THE REAL-ZEROS OF JONES POLYNOMIAL OF TORUS

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Dedicated to My Parents.

Abstract. This article proved two theorems and presented one conjecture about the real-zeros of Jones Polynomial of Torus. The distribution of zeros of Jones Polynomial is an interesting topic in knots theory of math and physics.

1. The Real-Zeros of Jones Polynomial of Torus

The real-zeros of Jones polynomial of torus $V_t(p, q)$ are distributed on x axis. In physics, since real-zeros of an equation usually represent observable values, it is interested to investigate them in advanced.

(1.1) $V_t(p, q) = 1 - t^{p+1} - t^{q+1} + t^{p+q}, t \in \mathbb{C}$

Theorem 1. For all Jones polynomials of Torus $V_t(p, q)$ there are two positive real-zeros with one is $+1$ and another is inside $1 < r < 2$.

Proof. Suppose a positive real-zero of $V_t(p, q)$ is $r$ where $0 < r < +\infty$, then there is a $r$ so that $V_t(p, q) = 0$. By Cauchy theorem, $||r|| < 1 + \max(||a_i||) \leq 2$. We can represent $V_t(p, q) = 0$ as $(1 - r^p)(1 - r^q) = (r - 1)(r^p + r^q)$. If we suppose $0 < r < 1$, then $0 < r^p < 1$ and $0 < r^q < 1$, the left side of equation is positive, but since $r - 1 < 0$, the right side is negative. So there is $1 < r < 2$.

By Descartes’ rule of signs, there are two signs change in the coefficients of equation (6), so there are two positive zeros. Obviously, $+1$ is a zero, another positive real-zero is in $1 < r < 2$.

Theorem 2. For all Jones polynomials of Torus $V_t(p, q)$, for both $p$ and $q$ are positive even integers, there are no negative zeros; for one of $p$ or $q$ is odd and another is even, there is one negative zero; for both $p$ and $q$ are odd, there are two negative zeros.

Proof. Replace $-t$ in Jones polynomial $V_t(p, q)$ as $V_{-t}(p, q)$ Descartes’s rule can be applied to decide all the negative real-zeros.

* Suppose $p = 2m, q = 2n$, then $V_{-t}(p, q) = 1 + t^{p+1} + t^{q+1} + t^{p+q}$. Because no signs change, there are no negative real-zeros.

* Suppose $p = 2m+1, q = 2n$, $V_{-t}(p, q) = 1 - t^{p+1} + t^{q+1} - t^{p+q}$. Because $V_{-t}(p, q) = V_{-t}(q, p)$, it is also true for $p = 2m, q = 2n+1$.
* Suppose $p = 2m + 1, q = 2n + 1$, then $V_{-t}(p, q) = 1 - t^{p+1} - t^{q+1} + t^{p+q}$. It is the case of positive real-zeros, so there are two negative real-zeros -1 and $r$ with $-1 < r < 0$. 

**Conjecture 1.** For $p \geq 2, q \geq 3$, $Ze(p, q)$ is the positive real-zero of $V_t(p, q)$ which is not equal to $+1$, then there is approximation formula below

\[
\frac{\ln(Ze(p, q) - 1)}{\ln(Ze(p, q + 1) - 1)} \approx \frac{\ln(p + q - 3)}{\ln(p + q - 2)}
\]

This conjecture shows the relationship of the real-zeros distribution of Jones Polynomial of Torus. Computer software has partially verified this conjecture.

**References**


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