# From Labor Theory of Value to Price Eigenfunction
## ---Microfoundation of General Equilibrium

Erman ZENG

Amoy Institute of Technovation

(XiaMen Municipal Productivity Promotion Center)

1300 JiMei Blvd., XiaMen, FuJian 361024, P.R.China

Email: zengerman@163.com

***Abstraction：*** The quantitative Marxian function system is developed on the basis of the labor theory of value as the micro foundation resulting labour value function, surplus value function, Marx production function. The heterogeneous capital aggregation problem is overcome by value transformation analysis of Leontief intermediate coefficient matrix leading to production price eigenvectors and Marx-Sraffa-Leontief General Equilibrium eigenvalues as well as the details about an economic system such as the reduced organic composite of the capital, the rate of profit, the surplus rate of value, the elasticity of capital output. The falling tendency of the rate of profit may not be true if the economy undergoes a general equilibrium.

***Key words:*** transformation problem, roundaboutness, Solow residue, Okishio theorem， value theory function, production function, General Equilibrium, Marx GE eigenvalue, Marx price eigenvector, input-output, division of labour, turnpike growth, Verdoorn's Law, Kaldor steady state, aggregation problem, two Cambridge debate

***JEL:*** E11, O47

This research attempts to place the formal Sraffian model with linear production sets into a general equilibrium framework and to derive a quantitative transformation theorem about Marxian theory of labor value and production price. Marxian reproduction solution established a dynamic general economic equilibrium which can be characterized by input-output ratio, namely, the reduced Organic Composite of Capital divided by the total productivity rate. Under the framework of the dynamic Marxian general equilibrium, it is possible to undergo an optimal socialism planning about an economic system by the regulation of the government input, entrepreneur taxation, and minimal wage rate. In the first part of this paper, a neoclassical framework is proposed which places the Marxian conceptions of both Constant Capital and Variable Capital into a Cobb-Douglas production function like model in order to obtain the mathematical formulations of Marx-Leontief-Sraffa input-output function system. In the second part, the value transformation for the Two Departments and the Three Industry Sectors as well as the Sraffa Model are proposed based upon the general equilibrium properties of the quantitative Marxian productivity theories and Leontief input-output economics.

## I. Marx-Leontief Input-Output Economics[1]

Marx's labor theory of value[2] pointed out that the value of each commodity (Q) contained three sources: the first part is "constant capital (C)", representing the value transferred from raw materials and machinery used up, the second part is "variable capital (V)" replacing the value of the labor power, and the third part is the surplus value (M) including net profit (P) and taxation (T). Therefore, the total value Q is expressed as a linear production function:

$$Q = C + V + M = C + Y = C + V + p'(C+V) = P'(C+V) = P'C\frac{g+1}{g} = \frac{P'}{\beta}C$$

$$M = P + T = p'(C+V) = \frac{p'}{\beta}C = m'V, g \equiv \frac{C}{V}$$

$$\beta := \frac{g}{g+1} = \frac{C}{C+V} = \frac{C}{Cv}, 1-\beta = \frac{1}{g+1} = \frac{V}{Cv} = \frac{p'}{m'}, g = \frac{\beta}{1-\beta}$$

$$\alpha := \frac{g}{g+\gamma} = \frac{p'C}{p'C+P'V} = p'\frac{C}{Y}, \gamma \equiv \frac{P'}{p'} = \frac{Q}{M}, \frac{Q}{C} - \frac{Y}{C} = 1 = \frac{P'}{\beta} - \frac{p'}{\alpha}$$

$$\frac{\alpha}{\beta} = \frac{g+1}{g+\gamma} = \frac{(g+1)p'}{p'g+P'} = \frac{m'}{m'+1} = \frac{M}{Y} = \frac{1}{1+\frac{1}{m'}} = \frac{p'}{p'+1-\beta} = \frac{p'+\alpha}{p'+1}, m' = \frac{\alpha}{\beta-\alpha}$$

$$1-\alpha = \frac{P'}{p'g+P'} = \frac{p'+1}{m'+1}, Cv = C+V, P' = \frac{Q}{C+V}, p' = \frac{M}{Cv} = \frac{m'}{1+g} = m'(1-\beta)$$

$$p' = \frac{\alpha(1-\beta)}{\beta-\alpha}, P' = p'+1 = \frac{\beta(1-\alpha)}{\beta-\alpha}$$

$$Q_1 = C_1 + \underline{V_1 + M_1} = C_1 + \underline{Y_1} = C_1 + \underline{C_2} = C$$
$$Q_2 = Y = \underline{C_2} + V_2 + M_2 = \underline{C_2} + Y_2 = \underline{Y_1} + Y_2 = Y$$
$$Q = C + V + M = C + Y = Q_1 + Q_2$$

Where

C = nK (K: capital, n: capital turnover rate), constant capital;

V = wL (L: labor, w: per-capita wages), variable capital

P': the productivity rate, $P' = Q/(C+V) = p'+1$

M: the surplus value

p': the rate of profit, $p' = M/(C+V) = m'/(g+1) = P'-1$

m': the rate of surplus value, m'= M/V

g: the organic composition of capital (OCC): $g = C/V = nK/(wL) = nk/w$

β: reduced OCC

f: the productivity growth rate,

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1 曾尔曼.《马克思生产力经济学导引》[M]，厦门大学出版社.(Erman ZENG: Introducing Marxian Productivity Economics, Xiamen Univ. Press), 2016

2 马克思.《资本论》I [M]. 北京：人民出版社，1975.

re-integration gives the Labor Value Function[3] Q as, :

$$\dot{Q}-\dot{C}=f-\dot{\beta},\dot{M}-\dot{C}=p-\dot{\beta},$$

$$f\equiv\frac{dP'}{P'dt},\dot{\beta}=\dot{g}-\frac{dg}{(g+1)dt}=(1-\beta)\dot{g}$$

$$if:\dot{Q}=\dot{C},then:\dot{\beta}=f=(1-\beta)\dot{g}*,$$

$$Q=B_0e^{ft}C^{\beta}V^{1-\beta},M=b_0e^{pt}C^{\beta}V^{1-\beta};$$

$$\dot{Y}-\dot{C}=p-\dot{\alpha},$$

$$p\equiv\frac{dp'}{p'dt}=\gamma f=\frac{1-\alpha}{\alpha}\beta\dot{g}*,$$

$$\gamma=\frac{P'}{p'}=\frac{Q}{M}=\frac{Q}{Y}\frac{Y}{M}=\frac{1-\alpha}{1-\beta}\frac{\beta}{\alpha}$$

$$\dot{\alpha}=\dot{g}-\frac{g}{g+\gamma}\dot{g}-\frac{\gamma}{g+\gamma}\dot{\gamma}=(1-\alpha)(\dot{g}-f+p)$$

$$\dot{Y}-\dot{C}=\alpha p-(1-\alpha)(\dot{g}-f)=(1-\alpha)(\beta\dot{g}*-\dot{g}+f)=(1-\alpha)(\dot{g}*-\dot{g})$$

$$\dot{Y}=F+\alpha\dot{C}+(1-\alpha)\dot{V},$$

$$F:=(1-\alpha)\dot{g}*=\frac{\alpha}{\beta}p;$$

$$Y=a_0e^{Ft}C^{\alpha}V^{1-\alpha}=A_0e^{mt}K^{\alpha}L^{1-\alpha}=A_0e^{Ft}K^{\alpha}L^{1-\alpha},$$

$$m=\alpha\dot{n}+(1-\alpha)\dot{w}+F=\alpha\dot{n}+(1-\alpha)\dot{w}+\frac{\alpha}{\beta}p\cong\frac{\alpha}{\beta}p=F$$

$$f=\dot{Q}-\dot{C}+(1-\beta)\dot{g}=(1-\beta)\dot{g}*,(\dot{Q}=\dot{C})$$

and the cost function is: $Cv=C+V=c_0C^{\beta}V^{1-\beta}$.

Under C-D production function situation where only K and L are taken into consideration, there is: m=F, named as the productivity coefficient.

$$\frac{Y}{M}=\frac{\beta}{\alpha}=\frac{Y/V}{m'}=\frac{y/w}{m'}$$

$$\dot{y}=\dot{w}+\dot{m}'+\dot{\beta}-\dot{\alpha}=\dot{w}+F+\alpha\dot{g}$$

$$if:\dot{m}'=0,then:\dot{\beta}-\dot{\alpha}=\dot{Y}-\dot{M}=-\frac{V}{Y}\dot{m}'=0,$$

$$\Rightarrow F=-\alpha\dot{g}=\frac{\alpha}{\beta}p,\dot{g}*=\frac{-\alpha\dot{g}}{1-\alpha},\dot{y}=\dot{w},\dot{Y}=\dot{V};$$

$$p=-\beta\dot{g}=\beta(\dot{Y}-\dot{C}),if:\dot{C}>\dot{Y}=\dot{V},g>0\Rightarrow F=p/\gamma<0$$

$$when:\dot{C}=\dot{Y}(d.g.e.):\dot{y}^{\uparrow}=\dot{w}+F+\alpha\dot{g}=\dot{w}+\frac{F}{1-\alpha}=\dot{w}^{\uparrow}+\frac{\alpha p^{\uparrow}}{(1-\alpha)\beta}$$

[3] 曾尔曼.《厦门科技》2014（3）31-35.

$$\dot{g}^* = \alpha(\dot{g}^* - \dot{g}) \Rightarrow p - F = (\alpha - \beta)\dot{g}, f - p = (\alpha - \beta)(\dot{g}^* - \dot{g})$$

$$\dot{\alpha} = p + (\dot{g} - \dot{g}^*)(1-\alpha) = \gamma\dot{g}^*(1-\beta) + (\dot{g} - \dot{g}^*)(1-\alpha);$$

$$\dot{\beta} = f + (\dot{g} - \dot{g}^*)(1-\beta) = \dot{g}(1-\beta)$$

$$\begin{pmatrix} \dot{\beta} \\ \dot{\alpha} \end{pmatrix} = \begin{pmatrix} \dot{g} & 0 \\ \gamma\dot{g}^* & \dot{g} - \dot{g}^* \end{pmatrix} \begin{pmatrix} 1-\beta \\ 1-\alpha \end{pmatrix}$$

$$1).\lambda_1 + \lambda_2 = \dot{g} + \dot{g} - \dot{g}^* = \dot{g}(1 + \frac{1}{1-\alpha}) \le 0 if F = -\alpha\dot{g} \ge 0, \lambda_1\lambda_2 = \frac{\dot{g}^2}{1-\alpha} \ge 0;$$

$$2).\lambda_1 + \lambda_2 = \dot{g} + \dot{g} - \dot{g}^* = \dot{g} < 0(\dot{g} = \dot{g}^*), \lambda_1\lambda_2 = 0$$

$$\dot{d}_L = \frac{-\alpha}{1-\alpha}\dot{\alpha} = -\alpha(\dot{g} - \dot{g}^*) - \beta\dot{g}^*$$

$$\dot{d}_R = -\beta\dot{g}$$

$$\begin{pmatrix} \dot{d}_L \\ \dot{d}_R \end{pmatrix} = \begin{pmatrix} \dot{g} - \dot{g}^* & \dot{g}^* \\ 0 & \dot{g} \end{pmatrix} \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}$$

The **division of labor** by Adam Smith[4] can be characterized as the labor output elasticity $(1-\alpha)$ of the Cobb-Douglas production function:

$d_L \equiv 1-\alpha = \frac{\gamma}{g+\gamma} = \frac{p'+1}{m'+1}$; similarly, the **roundaboutness** by Allyn Young[5] can be characterized as the variable capital output elasticity $(1-\beta)$ of the Marx production function: $d_R \equiv 1-\beta = \frac{1}{1+g} = \frac{p'}{m'}, \gamma_{\downarrow} = 1 + \frac{1}{p'^{\uparrow}} = \frac{\beta}{1-\beta}\frac{1-\alpha}{\alpha} = \frac{\frac{1}{(1-\beta)^{\uparrow}} - 1}{\frac{1}{1-\alpha^{\uparrow}} - 1}$,

$$\dot{\gamma} = \frac{d(P'/p')}{(P'/p')dt} = f - p = f(1-\gamma) = (1-\beta)\dot{g}[1 - \frac{\beta(1-\alpha)}{(1-\beta)\alpha}] = \dot{g}(1 - \frac{\beta}{\alpha}) < 0$$

$$\dot{d}_L = \dot{\gamma} - \frac{d(g+\gamma)}{(g+\gamma)dt} = \dot{\gamma} - \frac{g}{g+\gamma}\dot{g} - \frac{\gamma}{g+\gamma}\dot{\gamma} = \frac{g}{g+\gamma}(\dot{\gamma} - \dot{g}) = -\beta\dot{g}^* + \alpha(\dot{g}^* - \dot{g})$$

$$\dot{d}_R = -\frac{d(1+g)}{(1+g)dt} = -\frac{g}{1+g}\dot{g} = -\beta\dot{g}$$

$$\dot{d}_L - \dot{d}_R = (\alpha - \beta)(\dot{g}^* - \dot{g})$$

4 亚当·斯密著，郭大力 王亚南译. 国民财富的性质和原因的研究[M]. 北京：商务印书馆，1972.
5 Young AA. Increasing Returns and Economic Progress [J]. The Economic Journal, 1928, 38: 527-42.

$$p=\frac{1-\alpha}{\alpha}\beta\dot{g}^*=\frac{\beta}{\alpha}F=\gamma f,-\frac{1-\alpha}{\alpha}\dot{d}_L=-\frac{1-\beta}{\beta}\dot{d}_R\Leftrightarrow\dot{d}_R=\gamma\dot{d}_L>\dot{d}_L$$
$$p-\dot{\alpha}=(1-\alpha)(\dot{g}^*-\dot{g})=\dot{Y}-\dot{C},p\cong\dot{\alpha}$$
$$f-\dot{\beta}=(1-\beta)(\dot{g}^*-\dot{g})=\dot{Q}-\dot{C},f\cong\dot{\beta}$$
$$\dot{Y}^{\uparrow}=\dot{C}^{\uparrow}+p^{\uparrow}+\frac{1-\alpha}{\alpha}\dot{d}_L{}^{\uparrow},(\alpha\dot{\alpha}+d_L\dot{d}_L=0);$$
$$\dot{Q}^{\uparrow}=\dot{C}^{\uparrow}+f^{\uparrow}+\frac{1-\beta}{\beta}\dot{d}_R{}^{\uparrow};$$
$$\dot{M}^{\uparrow}=\dot{C}^{\uparrow}+p^{\uparrow}+\frac{1-\beta}{\beta}\dot{d}_R{}^{\uparrow}$$
$$\frac{V}{Y}=1-\frac{M}{Y}=1-\frac{\alpha}{\beta}\Rightarrow$$
$$w=y(1-\frac{C}{Y\beta}p')=y(1-\frac{p'}{p'+1-\beta})=y(1-\frac{1}{1+(1-\beta)(p')^{-1}})$$

**The rate of profit would not fall**:

$$F=(1-\alpha)\dot{g}^*=(1-\alpha)(\dot{Y}^*-\dot{V}^*)=(1-\alpha)(\dot{y}^*-\dot{w}^*)\geq 0$$
$$\Leftrightarrow\dot{y}^*-\dot{w}^*\geq 0\Rightarrow f=p/\gamma\geq 0$$

$$F=\dot{Y}-\dot{V}-\alpha\dot{g}=\frac{M}{V}(\dot{M}-\dot{Y})-\alpha\dot{g}=m'(\dot{\alpha}-\dot{\beta})-\alpha\dot{g}\cong-\alpha\dot{g}$$
$$=-\frac{g}{g+\gamma}\frac{dg}{gdt}=-\frac{\gamma}{g+\gamma}\frac{dg}{\gamma dt}\approx-(1-\alpha)\frac{d(g/\gamma)}{dt}=-(1-\alpha)\frac{d}{dt}[\frac{\alpha}{1-\alpha}]$$
$$=-\alpha[\dot{\alpha}-\dot{d}_L]=\alpha(\dot{\gamma}-\dot{g})=\frac{d(1-\alpha)}{(1-\alpha)dt}$$
$$=\dot{d}_L$$

$$\frac{m}{\dot{y}}=1-\alpha'\frac{\dot{k}}{\dot{y}}=1-\frac{rK}{Y}\frac{ydk/dt}{kdy/dt}=1-\frac{rC}{nY}\frac{Y\Delta k}{K\Delta y}=1-\frac{r\alpha}{np'}\frac{Y\Delta C}{C\Delta Y}=1-\frac{r\alpha}{np'}\frac{\dot{C}}{\dot{Y}}\approx 1-\alpha=d_L$$

**Okishio Theorem**[6] asserts that if real wages remain unchanged, the rate of profit necessarily rises in consequence of an cost-saving technology innovation.

$$m=\dot{y}-\alpha\dot{k}\Leftrightarrow m=\alpha\dot{n}+(1-\alpha)\dot{w}+\frac{\alpha}{\beta}p$$
$$m=\alpha\dot{n}+(1-\alpha)\dot{w}+F=\dot{n}^*+(1-\alpha)\dot{k}^*$$

**Schumpeterian innovation**[7] **function** or degree of innovation S can be

6 Okishio, N. "Technical Change and the Rate of Profit", Kobe Univ. Econ. Review, 7, 1961, pp. 85–99.
7 Schumpeter, J.A. *The theory of economic development: an inquiry into profits, capital, credit, interest, and the business cycle* translated from the German by Redvers Opie (1961) New York: OUP

characterized by γ/g :

$$S := \frac{\gamma}{g} = \frac{1-\alpha}{\alpha} = \frac{Q}{M}\frac{V}{C} = \frac{B_0 e^{ft} g^{\beta-1}}{b_0 e^{pt} g^{\beta}} = \frac{B_0 e^{(f-p)t}}{b_0 g} = \frac{B_0}{b_0} e^{(1-\frac{\beta}{\alpha})\dot{g}*t} g^{-1}$$

$$S^{\uparrow} = \frac{Q/C}{M/V} = \frac{\mathrm{Y}^{\uparrow}+\mathrm{C}}{\mathrm{C}}\frac{\mathrm{Y}^{\uparrow}-\mathrm{M}}{\mathrm{M}} = \frac{(Y+C)/C}{(Y-V)/V} = \frac{\frac{Y}{C}+1}{\frac{Y}{V}-1} = \frac{\frac{1}{C_{\downarrow}}+\frac{1}{Y}}{\frac{1}{V^{\uparrow}}-\frac{1}{Y}}$$

$$\dot{\mathrm{S}}^{\uparrow} = \dot{\mathrm{Q}}^{\uparrow} - \dot{\mathrm{M}}^{\downarrow} - \dot{\mathrm{g}}^{\downarrow} = \dot{\mathrm{Y}}^{\uparrow} + \dot{\mathrm{V}}^{\uparrow} - \dot{\mathrm{C}}^{\downarrow} - \dot{\mathrm{M}}$$

$$\frac{\partial S}{\partial C} = \frac{-C^{-2}}{\frac{1}{V}-\frac{1}{Y}} < 0, \frac{\partial S}{\partial V} = (\frac{1}{C}+\frac{1}{Y})(\frac{1}{V}-\frac{1}{Y})^{-2}V^{-2} > 0, \frac{\partial S}{\partial Y} = \frac{V}{CM}\frac{Q}{CM} > 0$$

**Verdoorn's Law**[8] can be explained by Marx production function:

$$\dot{Q} = f + \beta\dot{C} + (1-\beta)\dot{V} = \dot{C} + (1-\beta)(\dot{g}*-\dot{g}) = \dot{Q}_I + f - \dot{\beta}$$
$$\dot{Y} = F + \alpha\dot{C} + (1-\alpha)\dot{V} = \dot{C} + (1-\alpha)(\dot{g}*-\dot{g}) = \dot{y} + \dot{L} = \dot{Q}_{II}$$
$$\dot{y} = \frac{\alpha}{\beta}\dot{Q} + F + \dot{w} - \frac{\alpha}{\beta}(f+\dot{V})$$
$$= \frac{\alpha}{\beta}\dot{Q} + F + \dot{w}\,(1-\frac{\alpha}{\beta}) - \frac{\alpha}{\beta}(f+\dot{L}) = \frac{\alpha}{\beta}\dot{Q} + F + \frac{\alpha}{\beta}[(\frac{\beta}{\alpha}-1)\dot{w} - (f+\dot{L})]$$
$$= \frac{\alpha}{\beta}\dot{Q} + F + \frac{\alpha}{\beta}[(\frac{Y}{M}-1)\dot{w} - (f+\dot{L})] = \frac{\alpha}{\beta}\dot{Q} + F + \frac{\alpha}{\beta}(\frac{\dot{w}}{m'} - f - \dot{L})$$
$$\approx \frac{\alpha}{\beta}\dot{Q} + F$$

$$\dot{Y} = \dot{y} + \dot{L} = \frac{1}{Y}(Y_1\dot{Y}_1 + Y_2\dot{Y}_2) = \frac{1}{Y}(Y_1\dot{y}_1 + Y_2\dot{y}_2) + \frac{1}{Y}(Y_1\dot{L}_1 + Y_2\dot{L}_2)$$
$$\dot{y} = \frac{Y_1}{Y}\dot{y}_1 + (1-\frac{Y_1}{Y})\dot{y}_2 + \frac{Y_1}{Y}\dot{L}_1 + (1-\frac{Y_1}{Y})\dot{L}_2 - \dot{L}$$

**Kaldor stylied facts**[9] are all explained very well.

## II. Value Transformation[10]

If there is no currency inflation， and the values of commodities keep invariant,

[8] Verdoorn, J. P. (1993), "On the Factors Determining the Growth of Labor Productivity", *L. Pasinetti (ed.), Italian Economic Papers*, Oxford: Oxford University Press,

[9] Kaldor, Nicholas (1957). "A Model of Economic Growth". *The Economic Journal*. **67** (268): 591–624.

[10] Samuelson,PA.: Understanding the Marxian notion of exploitation: a summary of the so-called transformation problem between Marxian values and competitive prices, *Jour. Econ. Liter.* 1971, 9, 399-431

then we have:
C+V=Cv=const.,C'+V'=$P_1$C+$P_2$V, dCv=d(C'+V'),dC=dV=0
Total production value (C+V) equal total market price (C'+V'):

$$\mathrm{dCv} = \mathrm{d}(\mathrm{P_1C} + \mathrm{P_2V}) = \mathrm{CdP_1} + \mathrm{P_1dC} + \mathrm{P_2dV} + \mathrm{VdP_2} = \mathrm{CdP_1} + \mathrm{VdP_2}$$
$$= \mathrm{Cv}[\beta \mathrm{dP_1} + (1-\beta)\mathrm{dP_2}] = \mathrm{Cv}[\beta\delta\mathrm{P_1} + (1-\beta)\delta\mathrm{P_2}]$$
$$\cong \mathrm{Cv}[\beta\ln(1+\delta\mathrm{P_1}) + (1-\beta)\ln(1+\delta\mathrm{P_2})] = \mathrm{Cv}[\beta\ln\mathrm{P_1} + (1-\beta)\ln\mathrm{P_2}]$$
$$= Cv\ln(P_1^{\beta}P_2^{1-\beta})$$
$$= 0$$
$$\Rightarrow P_1^{\beta}P_2^{1-\beta} = 1$$
$$\beta\dot{P}_1 + (1-\beta)\dot{P}_2 = 0, or: \beta\delta\mathrm{P_1} + (1-\beta)\delta\mathrm{P_2} = 0$$

$$Q' = C' + V' + M' = P_1C + P_2V + P_3M = B(P_1C)^{\beta}(P_2V)^{1-\beta} = QP_1^{\beta}P_2^{1-\beta} = Q$$
$$M' = P_3M = b(P_1C)^{\beta}(P_2V)^{1-\beta} = MP_1^{\beta}P_2^{1-\beta} = M, P_3 = 1;$$
$$P_1C + P_2V = C + V;$$
$$M' = Q' - (P_1C + P_2V) = P_3M = M = p'(C+V) = r'(P_1C + P_2V)$$

Total profit equals total surplus values.

For a Marxian two production departments system containing production means and living materials:

$$C_1 + V_1 + M_1 = C_1 + Y_1 = Q_1 = C_1 + C_2 = C, (Y_1 = C_2 : Simple\,\mathrm{Re}production)$$
$$C_2 + V_2 + M_2 = C_2 + Y_2 = Q_2 = Y_1 + Y_2 = Y$$
$$Q_1 + Q_2 = Q$$
$$p_1C'_1 + p_2Y'_1 = p_1Q'_1$$
$$p_1C'_2 + p_2Y'_2 = p_2Q'_2$$
$$\begin{pmatrix} C'_1 & Y'_1 \\ C'_2 & Y'_2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} Q'_1 & 0 \\ 0 & Q'_2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$
$$or: \begin{pmatrix} \frac{C'_1}{Q'_1} & \frac{Y'_1}{Q'_1} \\ \frac{C'_2}{Q'_2} & \frac{Y'_2}{Q'_2} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \lambda\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, p_i \equiv \frac{1}{P_i}$$
$$\lambda_1 = 1, \lambda_2 = \frac{C_1}{Q_1} + \frac{Y_2}{Q_2} - 1 = \frac{C_1}{Q_1}\frac{Y_2}{Q_2} - \frac{C_2}{Q_2}\frac{Y_1}{Q_1}$$

Since the intermediate input matrix is non-negative, according to Perron-Frobenius[11] theorem there exists at least one positive eigenvalue,
λ: Leontief input-output eigenvalue,
$P_i$: Marx price eigenvector

[11] Seneta, E. (1973) Non-negative Matrices – An Introduction to Theory and Applications. London: George Allen and Unwin

$$\begin{pmatrix} \frac{C'_1}{Q'_1} & \frac{Y'_1}{Q'_1} \\ \frac{C'_2}{Q'_2} & \frac{Y'_2}{Q'_2} \end{pmatrix} \begin{pmatrix} \frac{1}{P_1} \\ \frac{1}{P_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{P_1} \\ \frac{1}{P_2} \end{pmatrix} = \lambda \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, p_i \equiv \frac{1}{P_i}$$

$$1)\lambda_1 = 1 : p_2 = 1, p_1 = \frac{Q'_2 - Y'_2}{C'_2},$$

$$2)\lambda_2 = \frac{C'_1}{Q'_1} + \frac{Y'_2}{Q'_2} - 1 : p_2 = 1, p_1 = \frac{C'_1 - Q'_1}{C'_2}\frac{Q'_2}{Q'_1};$$

$$\frac{dC'_1}{dt}p_1 + C'_1\frac{dp_1}{dt} + \frac{dY'_1}{dt}p_2 + Y'_1\frac{dp_2}{dt} = \frac{dQ'_1}{dt}p_1 + Q'_1\frac{dp_1}{dt};$$

$$\frac{dC'_2}{dt}p_1 + C'_2\frac{dp_1}{dt} + \frac{dY'_2}{dt}p_2 + Y'_2\frac{dp_2}{dt} = \frac{dQ'_2}{dt}p_2 + Q'_2\frac{dp_2}{dt} \Rightarrow$$

$$\frac{d(C'_1 - Q'_1)}{dt}p_1 + \frac{dY'_1}{dt}p_2 + Y'_1\frac{dp_2}{dt} = (Q'_1 - C'_1)\frac{dp_1}{dt} \Rightarrow Y'_1(\frac{dp_1}{dt} - \frac{dp_2}{dt}) = \frac{dY'_1}{dt}(p_2 - p_1)$$

$$\Rightarrow \dot{Y}'_1 = \frac{d(p_1 - p_2)}{(p_2 - p_1)dt} = -\frac{d}{dt}\ln(p_1 - p_2);$$

$$\frac{d(Q'_2 - Y'_2)}{dt}p_2 + (Q'_2 - Y'_2)\frac{dp_2}{dt} = \frac{dC'_2}{dt}p_1 + C'_2\frac{dp_1}{dt} \Rightarrow \frac{dC'_2}{dt}(p_2 - p_1) = C'_2(\frac{dp_1}{dt} - \frac{dp_2}{dt})$$

$$\Rightarrow \dot{C}'_2 = \frac{d(p_1 - p_2)}{(p_2 - p_1)dt} = \dot{Y}'_1$$

Or in a Three-Sector system including production means and living materials as well as capital goods described by J. Winternitz[12]:

$$(1): P_1C_1 + P_2V_1 + P_3M_1 = P_1Q_1 = P_1C = P_1(C_1 + C_2 + C_3)$$

$$(2): P_1C_2 + P_2V_2 + P_3M_2 = P_2Q_2 = P_2V = P_2(V_1 + V_2 + V_3)$$

$$(3): P_1C_3 + P_2V_3 + P_3M_3 = P_3Q_3 = P_3M = P_3(M_1 + M_2 + M_3)$$

$$\begin{pmatrix} C_1 & V_1 & M_1 \\ C_2 & V_2 & M_2 \\ C_3 & V_3 & M_3 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}, or:$$

$$\begin{pmatrix} \frac{C'_1}{Q'_1} & \frac{V'_1}{Q'_1} & \frac{M'_1}{Q'_1} \\ \frac{C'_2}{Q'_2} & \frac{V'_2}{Q'_2} & \frac{M'_2}{Q'_2} \\ \frac{C'_3}{Q'_3} & \frac{V'_3}{Q'_3} & \frac{M'_3}{Q'_3} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \lambda \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, p_i \equiv \frac{1}{P_i}$$

$$\lambda_1 = 1, \lambda_2 + \lambda_3 = \frac{C'_1}{Q'_1} + \frac{V'_2}{Q'_2} + \frac{M'_3}{Q'_3} - 1, \lambda_2 * \lambda_3 = \det\begin{vmatrix} \frac{C'_1}{Q'_1} & \frac{V'_1}{Q'_1} & \frac{M'_1}{Q'_1} \\ \frac{C'_2}{Q'_2} & \frac{V'_2}{Q'_2} & \frac{M'_2}{Q'_2} \\ \frac{C'_3}{Q'_3} & \frac{V'_3}{Q'_3} & \frac{M'_3}{Q'_3} \end{vmatrix}$$

[12] 'Values and Prices: a solution to the so-called transformation problem', Econ. Jour. 1948, 58, 276-280.

All the Marx-Leontief input-output functions are still in the same form with respect to the market price:

$$\frac{Q'}{P_Q} \equiv \frac{\sum_{i=1}^{3} Q'_i}{P_Q} = \sum_{i=1}^{3} \frac{Q'_i}{P_i}, p_Q = \frac{1}{P_Q} = \sum_{i=1}^{3} \frac{Q'_i}{Q' P_i};$$

$$\frac{Y'}{P_Y} \equiv \frac{\sum_{i=1}^{3} V'_i + M'_i}{P_Y} = \sum_{i=1}^{3} (\frac{V'_i}{P_2} + \frac{M'_i}{P_3}), p_Y = \frac{1}{P_Y} = \sum_{i=1}^{3} \frac{p_2 V'_i + p_3 M'_i}{Y'};$$

$$\frac{Q'}{P_Q} = Q = B_0 e^{ft} (\frac{C'}{P_C})^{\beta} \frac{V'}{P_V})^{1-\beta} \Rightarrow Q' = \frac{P_Q}{P_C^{\beta} P_V^{1-\beta}} B_0 e^{ft} C'^{\beta} V'^{1-\beta} = B'_0 e^{ft} C'^{\beta} V'^{1-\beta}, B'_0 = \frac{B_0}{Ip_Q};$$

$$Y' = a'_0 e^{Ft} C'^{\alpha} V'^{1-\alpha}, B'_0 = \frac{a_0}{Ip_Y};$$

$$M' = b'_0 e^{pt} C'^{\beta} V'^{1-\beta}, b'_0 = \frac{b_0}{Ip_M};$$

$$C'v = \frac{I}{P_C^{\beta} P_V^{1-\beta}} c_0 C'^{\beta} V'^{1-\beta} = c_0 C'^{\beta} V'^{1-\beta}, \frac{dCv}{Cv} = \delta I = \ln(1+\delta I) = I = P_C^{\beta} P_V^{1-\beta}$$

$$r' = \frac{M'}{C'+V'} = \frac{M}{Ip_M Cv} = p' \frac{P_M}{I}$$

$$\frac{C}{Q} = \frac{\beta}{P'} \rightarrow \left| \sum_j \frac{C'_{ij}}{P_j} \right| = \frac{\beta_i}{P'_i} \left| \frac{Q'_i}{P_i} \right| = \lambda \left| \frac{Q'_i}{P_i} \right| \Leftrightarrow \sum_j \left| \frac{C'_{ij}}{Q'_i} \right| \left| \frac{1}{P_j} \right| = \frac{\beta_i}{1+p'_i} \left| \frac{1}{P_i} \right| = \lambda \left| \frac{1}{P_i} \right| \Rightarrow$$

$$\sum_j \frac{C'_{ij}}{C'_i} (\frac{C'_i}{Q'_i}) \left( \frac{1}{P_i} \right) = \frac{\beta_i}{P'_i} \left( \frac{1}{P_i} \right) \Leftrightarrow \sum_j \left| \frac{C'_{ij}}{C'_i} \right| \left( \frac{1}{P_i} \right) = \lambda \left( \frac{1}{P_i} \right) (C'_i = \sum_j C'_{ij}, \lambda_1 = 1)$$

$$\Leftrightarrow \sum_j \left| \frac{C'_{ij}}{C'_i} \right| (p_j) = (p_j) : \Pr iceEigenfunction; (p_j) : \Pr iceEigenvector$$

$$\frac{C}{Y} = \frac{\alpha}{p'} \rightarrow \left| \sum_j \frac{C'_{ij}}{P_j} \right| = \frac{\alpha_i}{p'_i} \left| \frac{Q'_i - \sum_j C'_{ij}}{P_i} \right| = \frac{\alpha_i}{p'_i} \left| \frac{V'_i + M'_i}{P_i} \right| = \xi \left| \frac{Y'_i}{P_i} \right|$$

$$\sum_j \left| \frac{C'_{ij}}{Y'_i} \right| \left| \frac{1}{P_j} \right| = \frac{\alpha_i}{p'_i} \left| \frac{1}{P_i} \right| = \xi \left| \frac{1}{P_i} \right| \Leftrightarrow \sum_j \left| \frac{C'_{ij}}{C'_i} \right| \left( \frac{C'_i}{Y'_i} \right) \left| \frac{1}{P_j} \right| = \frac{\alpha_i}{p'_i} \left| \frac{1}{P_i} \right| \Rightarrow \sum_j \left| \frac{C'_{ij}}{C'_i} \right| \left| \frac{1}{P_j} \right| = \xi \left| \frac{1}{P_i} \right|$$

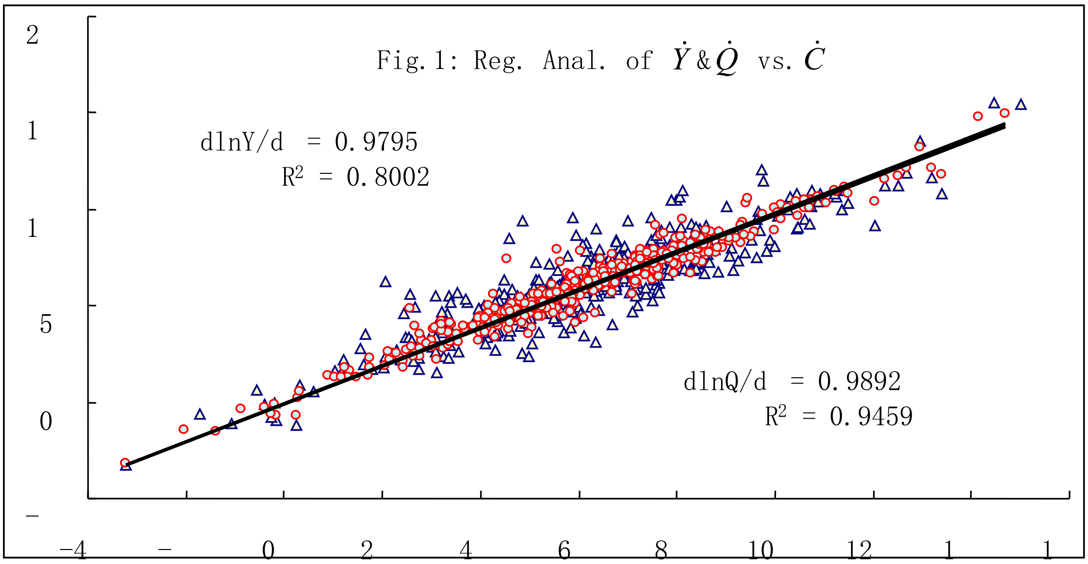
**The price parameter eigenfunction and eigenvalues as well as price eigenvectors are all obtained. $C'_{ij}/C'_i$ is Leontief intermediate coefficient matrix; Similarly,**

$$\frac{C}{M} = \frac{\beta}{p'} \rightarrow \left| \sum_j \frac{C'_{ij}}{P_j} \right| = \frac{\beta_i}{p'_i} \left| \frac{M'_i}{P_i} \right| \Leftrightarrow \sum_j \left| \frac{C'_{ij}}{M'_i} \right| \left| \frac{1}{P_j} \right| = \frac{\beta_i}{p'_i} \left| \frac{1}{P_i} \right| = \mu \left| \frac{1}{P_i} \right|$$

$$\frac{C}{V}=g=\frac{\beta}{1-\beta}\rightarrow\left|\sum_j\frac{C'_{ij}}{P_j}\right|=\frac{\beta_i}{1-\beta_i}\left|\frac{V'_i}{P_i}\right|\Leftrightarrow\sum_j\left|\frac{C'_{ij}}{V'_i}\right|\left|\frac{1}{P_j}\right|=\frac{\beta_i}{1-\beta_i}\left|\frac{1}{P_i}\right|=\upsilon\left|\frac{1}{P_i}\right|;$$

$$or:\frac{C}{C+V}=\frac{C}{Q-M}=\beta,\sum_j\left|\frac{C'_{ij}}{Q'_i-M'_i}\right|\left|\frac{1}{P_j}\right|=\beta_i\left|\frac{1}{P_i}\right|$$

The regression analysis of the USA manufacture industry data[13] (>400 sectors) from 1958-96 supported the above results：dlnY'/dt=0.98dlnC'/dt, dlnQ'/dt=0.99dlnC'/dt：




According to the USA 1997-2014 input-output data[14] (15x15 to 3x3, Table 4), the eigenvalue of ζ (=C'/Q') equals to 0.48：

| Table 4. | | 11 | 21 | 22 | 23 | 31G | 42 | 44RT | 48TW | 51 | FIRE | PROF | 6 | 7 | 81 | G |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **λ** | Agric | Minin | Utili | Const | Manuf | Whole | Retai | Trans | Infor | Finan | Profe | Educa | Arts, | Other | Govern. |
| 2014 | **0.485** | 0.4364 | 0.1773 | 0.1735 | 0.2956 | **0.4964** | 0.1413 | 0.1544 | 0.3348 | 0.2270 | **0.1185** | 0.1586 | 0.1909 | 0.2433 | 0.1786 | 0.2000 |
| 2013 | **0.477** | 0.4171 | 0.1636 | 0.1727 | 0.3065 | **0.4977** | 0.1414 | 0.1605 | 0.3389 | 0.2283 | **0.1221** | 0.1606 | 0.1946 | 0.2455 | 0.1791 | 0.2102 |
| 2012 | **0.481** | 0.4807 | 0.1558 | 0.1500 | 0.3017 | **0.4942** | 0.1327 | 0.1467 | 0.3235 | 0.2247 | **0.1045** | 0.1469 | 0.1834 | 0.2355 | 0.1662 | 0.2083 |
| 2011 | **0.478** | 0.4452 | 0.1551 | 0.1567 | 0.3110 | **0.4967** | 0.1417 | 0.1527 | 0.3291 | 0.2192 | **0.1160** | 0.1526 | 0.1927 | 0.2482 | 0.1723 | 0.2117 |
| 2010 | **0.463** | 0.4784 | 0.1548 | 0.1738 | 0.3136 | **0.4851** | 0.1306 | 0.1546 | 0.3007 | 0.2034 | **0.1255** | 0.1530 | 0.1926 | 0.2471 | 0.1717 | 0.2092 |
| 2009 | **0.447** | 0.5229 | 0.1260 | 0.1628 | 0.3206 | **0.4783** | 0.1000 | 0.1361 | 0.2798 | 0.2027 | **0.1293** | 0.1510 | 0.1890 | 0.2444 | 0.1622 | 0.2085 |
| 2008 | **0.492** | −0.4683 | −0.1664 | −0.2360 | −0.3257 | **−0.482** | −0.1294 | −0.1367 | −0.3143 | *−0.1738* | **−0.132** | *−0.135* | −0.1836 | −0.2335 | −0.1672 | −0.2044 |
| 2007 | **0.49** | −0.4614 | −0.1786 | −0.2261 | −0.3201 | **−0.494** | −0.1262 | −0.1378 | −0.3116 | −0.1845 | **−0.131** | −0.1437 | −0.1865 | −0.2293 | −0.1600 | −0.2008 |
| 2006 | **0.484** | 0.4518 | 0.2100 | 0.2120 | 0.3321 | **0.4931** | 0.1296 | 0.1350 | 0.2896 | 0.2007 | **0.1371** | 0.1420 | 0.1913 | 0.2301 | 0.1579 | 0.2020 |
| 2005 | **0.493** | 0.4320 | 0.2351 | 0.2509 | 0.3280 | **0.4815** | 0.1367 | 0.1401 | 0.2848 | 0.1790 | **0.1519** | 0.1463 | 0.2016 | 0.2350 | 0.1591 | 0.1964 |
| 2004 | **0.473** | −0.4090 | −0.2505 | −0.2055 | −0.3496 | **−0.501** | −0.1344 | −0.1435 | −0.2692 | −0.1964 | **−0.138** | −0.1435 | −0.1982 | −0.2410 | −0.1633 | −0.2003 |
| 2003 | **0.466** | 0.4398 | 0.2467 | 0.2125 | 0.3430 | **0.4957** | 0.1275 | 0.1304 | 0.2499 | 0.2176 | **0.1209** | 0.1381 | 0.1992 | 0.2404 | 0.1582 | 0.1946 |
| 2002 | **0.467** | −0.4634 | −0.2187 | −0.1925 | −0.3412 | **−0.505** | −0.1324 | −0.1277 | −0.2439 | −0.2236 | **−0.109** | −0.1345 | −0.1986 | −0.2391 | −0.1484 | −0.1915 |
| 2001 | **0.477** | −0.4479 | −0.2341 | −0.2544 | −0.3252 | **−0.493** | −0.1165 | −0.1196 | −0.2293 | −0.2476 | **−0.110** | −0.1391 | −0.1973 | −0.2406 | −0.1553 | −0.1909 |
| 2000 | **0.489** | −0.4301 | −0.2612 | −0.2337 | −0.3268 | **−0.492** | −0.1226 | −0.1312 | −0.2481 | −0.2556 | **−0.123** | −0.1462 | −0.1955 | −0.2373 | −0.1354 | −0.1854 |
| 1999 | **0.486** | −0.4651 | −0.2341 | −0.1781 | −0.3439 | **−0.510** | −0.1184 | −0.1259 | −0.2355 | −0.2102 | **−0.111** | −0.1455 | −0.1949 | −0.2532 | −0.1409 | −0.1811 |
| 1998 | **0.487** | −0.4431 | −0.2400 | −0.1565 | −0.3518 | **−0.523** | −0.1064 | −0.1147 | −0.2250 | −0.2066 | **−0.110** | −0.1470 | −0.2020 | −0.2792 | −0.1440 | −0.1819 |
| 1997 | **0.489** | −0.4310 | −0.2342 | −0.1379 | −0.3607 | **−0.523** | −0.1119 | −0.1220 | −0.2564 | −0.2022 | **−0.103** | −0.1345 | −0.1971 | −0.2869 | −0.1404 | −0.1818 |

[13] NBER-CES Manufacturing Industry Database [EB/OL].(2011-02-02)[2012-10-11]. www.nber.org.
[14] http://www.bea.gov/industry/io_annual.htm

## III . Optimal Economic Planning

According to Marxian general equilibrium, which is indeed of macro dynamic, the growth rates of both departments should equal:

$$Y'/C' = b'_0 e^{Ft} g'^{\alpha-1} \Rightarrow \dot{Y}' - \dot{C}' = (1-\alpha)(\dot{g}'^* - \dot{g}');$$

$$Q'/C' = B'_0 e^{ft} g'^{\beta-1} \Rightarrow \dot{Q}' - \dot{C}' = (1-\beta)(\dot{g}'^* - \dot{g}');$$

$$\therefore \dot{g}' = \dot{g}'^* \Rightarrow \dot{Y}'^* = \dot{C}'^* = \dot{Q}'^*$$

Therefore, there could be a policy regulation among the government input, namely constant capital ($C'=C-\delta_k$), and the income ($Y'=Y+\delta_k$) distribution—the wage rate ($\delta_1/L=\delta_w$), and the surplus value ($\delta_M=\delta_2+\delta$) including government taxation $\delta$ and entrepreneur profit $\delta_2$:

$$\dot{Y} = \frac{V\dot{V} + M\dot{M}}{Y} = \frac{Y-M}{Y}\dot{V} + \frac{P\dot{P} + T\dot{T}}{Y}, M = P + T = T(\kappa+1), \kappa \equiv P/T$$

$$\dot{Y} = (1-\frac{\alpha}{\beta})\dot{V} + \frac{\alpha}{\beta}(\frac{\dot{P}}{1+\kappa^{-1}} + \frac{\dot{T}}{1+\kappa});$$

$$Q\dot{Q} = C\dot{C} + Y\dot{Y}, C' = C - \delta_k, Y' = Y + \delta_k = V' + M' = V' + P' + T',$$

$$\dot{Q}' = \frac{C'}{Q}\dot{C} + \frac{Y'}{Q}\dot{Y} = \dot{C}' = \dot{Y}' \Rightarrow \dot{Q}' = \frac{C_t - \delta_k}{Q_t}\dot{C} + \frac{Y+\delta_k}{Q_t}\dot{Y} = \dot{Q} + \frac{\dot{Y}-\dot{C}}{Q_t}\delta_k = \dot{C}' = \dot{C} - \frac{\delta_k}{C_t}$$

$$\Rightarrow \delta_k = \frac{\dot{C} - \dot{Q}}{\frac{\dot{Y}-\dot{C}}{Q_t} + \frac{1}{C_t}};$$

$$\dot{Y}' = \frac{\Delta Y_t}{Y_t \Delta t} = (1-\frac{\alpha}{\beta})\frac{\Delta V + \delta_1}{V\Delta t} + \frac{\alpha}{\beta}(\frac{\frac{\Delta P_t + \delta_2}{P_t \Delta t}}{1+\kappa_t^{-1}} + \frac{\frac{\Delta T_t + \delta}{T_t \Delta t}}{1+\kappa_t}),$$

$$\delta_1 + \delta_2 + \delta = \delta_k$$

$$\frac{\delta}{T_t} = \dot{Y} - \dot{T} + \beta \dot{g}^*(\frac{1}{\alpha} - 1 - \frac{1}{\beta}) = \dot{Y} - \dot{T} + p\frac{\alpha}{1-\alpha}(\frac{1}{\alpha} - 1 - \frac{1}{\beta})$$

$$\to \delta = T_t[\dot{Y} - \dot{T} + p(1 - \frac{\alpha}{\beta(1-\alpha)})](if: \alpha > 0.5, \frac{1}{\alpha} - 1 - \frac{1}{\beta} < 0);$$

$$\dot{Y}' \underline{\underline{\Delta t = 1}} (1-\frac{\alpha}{\beta})(\frac{\delta_1}{V_t} + \dot{V}) + \frac{\alpha}{\beta}(\frac{\dot{P} + \frac{\delta_2}{P_t}}{1+\kappa_t^{-1}} + \frac{\dot{T} + \frac{\delta}{T_t}}{1+\kappa_t})$$

$$= (1-\frac{\alpha}{\beta})(\dot{V} + \frac{\delta_1}{V_t}) + \frac{\alpha}{\beta}(\frac{\dot{P}}{1+\kappa_t^{-1}} + \frac{\dot{T}}{1+\kappa_t}) + \frac{\alpha}{\beta}(\frac{\delta_2 + \delta}{M_t})$$

$$= \dot{Y} + (1-\frac{\alpha}{\beta})\frac{\delta_1}{V_t} + \frac{\alpha}{\beta}(\frac{\delta_k - \delta_1}{M_t}) = \dot{C}' = \dot{C} - \frac{\delta_k}{C_t} \to \delta_1 = \frac{\dot{C} - \dot{Y} - \delta_k(\frac{1}{C_t} + \frac{\alpha}{\beta}\frac{1}{M_t})}{(1-\frac{\alpha}{\beta})\frac{1}{V_t} - \frac{\alpha}{\beta}(\frac{1}{M_t})}$$

## IV. Conclusion

Marxian reproduction solution established an economic equilibrium, which can be characterized by input-(total) output ratio, namely, the reduced Organic Composite of Capital divided by the total productivity. The labor value can be determined from the production price by the use of the input-output matrix analysis. The value ROP and the price ROP analyses of input-output data provide an accurate description on the OCC change, which reflects the industry structure adjustment. Under the framework of the dynamic Marxian general equilibrium, it is possible to undergo an optimization about an economic system by the regulation of the input, taxation, and minimal wage rate, so as to realize the development of the productivity of the society. In short, this study is aimed to obtain a quantitative description of Marxian capital theory including Marx labour value function and Marx surplus value function as well as Marx production function. The labor output elasticity ( $1-\alpha$ ) of Cobb-Douglas production function is defined as the parameter for the division of labor. The productivity parameter in Marx production function is defined as the product of the change rate of the organic composite of capital with the coefficient of the division of labor.