DETERMINING THE MASS OF THE DARK MATTER PARTICLE

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Abstract

It is clear that by measuring the density profile ρ(r) of a dark matter galactic halo we are also indirectly measuring the mass of the dark matter particles (WIMPs.) So in this paper we will derive such mass value m_{wimp} from an actual measured ρ(r) function.

It is also undeniable that gravitational fields affect the form and shape of the wave functions Ψ(r,t) of quantum particles, since such wave functions must live in the space-time defined by the metric $g_{\mu\nu}$ consequence of the distribution of mass. In addition, in the case of dark matter, where no other interaction is assumed acting, such gravitational fields are the only potential energy present in the Hamiltonian to determine the shape of the quantum wave function $\Psi_{wimp}(r,t)$.

Therefore to determine such wave function of dark matter particles inside galactic halos we will consider a quantum equation containing in the Hamiltonian the gravitational potential energy due to the totality of the galactic matter, both baryonic and dark. At some point we will propose this, our derived quantum equation, as the sought after universal law for dark matter halos.

Along this process we will pay some special attention to the fact that we will be solving a quantum equation, including boundary conditions, etc, under very unusual circumstances in terms of lengths of galactic proportions, extremely large information travel time across the whole halo quantum system, etc.

Finally, taking the Milky Way as our case of study, we will solve our quantum equation to determine the mass m_{wimp} of this elusive dark matter particle.

1.- INTRODUCTION.

In this research work we will focus on developing a quantum equation for dark matter, on its application to galactic halos and on the evaluation of such equation to be the sought after halos' universal law. Afterwards, we will use such equation to estimate the mass of the dark matter particle.

The best evidence for invisible matter in galaxies is given by the rotation curves of their visible matter: their rotation speed do not fall off fast enough at large distances from the center of the galaxy to match the prediction of Newton's gravitational law under visible mass, as in Fig 1.

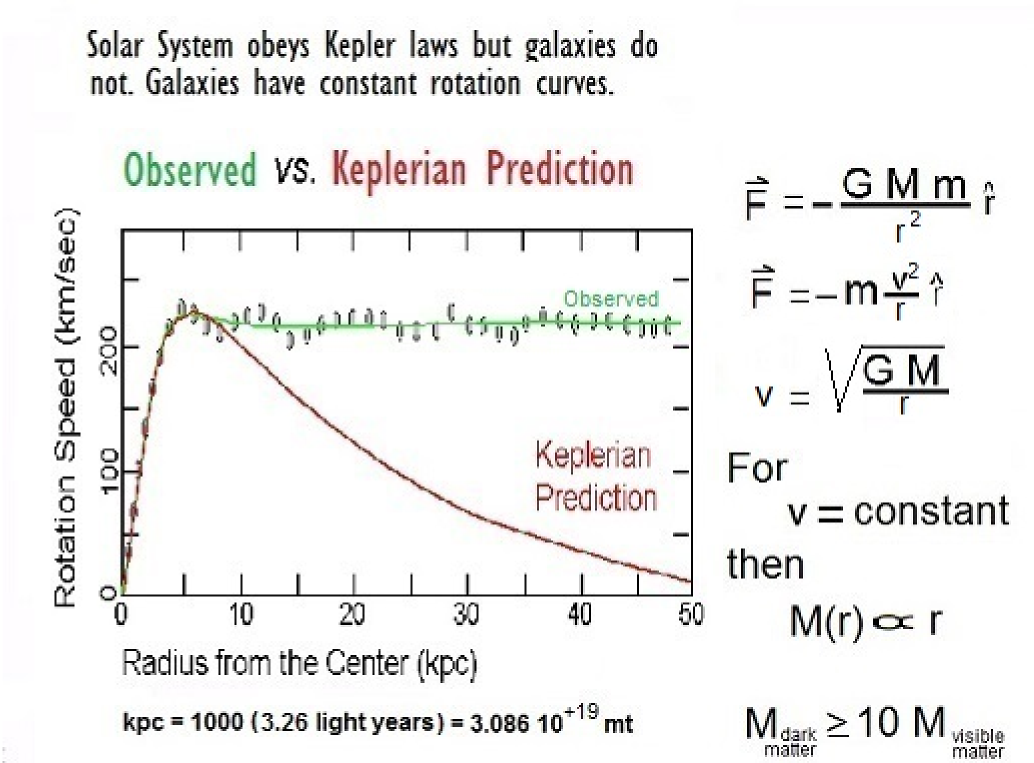


Fig 1

Such discrepancy could be explained either by assuming that Newton's law of gravitation is not valid at large distances, or by assuming that in addition to the visible, or baryonic, matter, there is a distribution of invisible matter, better known as dark matter. In the latter case, and for the flat rotation curves that are observed, the density of dark matter has to fall off like $1/r^{\alpha}$ at large distances, with $\alpha > 2$.

Given the overwhelming evidence for dark matter at all scales (from the flatness of the universe, combined with gravitational lensing down to the formation of galaxies and the large-scale structure with dark matter,) we will assume that dark matter does exist, and that at its most fundamental level it consists of a single, yet unknown and undetermined, fundamental "particle" (or quantum field, or entity, as we would prefer to call it,) widely referred to as WIMP.

Such fundamental quantum entity has the property that it does not interact with light nor any other electromagnetic radiation. We will also include the weak nuclear force in the no-interaction list, as it has been confirmed by the failure of detection in the latest experiments. Therefore we are assuming that dark matter, in good extent, can only be detected by measuring its gravitational effects.

We are also assuming that gravitational field $g_{\mu\nu}$ to be classical all the way down to Plank distances since all attempts to develop and test a quantum theory of gravitation, with its corresponding quantum of interaction, have not been successful. But in any case, regarding the dimensions of the spatial component of the quantum wave function $\Psi_{wimp}(r,t)$ for WIMPs, quite opposite, what we expect are very low energies and wavelengths of galactic proportions.

In the quantum world, it is interactions what drive fundamental quantum entities into adopting definite-position, or "particle," states $\delta(r_o)$ (using Dirac's delta function,) at the moment of the interactions. However, without further interactions these "particle" states are highly temporary since according to Heisenberg's uncertainty principle such state will naturally evolve asymptotically (as $\Delta E >\sim \hbar/\Delta t$,) toward some non-localized definite energy state φ_n with E_n = const. This can also be seen from a purely mathematical point of view: since such $\delta(r_o)$ packet is built as a superposition of waves of all frequencies, it will spread as it travels, due to a non-linear dispersion relation $\omega(\mathbf{k})$.

In our baryonic world of frequent interactions (electromagnetic, etc) such evolution into a definite energy state φ_n as $\Delta t \rightarrow \infty$ will be promptly interrupted by a subsequent interaction which will localize (collapse) the quantum entity back into another "particle" $\delta(r')$ state. Thus, the cycle from localized-to-delocalized states starts all over again. This is the way our macroscopic baryonic thermal reality of huge conglomerates of interacting fundamental quantum entities is continuously created, irreversibly collapsed, and then recreated again. No room for macroscopic Schrödinger's cats in here. Also, such irreversibility at quantum level has much more to do with the irreversible character of the "arrow of time" than any thermodynamic consideration.

Therefore, as far as the no-interactions assumptions is valid for WIMPs, we expect them to always be in non-localized states. Also, no electromagnetic/weak interactions means the total absence of thermal "collisions," implying that the temperature of dark matter, such as we understand such concept in baryonic thermodynamics, would be extremely low ($T \sim \varepsilon$ ^oK,) leading dark matter quantum entities into forming Bose-Einstein condensate structures, if we were also to assume their spin to be an integer. In other words, our thermic universe does not exist (or barely) for dark matter.

Such integer spin assumption is less arbitrary than what it seems. According to some authors, due to Pauli exclusion principle, quantum entities with spin 1/2 could not provide enough mass to explain the observed visible mass rotation curves.[49]

Another point worth mentioning about quantum interpretation is the too frequent insistence that, even once the fundamental quantum particle is in a stationary definite energy state $\varphi_n(x,y,z)$, a classical point-like particle is still moving around somewhere inside the $\varphi_n(x,y,z)$ "cloud." This is a wrong interpretation. Mathematically speaking, that would be equivalent to thinking that embedded inside a stationary state function $\varphi_n(x,y,z)$ there is still another time-dependent non-spreading $\delta(r,t)$ function unaccounted for by the theory. The fact is that, once the quantum particle is in a stationary state $\varphi_n(x,y,z)$, with no definite position, it has no position at all. In other words, it is nowhere (not everywhere, as some people mistakenly like to say, but literally nowhere.) All quantum paradoxes, from the double slit experiment paradox about "what slit did the particle actually go through," to Einstein's "spooky action at a distance" can be better understood by keeping this "nowhere" fact it mind. This will also help to put in a proper context those frequent estimations of a number of dark matter particles "crossing" a given area, or detector, per second.

Of course, this concept of "nowhere" (or no-value) also applies to all magnitudes with no definite eigenvalue in any current quantum state. In fact, the whole probabilistic nature of quantum mechanics is a direct logical consequence of magnitudes having no-values at all, because only the total absence of values is logically consistent with the adoption of a particular one in a truly probabilistic manner (when measured.)

And the fact that the wave function of a current quantum state can always be expressed as a superposition of any complete set of single eigen-value functions is nothing more than a mathematical coincidence product of the beautiful consistency of the quantum formalism, so trying to interpret that as that the current quantum state possesses at the same time all the eigen-values in a complete set is absurd, as absurd as it is to claim that an electrical outlet of 120 volts measures at the same time the values of 1 volt, and 2 volts, and 3 volts... and n volts just because 1+2+3...+n = 120.

2.- Extending the Schrödinger equation to galactic scales.

In this research work we will present our rather unusual gravitation potential time-independent Schrödinger-type of equation to be applied to a single WIMP in a galactic halo structure, and will solve such equation to estimate the mass of such dark matter quantum entity. In this equation a single WIMP quantum entity will be subjected solely to the gravitational potential produced by the galactic matter, both dark matter and visible matter, the latter being estimated to be only between 10% to 20% of the total mass of the galaxy.

In principle, due to the galaxy's baryonic mass dynamics, this problem would require to deal with the time-dependent Schrödinger equation (1-a), with a time-dependent gravitational energy field $U_g(r,t)$:

$$[\frac{-\hbar^2}{2m}\nabla^2+U_g(r,t)]\Psi(r,t)=i\hbar\frac{\partial}{\partial t}\Psi(r,t) \quad (1\text{-a})$$

However, due to the high ratio of dark matter over baryonic matter in a galactic average, we will approximate equation 1-a into a time-average time independent Schrödinger-type of equation:

$$[\frac{-\hbar^2}{2m}\nabla^2+\langle U_g\rangle(r)]\Psi(r) = E\,\Psi(r) \quad (1\text{-b})$$

Also, with this approach we are extending the range of validity of Schrödinger equations (1) from the well tested atomic/human distances to thousands of parsecs (1 parsec = 3.26 light years = $3.086 \times 10^{+16}$ meters.)

This will require an assumption of a strong degree of "immutability" (stability) on the part of the dark matter halo dark matter systems, since by being non-relativistic, the Schrödinger equations (1-a) and (1-b) are implicitly assuming the speed of light to be $\mathbf{c} \rightarrow \infty$ (this is,

instantaneous transport of information.) Such approximation might bear no measurable consequences at atomic or human distances, but that might not be necessarily the case when galactic distances are involved.

Roughly speaking, and taking the Milky Way as an example, with a diameter of 100,000 light years, if we assume that its halo's effective diameter is 100 times that value, and giving an extra margin of again 100 times that value, then we would need $10^5 x 10^2 x 10^2 = 10^9$ years of "immutability" for the Milky Way, which still is 1/12 of its estimated age.

3.- Could a General Relativity quantum equation be more suitable to describe WIMPs wave function?

Due to its appealing generality, the first possibility that we considered to determine the wave function $\Psi_{WIMP}(r,t)$ of a single WIMP quantum entity inside a galactic halo was a general relativity extension to the Klein Gordon equation for a single free "particle," free meaning not subjected to any non-gravitational potential:

$$\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu)\psi_{wimp}-(\frac{mc^2}{\hbar c})^2\psi_{wimp}=0 \qquad (2)$$

Such approach must be valid since the Klein-Gordon equation refers to a free particle in a gravitational field $g_{\mu\nu}$ where the meaning of being free is simply moving by inertia along a geodesic of the space-time defined by $g_{\mu\nu}$.

Of course, a delocalized quantum entity occupying a given region would overlap all neighboring geodesics in that space-time region, and such region could be quite extended. As far as we know, no proper theoretical consideration has been given to the gravitational field created by delocalized matter which, regardless how minute their field may be, it must exist as a matter of theoretical principle.

And for the gravitational field $g_{\mu\nu}$ inside the halo to be used in equation (2) we contemplated using a Schwarzschild inner solution with variable halo mass density profile $\rho(r)$, as determined by astronomical observations.

Among some other considerations, the Klein-Gordon equation refers to particles of spin=0, which is precisely the justified assumption we are making. Also, since this is a low energy situation, we could use equation (2) disregarding the instabilities of the Klein-Gordon (and Dirac) equations due to the process of creation and annihilation of virtual particles in empty space due to high energies.

However, with all things considered, despite it's seductive generality, we soon discarded such approach since in our present case of dark matter halos, we will be dealing, not only with very low energy and small gravitational fields, but also with an assumption of a high degree of "immutability" of the halo system (being structures settled long time ago,) as we discussed above, and those conditions do not justify the obvious complexity of a General Relativity approach. Besides, we don't believe that a such approach would provide additional insight on the physics of our problem. Therefore, we will develop a non-relativistic approach.

Finally, we can notice that equation (2) has a mass dependency of m^2, dependency that in the case of Dirac's equation led to his prediction of the existence of antimatter. We'll encounter a similar m^2 dependency in our non-relativistic quantum equation for WIMPs.

4.- The universal (quantum) law for dark matter halos.

We will look solutions for a tri-dimensional time-independent Schrödinger type of equation for a single WIMP quantum entity of mass m inside a galactic dark matter halo.

In such equation we will use the gravitational potential $V(r, \theta, \phi)$ (or rather, the gravitational potential energy $Ep = m_{wimp}V(r, \theta, \phi)$) produced by the distribution of mass of the galactic halo itself around the center of the galaxy (at r=0) as part of the Hamiltonian. This distribution of dark matter mass in halos will be taken from observed measurements and fittings of the density profiles ρ of actual halos (rather than numerical classical simulations,) and this quantum equation will be our proposed universal law for dark matter halos.

In principle the specification of the mass distribution of a tri-dimensional halo would require three space coordinates, such as $\rho = \rho(r, \theta, \phi)$. However, such degree of three-dimensional detail of halos structures we have found is beyond what current observations are able to provide, which is mostly the radial dependency of density profiles.

Dark matter profiles are difficult to measure, and an impressive amount of work have been conducted to estimate the distribution of dark matter around galaxies.

Estimation procedures go from measuring the discrepancy between the actual rotation speeds of galactic stars compared to the ones predicted by Newton's law assuming the presence of visible matter alone (as we summarized in Fig 1), through the measuring of the bending of the light of stars due to, presumably, the distribution of dark matter (gravitational lensing,) to even computer simulation based on classical mechanics.

One of the limitation of estimating ρ(r) based solely on the rotation speeds is that dark matter haloes extent quite beyond the limits of the visible galaxy (Fig 2.) It is estimated that dark matter constitutes 80-90% of the mass of galaxies.

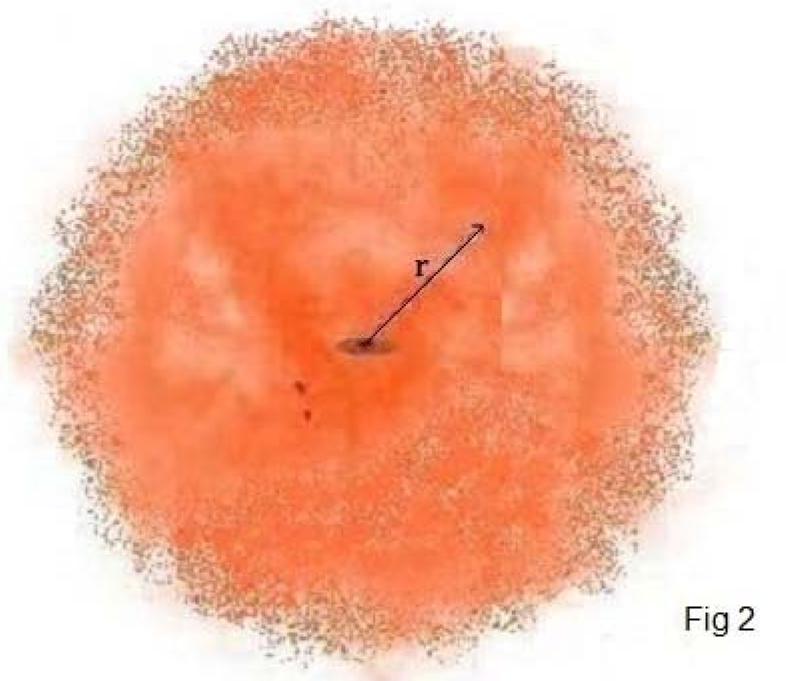


Fig 2

Today it is commonly believed that density profiles of dark matter halos can be well fitted by the universal density function, equation (3) below [30].

$$\rho(r) = \rho_0 \cdot \left(\frac{r}{r_0}\right)^{-\gamma} \cdot \left[\frac{1+\left(\frac{r}{a}\right)^{\alpha}}{1+\left(\frac{r_0}{a}\right)^{\alpha}}\right]^{\frac{\gamma-\beta}{\alpha}}$$

$$r = \sqrt{x^2 + \frac{y^2}{\epsilon_{xy}^2} + \frac{z^2}{\epsilon_z^2}} \qquad (3)$$

where r is the distance from the center of the galaxy and constants ρ_o, r_o, a, α, β, γ, ε_{xy} and ε_z are to be adjusted for each case. Constant ε_{xy} and ε_z will accommodate for oblate anisotropic dark matter halos.

Correspondence of equation (3) to some well known particular profiles forms are listed in the following table:

Profile	α	β	γ	a [kpc]	Reference
NFW	1.0	3.0	1.0	20.0	Navarro et al. (1997)
BE	1.0	3.0	0.3	10.2	Binney and Evans (2001)
Moore	1.5	3.0	1.5	30.0	Moore et al. (1999)
PISO	2.0	2.0	0.0	5.0	de Boer et al. (2005)
240	2.0	4.0	0.0	4.0	

So, our general quantum equation to be solved would be:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + mV(\mathbf{r},\theta,\phi)\right]\psi_{wimp}(\mathbf{r},\theta,\phi) = E\psi_{wimp}(\mathbf{r},\theta,\phi) \quad \text{(4-a)}$$

At the very core of our problem is the physical fact that in our quantum model the mass distribution of the halo, and therefore the potential $V(r,\theta,\phi)$, must strongly depend on the functional shape of the solution $\Psi_{WIMP}(r,\theta,\phi)$ of equation (4-a) (via $|\Psi_{WIMP}(r,\theta,\phi)|^2$,) making that equation highly non-linear. To emphasize this fact, lets rewrite it as equation (4-b):

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + mV(\mathbf{r},\theta,\phi,\psi_{DMC})\right]\psi_{wimp}(\mathbf{r},\theta,\phi) = E\psi_{wimp}(\mathbf{r},\theta,\phi) \quad \text{(4-b)}$$

Now, even for the cases of spherically symmetric density profiles ρ(r) (and likewise potentials V(r),) non-spherical symmetric solutions $\Psi_{WIMP}(r,\theta,\phi)$ for equation (4) would still be mathematically possible (due to the possibility of orbital angular momentum $\neq$ 0.) However, in our case we will discard such non-spherically symmetric solutions, since they would imply non spherical symmetric density profiles, contradicting our fundamental operational assumption of $\rho(r, \theta, \phi) = \rho(r)$.

Only the solutions that meet the condition that their $|\Psi_{WIMP}(r, \theta, \phi)|^2$ are proportional to ρ(r) will be physically consistent with our assumption of spherically symmetric density profiles ρ(r), as most models for density profiles ρ(r) assume. Such consideration will guide us to obtain a physically meaningful solution for equation (4-a) or (4-b), avoiding all its implicit non-linearities. Therefor the stationary state equation to be solved will be:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + mV(r)\right]\psi_{\text{wimp}}(\mathbf{r},\theta,\phi) = E\psi_{\text{wimp}}(\mathbf{r},\theta,\phi) \quad (5)$$

where V(r) is the galactic gravitational central potential given by:

$$V(r) = V_{\substack{\text{mass}\\\text{inside}}}(r) + V_{\substack{\text{mass}\\\text{outside}}}(r) \quad (6)$$

$$V(r) = -\frac{GM_i(r)}{r} - 4\pi G\int_r^{\infty}\rho(s)\, s\, ds$$

where G as Newton's gravitational constant and M_i(r) being the total mass enclosed within a given radius r:

$$M_i(r) = 4\pi\int_0^r \rho(s)\, s^2\, ds \quad (7)$$

To find solutions for equation (5) we will employ the standard variable separation technique. So, we will look for solutions of the following form:

$$\psi_w(\mathbf{r},\theta,\phi) \equiv R(\mathbf{r})\Theta(\theta)\Phi(\phi)$$
$$\psi_w(\mathbf{r},\theta,\phi) \equiv R(\mathbf{r})Y(\theta,\phi) \qquad (8)$$

where the angular component $Y(\theta, \phi)$ are the well known spherical harmonics, a complete set of orthogonal functions defined on the surface of a sphere of any radius. Therefore they can be used to expand functions defined on any sphere. This set of functions appear in many areas of physics as solutions for Laplace's equation and they are usually well explained in many introductory textbooks [38]

Each one of the functions in the complete set $Y(\theta\ \phi)$ is characterized by two variable separation constants ℓ and m_z, whose values are restricted by boundary conditions to be $\ell = 0, 1, 2...$ and $m_z = -\ell,... 0, ... +\ell$.

Therefore, to be more specific, we better denoted by $Y_{\ell\, mz}(\theta\ \phi)$ each function of this $Y(\theta, \phi)$ complete set. This set of functions can equally be used for any spherically symmetric central potential.

For the reasons explained above regarding the non-linearity of equations (4-b), we will discard all the $Y(\theta\ \phi)$ that do not conform to spherical symmetry. Therefore we will make $\ell = 0$ and $m_z = 0$, for which case we have:

$$Y_{0\,0}(\theta\ \phi) = \text{constant} = (4\pi)^{-1/2}$$

At this point there are a couple of points that we should mention: Oblate structures have been observed in some halos, as equation (3) explicitly stipulates. We can assume that such oblate shapes are the result of anisotropies in the baryonic mass distribution when the halo was in its formation stages, but also the possibility of WIMPs being at energy levels higher than the ground state shouldn't be ruled out. Due to the lack of any radiating mechanism (assuming a classical gravitational reality,) such WIMPs would remain at their energy level.

Also, as we already mentioned, due to our assumption of total lack of non-gravitational interactions for the WIMP quantum entities, and under our assumption of they having integer spin, the only state of matter available to dark matter would be that of a Bose–Einstein condensate (BEC): a dilute gas of bosons cooled to temperatures very close to absolute zero in which a large fraction of bosons occupy the lowest quantum state. Therefore we will look and focus on such lowest quantum state $\Psi_{WIMP}(r,\theta,\phi)$.

In addition, the existence of a small fraction of dark matter in higher energy states, such as having $\ell \neq 0$ within their energy level, cannot be completely ruled out, either in the lowest or even higher quantum states, since such possibility could explain the presence of some non-spherical substructures within halos.

5.- The radial equation.

After introducing definition (8) into equation (5), at the end of the variable separation process we get equation (9) for the radial component R(r)

$$\left[-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{\hbar^2 \boldsymbol{\ell}(\boldsymbol{\ell}+1)}{2mr^2}+mV(r)\right]R(r)=ER(r) \qquad (9)$$

where in our case the central potential V(r) is as in equation (10):

$$\begin{aligned} V(r) &= -\,A(r) - B(r) \\ A(r) &\equiv \frac{4\pi G\int_0^r \rho(s)\, s^2\, ds}{r} \qquad (10) \\ B(r) &\equiv 4\pi G\int_r^\infty \rho(s)\, s\, ds \end{aligned}$$

Introducing function **u(r) ≡ r R(r)**, making **ℓ = 0** and substituting our definition (10) for potential **V(r)** into (9) we get equation (11)

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - m\,A(r) - m\,B(r)\right]u(r) = E\,u(r)$$

$$\frac{\hbar^2}{2}\frac{d^2 u(r)}{dr^2} + m^2\left[A(r) + B(r)\right]u(r) = -mE\,u(r) \qquad (11)$$

$$\frac{\hbar^2}{2}\frac{d^2 u(r)}{dr^2} + m\left[m\left(A(r)+B(r)\right) + E\right]u(r) = 0$$

One thing to notice is the quadratic dependency on m (or m_{WIMP}) shown in equation (11). Also, functions A(r) and B(r) are in fact A(r,ρ(r)) and B(r,ρ(r)) where ρ(r) is the density profile of the dark matter halo in question.

Let's use the universal density profile, equation (3), assuming spherical symmetry and with α = 2, β = 4 and γ = 0. This yields to equation (12):

$$\rho_{\alpha\beta\gamma}(r) = \rho_{240}(r) = \rho_o \cdot \left[\frac{1+\left(\frac{r}{a}\right)^2}{1+\left(\frac{r_o}{a}\right)^2}\right]^{-2} \qquad (12)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

Inserting equation (12) into equation (10) and performing the integrals we obtain:

$$V(r) = -A(r) - B(r) = -(A(r) + B(r))$$

$$A(r)+B(r) = 2\pi G \rho_0 a^2 \left(1+\frac{r_0^2}{a^2}\right)^2 \frac{\arctan(r/a)}{(r/a)}$$

$$V(r) = -2\pi G \rho_0 a^2 \left(1+\frac{r_0^2}{a^2}\right)^2 \frac{\arctan(r/a)}{(r/a)} \qquad (13)$$

$$\left.\begin{aligned} V(r) &= -P\,\frac{\arctan(r/a)}{(r/a)} \\ \mathbf{E_p}(r) &= -m P\,\frac{\arctan(r/a)}{(r/a)} \end{aligned}\right\} \text{with } P \equiv 2\pi G \rho_0 a^2 \left(1+\frac{r_0^2}{a^2}\right)^2$$

Plugging this result in equation (13) into our radial quantum equation (11), we obtain:

$$\frac{d^2\mathbf{u(r)}}{dr^2} + \frac{2m}{\hbar^2}\left[m\,P\,\frac{\arctan(r/a)}{(r/a)} + E\right]\mathbf{u(r)} = 0 \qquad (14)$$

$$\text{with } P \equiv 2\pi G \rho_0 a^2 \left(1+\frac{r_0^2}{a^2}\right)^2$$

Besides the wave state function **u(r)** and the available eigenvalues of total energy E, this equation presents an additional challenging unknown: the mass m of the WIMP quantum entity, which is a major objective of this research work. This is an atypical situation that presents an additional difficulty compared to most cases of finding solutions for Schrödinger equations, where the mass of the quantum entity involved is normally well known.

In this regard we will make sound educated guesses of initial seed values for these parameters E and m before entering a numerical iteration process to solve equation (14), finding u(r) and final good-enough values for E and specially m.

6.- The scale and size of our quantum problem: the Milky Way.

To have a better perception of the scale of our problem at hand, we will introduce some values and plot some functions for our case of study, which in this first paper will be our galaxy: the Milky Way.

A **parsec** is a typical unit used in astronomical problems, and **one** $\mathrm{parsec} := 3.0857\cdot 10^{16}\mathrm{m}$ According to NASA, the radius of the Milky Way is $r_{mw} := 15000\mathrm{parsec} \quad \rightarrow \quad r_{mw} = 4.629\times 10^{20}\mathrm{m}$

As for the distance from our solar system to the center of the Milky Way, or parameter $\mathbf{r_o}$, we have:

$$ro := 8.33\cdot 10^{3}\cdot \mathrm{parsec} \quad \text{or} \quad ro = 2.57\times 10^{20}\mathrm{m}$$

As parameter $\boldsymbol{\rho_o}$, the mass density at $\mathbf{r_o}$, our solar system region, we will use a value which includes visible as well as dark matter, as measured by the Hipparcos satellite: $\rho_o = 0.102\ M_{sun} / \mathrm{parsec}^3$ [30].

Thus with a solar mass value of $M_{sun} := 1.989\,10^{30}\mathrm{kg}$ we have

$$\rho o := \frac{0.102\cdot M_{sun}}{\mathrm{parsec}^3} \quad \rightarrow \quad \rho o = 6.905\times 10^{-21}\frac{\mathrm{kg}}{\mathrm{m}^3}$$

Also, from the table of density profiles above we have that for ρ240(r) profile $a := 4000\cdot \mathrm{parsec} \quad \rightarrow \quad a = 1.234\times 10^{20}\mathrm{m}$

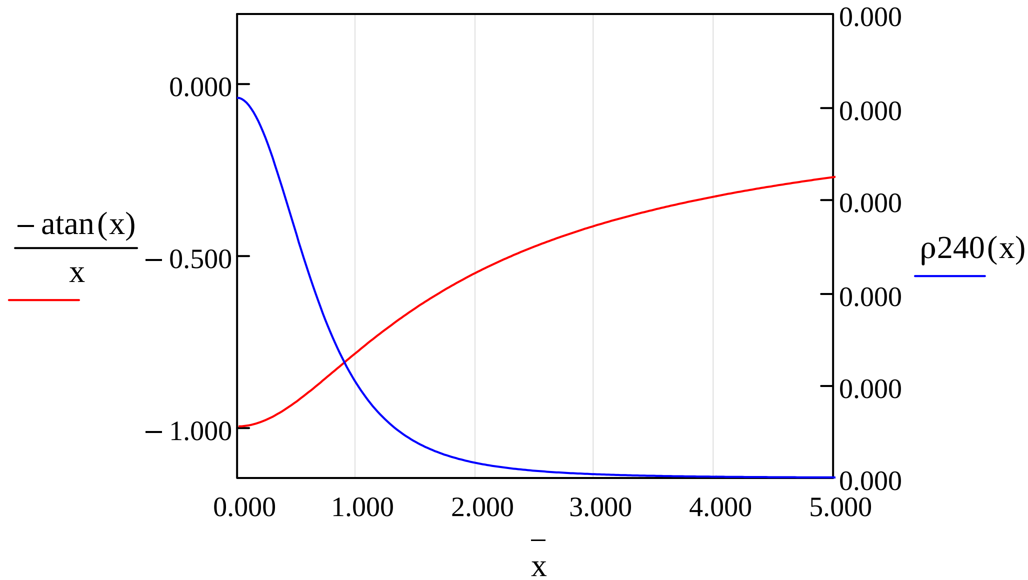
We need Newton's gravitational constant $G := 6.67259\cdot 10^{-11}\cdot\frac{\mathrm{m}^3}{\mathrm{kg}\cdot \mathrm{sec}^2}$ to compute the potential constant P as:

$$P := 2\cdot\pi\cdot G\cdot\rho o\cdot a^2\cdot\left(1+\frac{ro^2}{a^2}\right)^2 \quad \rightarrow \quad P = 1.256\times 10^{12}\frac{\mathrm{m}^2}{\mathrm{s}^2}$$

As for our density profile we now define the function ρ240(x) below as:

$$\rho240(x) := \rho o \frac{\left(1 + \frac{ro^2}{a^2}\right)^2}{\left[1 + (x)^2\right]^2} \quad = \quad \rho240\left(\frac{r}{a}\right)$$

This functional forms of ρ240(x) and that of our central potential in equation (14), -arctan(x)/x, are graphed below, with x = r/a > 0 as independent variable.



Distance in units of a (x = r/a)

Later, to meet continuity and boundary conditions we will mirror our potential, and its solutions, onto the region r < 0, although, of course, only the region r >= 0 will have physical meaning.

Finally, as a consistency check, we can compute the total mass of the Milky Way that our ρ240(r) yields to by integrating it from r = 0 to ∞ and compare it with various tabulated estimates. Thus, based on our ρ240(r), the total mass of the Milky way is

$$M\rho\infty = 4\cdot\pi\cdot\int_0^\infty \rho240\left(\frac{r}{a}\right)\cdot r^2\,dr \qquad \rightarrow$$

$$\rightarrow \quad M\rho\infty := \pi^2\cdot\rho o\cdot\left(1+\frac{ro^2}{a^2}\right)^2\cdot a^3 \qquad \rightarrow$$

$$M\rho\infty = 3.65\times10^{42}\,kg \quad \text{or} \quad \frac{M\rho\infty}{M_{sun}} = 1.835\times10^{12} \quad \text{solar masses.}$$

This our value of Mρ∞ is around two times the smallest we could find (a recent estimate of 0.9×10^{12} solar masses extended out to 300 kpc, based on the method of hierarchical Bayesian statistical analysis.) However, other recent studies indicate a range in mass as large as 4.5×10^12. In other words, our integrated Mρ∞ estimate is well within the middle of the acceptable range.

In any case, we adjudicate the predicting power of our model not onto whether or not it exactly conforms numerically to a particular galactic case, which by necessity of imperfect measurements is going to drag a high degree of uncertainty, but to the fact that the values that we use consistently describe a possible dark-matter galactic structure that because of our model following very fundamental laws of physics, it could exist in a galaxy somewhere else in the universe.

7.- Asymptotic solutions for our radial equation.

Examining the asymptotic behavior of equation (14) as $r \rightarrow 0$ and as $r \rightarrow \infty$ we get the following equations for the asymptotic functions $u_o(r)$ and $u_\infty(r)$:

As $r \to 0$:

$$\frac{d^2 \mathbf{u_o(r)}}{dr^2} + \frac{2m}{\hbar^2}\left[m\mathrm{P} + E\right]\mathbf{u_o(r)} = 0 \tag{15}$$

As $r \to \infty$:

$$\frac{d^2 \mathbf{u_\infty(r)}}{dr^2} + \frac{2m}{\hbar^2} E\,\mathbf{u_\infty(r)} = 0$$

In our particular case, we are interested in bound states, which means total E < 0. Since that E > - m P for all cases, then for our bound case - m P < E < 0. After imposing convergence conditions, the solutions for these asymptotic equations are:

For $r \to 0$:

$$\mathbf{u_o(r)} = \mathrm{A}\sin(kr) + \mathrm{B}\cos(kr)$$

$$\text{with } k = \pm\sqrt{\frac{2m}{\hbar^2}\left[m\mathrm{P} + E\right]} \tag{16}$$

For $r \to \infty$:

$$\mathbf{u_\infty(r)} = \mathrm{C}\,\mathbf{e}^{-qr} \quad \text{with } \mathbf{q} = \pm\sqrt{-\frac{2m}{\hbar^2}E}$$

Both k and q are real since $-m\mathrm{P} < E < 0$ for bound states.

Also $k^2 + q^2 = \frac{2\mathrm{m}}{\hbar^2}\mathrm{P}$ with $\mathrm{P} = 2\pi G \rho_0 a^2 (1 + \frac{r_0^2}{a^2})^2$

Now we are going to test our claim that the actual ρ(r) in a real physical situation will be directly proportional to the probability density involving the wave function solution $\Psi_{WIMP}(r, \theta, \phi)$ to that situation. In other words, $\rho(r) \propto |\Psi_{WIMP}(r, \theta, \phi)|^2$ Thus, with **D** as an arbitrary normalizing factor, our claim is that $u(r) \propto (\rho240(r))^{1/2}$ or

$$u(r) = \mathbf{D}\left[\frac{1+\left(\frac{r}{a}\right)^2}{1+\left(\frac{ro}{a}\right)^2}\right]^{-1} = \mathbf{D}\left[\frac{1+\left(\frac{ro}{a}\right)^2}{1+\left(\frac{r}{a}\right)^2}\right] \qquad (17)$$

Now, why $u(r) \propto \rho240(r)^{1/2}$ and not $R(r) \propto \rho240(r)^{1/2}$? Well, $\rho240(r)$

must be normalizable (as we already did by obtaining $M\rho\infty$) and the radial component of $\Psi(r, \theta, \phi)$ that is normalizable is not R(r) but r R(r), which is by definition u(r).

We must also keep in mind that, considering the reliability of the processes available today for estimating and measuring dark matter halo densities ρ(r), this proportionality assumption will be less reliable for testing purposes as the distance r increases.

Thus, substituting equation (17) assumption into either of the $u_o(r)$, equations in (15) or (16) above, imposing further continuity and boundary conditions based on our mirroring of our potential and its solutions onto the region $r < 0$, and making $r \rightarrow 0$ yet again, we obtain the relation among the problem's unknown constants:

$$\mathbf{E} = \frac{\hbar^2}{m a^2} - m P \qquad (18)$$

8.- Exploring a Harmonic Oscillator approximation near the origin.

Before starting our iteration process to solve equation (14) numerically, we want to estimate the starting seeds for the energy eigenvalue E, and later the WIMP mass m.

For that we will be applying a harmonic oscillator approximation solution to our potential around r ~ 0. Once this pre-iteration numerical seeds for E is determined, we will use equation (20) to estimate a pre-iteration value for $m = m_{WIMP}$.

For that process, a graphic comparison of our halo's potential atan(x)/x, (in units of the product m·P,) vs the harmonic oscillator potential $1/2\ k\ x^2$, with $x = r/a$ for both, may be helpful.

Our halo potential function $E_p(r) = -\ m\ P\ \mathrm{atan}(r/a) / (r/a)$ can expanded, for $x = r/a < 1$, as

$$\arctan(x) / x = (1 - x^2/3 + x^4/5 - x^6/7 + \ldots) \quad \rightarrow$$

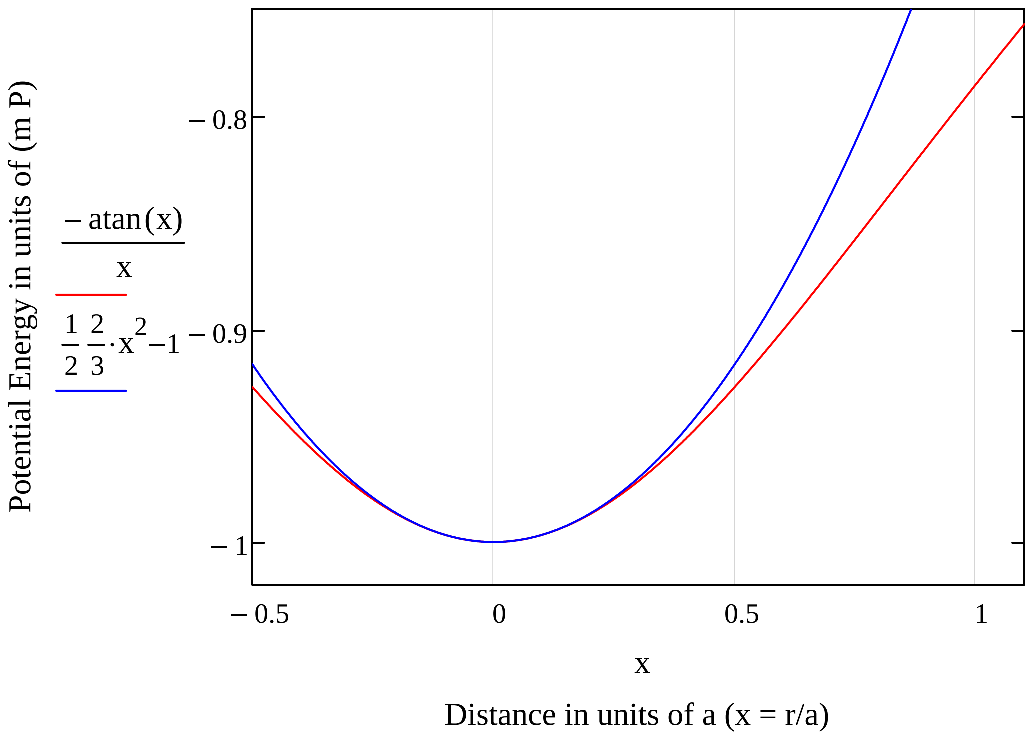
$$E_p(r) = -\ m\ P\ (1 - x^2/3 + x^4/5 - x^6/7 + \ldots) \quad \text{or}$$

$$E_p(r) = -m\ P + m\ P\ r^2/(3\ a^2) - O(r^4)\ldots$$

In the harmonic oscillator model the potential is $V(r) = Vo + (1/2)\ k\ r^2$ so by expressing $E_p(r)$ above in the same $V(r)$ form we can identify an expression for k to be used in our harmonic oscillator approximation:

$$E_p(r) = 1/2\ \mathbf{[2\ m\ P/(3\ a^2)]}\ r^2 - m\ P \quad \rightarrow \quad \mathbf{k = 2\ m\ P/(3\ a^2)}$$

These comparisons are graphed below:



We will use a 1-D (and not a 3-D) oscillator model to approximate our potential (and provide the initial seeds for numerical integration) because equation (14) describes an unidimensional situation, where

the other two dimension have already been taken into account by the $Y_{\ell\, mz}(\theta\ \phi)$ part of the solution of $\Psi_{WIMP}(r, \theta, \phi)$. This is, one degree of freedom vs. the three degrees of freedom of the 3-D harmonic oscillator where

$$V(x,y,z)= 1/2\ k\ (x^2 + y^2 + z^2)^2$$

In other words, that variable "r" in equations (13), (14), etc, does not refer to the 3-D vector **r** but rather to a unidimensional parameter by the name of r.

We summarize the well established solutions for the one-dimensional quantum harmonic oscillator as follows, in Fig 3 below.

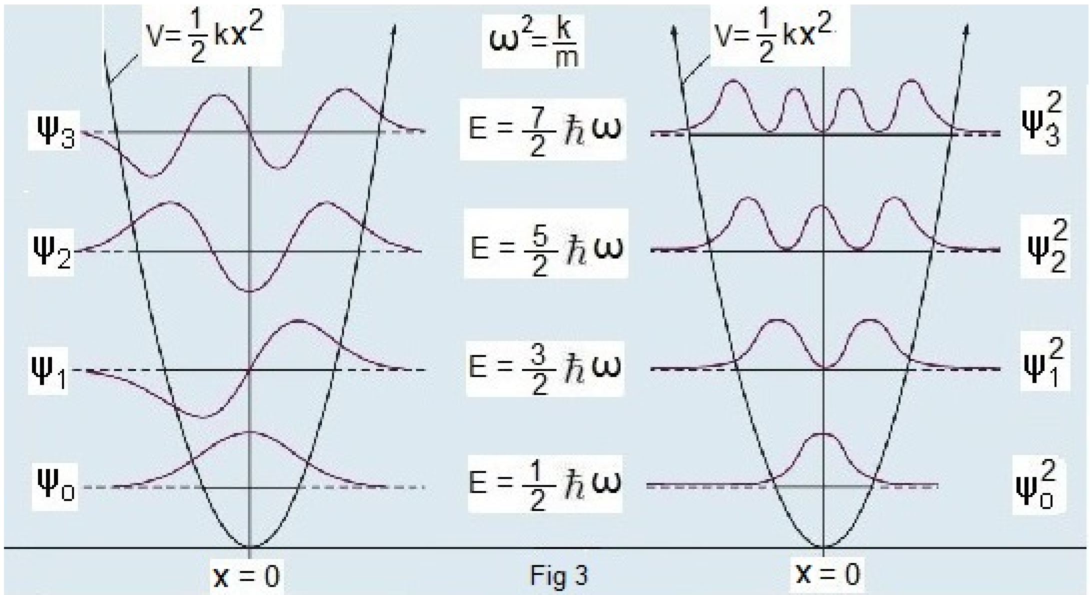


Fig 3

and its normalized ground state solution $\psi_o(x)$ as:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-m\omega x^2/2\hbar} \qquad (19)$$

So now we can evaluate ω as $\omega^2 = k/m = 2\,P/(3\,a^2) \rightarrow$

$$\omega := \sqrt{\frac{2\cdot P}{3\cdot a^2}} \quad \text{or} \quad \omega = 7.414 \times 10^{-15}\frac{1}{s}$$

which in our case it happens to be independent of the mass m, an expected condition since that's the nature of gravity (principle of equivalence.)

As we already explained and justified above, we are assuming that our system is in the ground state, so in our harmonic oscillator approximation this means, with Planck's constant $\hbar := 6.6260693 \cdot 10^{-34} \cdot \text{joule} \cdot \text{sec}$

that

$$Eo := \left(\frac{1}{2}\right) \cdot \hbar \cdot \omega \quad \text{or} \quad Eo = 2.456 \times 10^{-48} \, J$$

In our case the "zero energy point" is at -m P, so in this harmonic oscillator approximation the total energy at the ground state level is, $E = E_o - m P$. So substituting this in equation (18) and solving for the WIMP mass m we have:

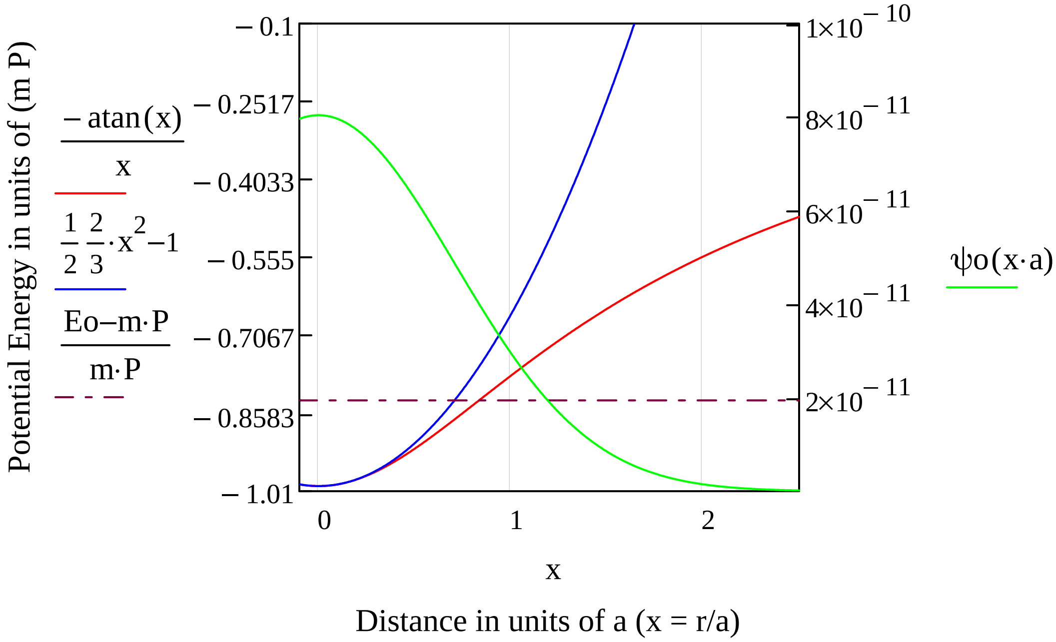
$$E = Eo - m P = \hbar^2 / (m a^2) - m P \rightarrow Eo = \hbar^2 / (m a^2) \rightarrow$$

$$m := \frac{\hbar^2}{Eo \cdot a^2} \quad \rightarrow \quad m = 1.173 \times 10^{-59} \, kg \quad \text{and}$$

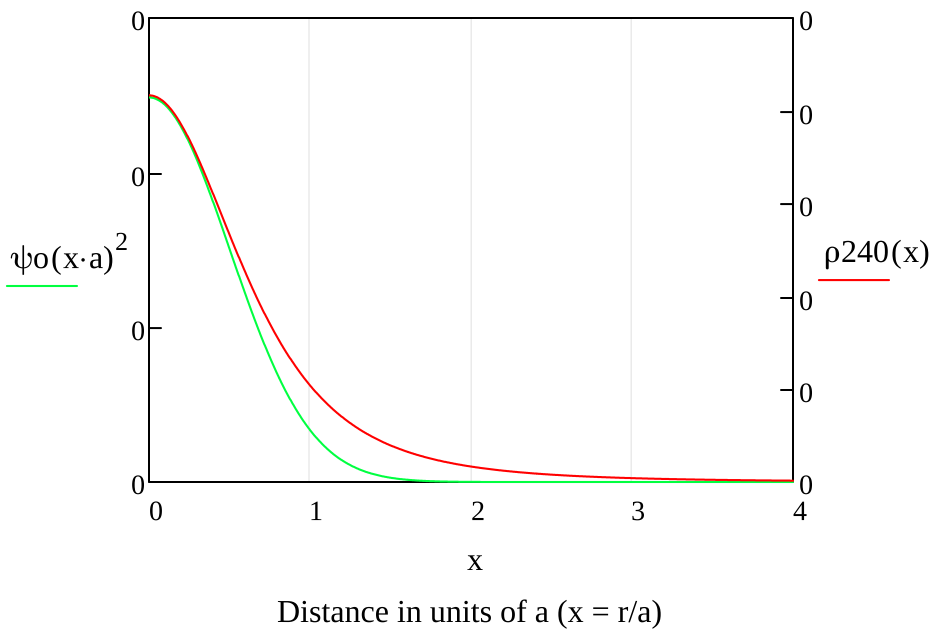
$$E := Eo - m \cdot P \quad \rightarrow \quad E = -1.228 \times 10^{-47} \, J$$

which yields to function $\psi o(x) := \left(\frac{m \cdot \omega}{\pi \cdot \hbar}\right)^{\frac{1}{4}} \cdot e^{\frac{-m \cdot \omega \cdot x^2}{2 \cdot \hbar}}$

Similar to Fig 3 above, and as an illustration, we will over-impose wave function $\psi_o(x)$ over both our harmonic oscillator approximation and actual gravitational potential energy functions. We will include an horizontal line marking the energy level of the fundamental state, (Eo - m P)/m P, in units of m P.



At this point we can proceed to graphically compare ρ240(r) with $\psi_o(r)^2$ to test how close the harmonic oscillator solution matches our claim of proportionality to ρ240(r). As we can see, we have a reasonable good match (scales of both Y axes have been arbitrarily adjusted to highlight the match):



9.- Solving radial equation (14) by numerical integration.

As we already stated, the assumption of proportionality between the measured ρ240(r) and $|\Psi_{WIMP}(r, \theta, \phi)|^2$, or $u(r) \propto \rho240(r)^{1/2}$, will be less reliable as the distance r increases. Therefore we better approach the solution of equation (14) as an initial value problem. So, as such initial conditions we would have

$$u(r)\big|_{r=0} = D \qquad \frac{du(r)}{dr}\bigg|_{r=0} = 0 \qquad (20)$$

where D (= u(0)) will be a normalization constant, of minor importance in our case since our main interest lies on comparisons.

We found out that our numerical solutions for equation (14) catastrophically fails if parameters m and E deviate too much from the values yielded by the harmonic oscillator approximation near de origin. However, for values very close to them, solutions can be found that quite reasonably match our proportionality claim for 0 < r < ~6.5 x a.

To perform such numerical iteration we will use constants k1 and k2 to adjust the values of parameters mass m and energy E that will produced the desired match when plugging them into equation (14) and solving it numerically.

Thus, with the values found for harmonic oscillatoı $E = -1.228 \times 10^{-47}\,\mathrm{J}$

$m = 1.173 \times 10^{-59}\,\mathrm{kg}$ and after few trials adjusting k1 and k2 at

the end of the process we find the value of mass k_1 m and energy k_2 E that best led to a solution of equation (14) that matched the density profile function ρ240(r) the closest.

To this purpose we change integration variable in equation (14), from r to x = r/a. We also substitute m for k_1 m and energy E for k_2 E, obtaining equation (23) below. So Given

$$u''(x) + \frac{2 \cdot m_k \cdot a^2}{\hbar^2} \cdot \left[m_k \cdot P \cdot \left(\frac{\operatorname{atan}(x)}{x} \right) + E_k \right] \cdot u(x) = 0 \qquad \textbf{(23)}$$

we numerically integrate from $x = 0$ up to $x = r/a \sim 7.5$. To avoid divergences when integrating from $r/a = x = 0$ we make

$d := 10^{-10}$ and set initial conditions as $u(d) = 1$ $\quad u'(d) = 0$

We used an ODE numerical integration function available in our software package (MathCad 15,) trying all its different algorithms available. For all of them the graphic results were very much the same.

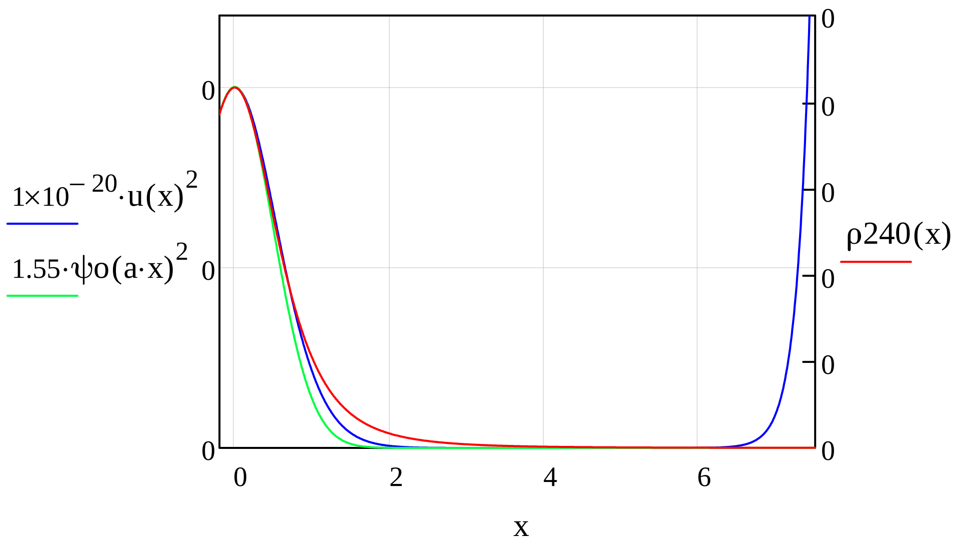
So, parameter values $k_1 := 1.03060$ and $k_2 := 1.0700754285$

make

$$m_k := k_1 \cdot m \quad \rightarrow \quad \boxed{m_k = 1.209 \times 10^{-59}\,\mathrm{kg}}$$

$$E_k := k_2 \cdot E \quad \rightarrow \quad E_k = -1.314 \times 10^{-47}\,\mathrm{J}$$

Thus we integrate $u := \mathrm{Odesolve}(x, 7.5)$ to obtain the following graph:



Distance in units of a (x = r/a)

Again, constants and scales of both Y axes have been arbitrarily adjusted to highlight the match.

As we can see, our solution for equation (14) (or 23) shows a closer match to **ρ240(x)** than the harmonic oscillator approximation.

Also, we notice that our numerical solution grows exponentially for x = r/a > ~ 6.5. This behavior is typically found when solving Schrödinger equations numerically, and it has to do with the inherent imprecision of the value of the energy eigenvalue obtained numerically.

Typically, an analytical solution for our equation (23) will be expressed as

$$u(r) = u_o(r)\ f(r)\ u_\infty(r)$$

where $u_o(r)$ and $u_\infty(r)$ are the functions obtained in equations (15) and (16), and function f(r) is expected to be well behaved only for certain values of $E = E_n$ for n=0,1,2...

In our case though, the facts that

1) Our adjustment constants **k1** and **k2** are very close to 1.0, which is the harmonic oscillator case

2) That the solutions for equation (23) badly fail when these **k1** and **k2** deviate just slightly from the values obtained above, close to 1.0,

3) That such critical failure constitutes evidence that our model is implicitly backed by the harmonic oscillator model make us feel confident that our values as obtained for $E = E_k$, and more importantly, that $m_{WIMP} = m_k$ which is the main objective of this research paper, must be excellent approximations.

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