dx}\left[\sigma(x)\rho(x)\right] = \tau(x)\rho(x) \qquad (20)$$

and A_n is normalization constant.

With the following definitions

$\tilde{\tau}(x) = 2\mu,\ \tilde{\sigma}(x) = 2E - \omega^2 x^2,$ and $\sigma(x) = 1 + \mu x$

the function $\pi(x)$ which satisfies the requirement that the derivative of $\tau(x)$ should be negative may be written as

$$\pi(x) = -\frac{1}{2}\mu - \omega x - \frac{\omega}{\mu} + \frac{\sqrt{4\omega^2 + \mu^2(\mu^2 - 8E)}}{2\mu} \qquad (21)$$

with

$$k = \frac{2\omega^2 \pm \omega\sqrt{4\omega^2 + \mu^2(\mu^2 - 8E)}}{\mu^2} \qquad (22)$$

and

$$\tau(x) = \mu - 2\omega x - \frac{2\omega}{\mu} + \frac{\sqrt{4\omega^2 + \mu^2(\mu^2 - 8E)}}{\mu} \tag{23}$$

Comparing the equations (17) and (18) one may deduce

$$k = \omega(2n+1) \tag{24}$$

so that the desired discrete energy eigenvalues become

$$E_n = \omega(n + \frac{1}{2}) - \frac{1}{2}n(n+1)\mu^2 \;,\; n = 0,\ 1,\ 2, 3, 4, \ldots \tag{25}$$

3.2. Discrete wave functions

The substitution of $\sigma(x)$ and $\tau(x)$ into (20) yields

$$\rho(x) = e^{-\frac{2\omega}{\mu}x}(1+\mu x)^{2n+1-\frac{2\omega}{\mu^2}} \tag{26}$$

so that

$$y_n(x) = A_n(1+\mu x)^{\frac{2\omega}{\mu^2}-(2n+1)} e^{\frac{2\omega}{\mu}x} \frac{d^n}{dx^n}[(1+\mu x)^{3n+1-\frac{2\omega}{\mu^2}} e^{-\frac{2\omega}{\mu}x}] \tag{27}$$

and

$$\phi(x) = e^{-\frac{\omega}{\mu}x}(1+\mu x)^{n-\frac{\omega}{\mu^2}} \tag{28}$$

with A_n a normalization constant. Thus the non-normalized wave function $\psi_n(x)$ may be written as

$$\psi_n(x) = A_n\, e^{-\frac{\omega}{\mu}x}(1+\mu x)^{n-\frac{\omega}{\mu^2}}(1+\mu x)^{\frac{2\omega}{\mu^2}-(2n+1)} e^{\frac{2\omega}{\mu}x} \frac{d^n}{dx^n}[(1+\mu x)^{3n+1-\frac{2\omega}{\mu^2}} e^{-\frac{2\omega}{\mu}x}] \tag{29}$$

where μ is a constant. The wave function $\psi_n(x)$ may also be written in the form

$$\psi_n(x) = C_n e^{-\frac{\omega}{\mu^2}(1+\mu x)}(1+\mu x)^{n-\frac{\omega}{\mu^2}} L_n^{2n+1-\frac{\omega}{\mu^2}}(1+\mu x) \tag{30}$$

where $L_n^{2n+1-\frac{\omega}{\mu^2}}(1+\mu x)$ designates the associated Laguerre polynomial, and C_n the new normalization constant.

4. Numerical results using Matrix Diagonal Method

In this section, the matrix diagonalisation method is presented to cross check the previous analytical calculation [18]. Let us consider the Hamiltonian

$$H = \frac{1}{2}\left[p(1+\mu x)^2 p + x^2\right] \tag{31}$$

in place of equation (8) with $\gamma = 2$, $m_0 = 1$, and $\omega = 1$. The very purpose of writing the Hamiltonian in this form is that it must be invariant with reference to exchange of position and momentum part [18, 19]. Here the eigenvalue relation is solved as

$$H|\psi\rangle = E|\psi\rangle \tag{32}$$

where

$$\psi = \sum A_m |m\rangle \tag{33}$$

and $|m\rangle$ is the mth state harmonic oscillator eigenfunction satisfying the relation

$$\left[p^2 + x^2\right]|m\rangle = (2m+1)|m\rangle \tag{34}$$

Now, the following recursion relation is solved [19, 20].

$$P_m A_{m-4} + Q_m A_{m-2} + R_m A_{m-1} + S_m A_m + T_m A_{m+1} + U_m A_{m+2} + V_{m+4} = 0 \tag{35}$$

Here

$$P_m = \langle m|H|m-4\rangle \tag{36}$$

$$Q_m = \langle m|H|m-2\rangle \tag{37}$$

$$R_m = \langle m|H|m-1\rangle \tag{38}$$

$$T_m = \langle m|H|m+1\rangle \tag{39}$$

$$U_m = \langle m|H|m+2\rangle \tag{40}$$

$$V_m = \langle m|H|m+4\rangle \tag{41}$$

and

$$S_m = m + \frac{1}{2} + \frac{\mu^2}{8}\left(2m^2 + 2m + 3\right) - E \tag{42}$$

It should be remembered that $m - k \geq 0$ for $k = 2,4$. Further one has to carefully go step by step to achieve desired convergency in eigenvalues. The eigenvalues for different $\mu = 0.25; 0.5$, are tabulated in table-1.

Table-1: First four eigenvalues of $H = \frac{1}{2}[p(1+\mu x)^2 p + x^2]$.

μ	Numerical Results using MDM	Analytical Results using Eq.(25)
0.25	0.5 1.4375 2.3125 3.125	0.5 1.4375 2.3125 3.125
0.5	0.5 1.25 1.75 2.020942	0.5 1.25 1.75 2.020942

Further in order to make study complete, $|\psi_n|^2$ is plotted in Figure-1 for $\mu = 0.25$.

5. Discussion

The Schrödinger equation with position-dependent mass has shown a more adequate ability to describe the quantum features of a rich variety of physical systems. In this work the exact quantum mechanics of a quadratic Liénard type oscillator equation that exhibits exact harmonic periodic solutions is performed. More precisely, the exact quantization of harmonic potential with position-dependent mass has been carried out. By application of the Nikiforov-Uvarov approach, the discrete eigensolutions and the corresponding energy eigenvalues are obtained. The eigensolutions are expressed in function of associated Laguerre polynomials. The discrete bound state solutions with negative energy eigenvalues, according to (25) are ensured when the nonlinearity parameter $\mu \succ \omega$. As $\mu \to 0,$ $\omega = \omega_0$, as previously mentioned, and the classical quadratic Liénard type equation (2) reduces to the classical linear harmonic oscillator so that one may notice that the energy eigenvalues E_n reduce as expected to those of the quantum harmonic oscillator, viz

$$E_n = (n + \frac{1}{2})\omega_0$$

For $n = 0,$ the equation (25) shows that the ground state energy is that of the quantum harmonic oscillator. If the Hermite polynomials can easily be transformed into Laguerre polynomials, conversely the transformation of Laguerre polynomials into Hermite polynomials is not generally possible, so that investigating the limit as $\mu \to 0$ of the current wave function $\psi_n(x)$ in terms of the Hermite polynomials is not simple to be performed. It is worth to mention that numerical results match very well analytical predictions.

6. Conclusion

The exact quantization of a quadratic Liénard type oscillator equation having exact harmonic periodic solution but with amplitude dependent frequency is developed in this work. The discrete bound state solutions to the resulting Schrödinger equation are expressed as a function of associated Laguerre polynomials. The associated discrete negative energy eigenvalues are found to be ensured by the magnitude of the nonlinearity parameter using the Nikiforov-Uvarov theory. The work shows that the discrete energy spectrum of the quantum harmonic oscillator may be recovered for the zero value of the nonlinearity parameter. The numerical results are found to be in consistent agreement with analytical predictions.

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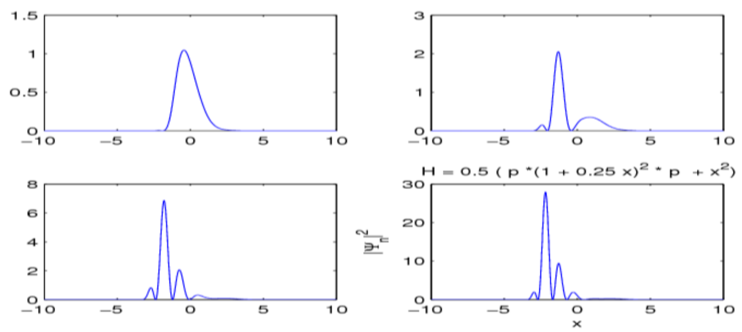

Figure 1 $|\psi_n|^2$ of $H = \frac{1}{2}[p(1+\mu x)^2 p + x^2]$.