## Part D.
## VBGC 1.2 - The extension and generalization of BGC as applied on i-primeths ($^{o}\wp^{*}$)

**VBGC 1.2 (version 1.2, the same with the version of this article) – main statement**:

1. **Defining i-primeths as:** $^{0}P_{x}=P\left(\underset{\substack{0\,iterations\\ of\,P\,on\,P}}{\underline{x}}\right)$, $^{1}P_{x}=P\left(\underset{\substack{1\,iteration\\ of\,P\,on\,P}}{\underline{P(x)}}\right)$, $^{2}P_{x}=P\left(\underset{\substack{2\,iterations\\ of\,P\,on\,P}}{\underline{P(P(x))}}\right)\ldots$

   $^{i}P_{x}=P\left(\underset{(i\geq 0)\,iterations}{\underline{P(P(\ldots P(x)))}}\right)$, with $P(x)$ being the x-th prime in the set of standard primes (usually denoted as $P(x)$ or $P_x$ and equivalent to $^{0}P_{x}$ alias "0-primeths") and the generic $^{i}P_{x}$ being named the generic set of i-primeths (with" i" being the "iterative"/recursive order of that i-primeth which measures the number of P-on-P iterations associated with that specific i-primeth subset).

   a. I have used the notation $^{0}P_{x}$ and $^{i}P_{x}$ instead of the standard notation $P^{1}(x)=P(x)\left[={}^{0}P_{x}\right]$ and $P^{i}(x)=\underset{i\,nested\;functions\,P}{\underline{P(P..P(x))}}\left[={}^{(i-1)}P_{x}\right]$, so that to strictly measure the number of P-on-P recursive steps (iterations) to produce a generic set $^{i}P$ from $^{0}P$ AND ALSO to not generate the confusion between $P^{i}(x)=\underset{i\,nested\;functions\,P}{\underline{P(P..P(x))}}$ and the exponential product $\left[P(x)\right]^{i}=\underset{i\,times}{\underline{P(x)\cdot P(x)\cdot\ldots P(x)}}$.

   b. It is also true that producing the elements of the (prime) function $P(x)$ from the natural set $\mathbb{N}^{*}$ is also like selecting just the naturals with prime indexes from $\mathbb{N}^{*}$, so that $^{0}P$ can be theoretically identified with $\mathbb{N}^{*}$ and the set of primes $\wp^{*}$ can be identified with $^{1}P$: however, $\mathbb{N}^{*}$ is not a set of primes and that is why I have avoided to note $\mathbb{N}^{*}$ with $^{0}P$ but to $^{(-1)}P$ (like the result of an inverse iteration) AND ALSO decided to count the sets of i-primeths starting from 0 (so that $^{0}P_{x}=P(x)$) in the purpose to strictly measure the number of P-on-P iterations starting from 1, so that $^{1}P_{x}=P\left(\underset{1iteration}{\underline{P(x)}}\right)$.

2. **<u>The inductive variant of VBGC states that</u>: "<u>Any even positive integer</u>** $\boxed{2m>2\cdot 2^{(a+1)(b+2)(a+b+1)}}$ **<u>can be written as the sum of at least one pair of distinct i-primeths</u>** $\boxed{^{a}P_{x}>{}^{b}P_{y}}$ **<u>, with the positive integers pair</u>** $\boxed{(a,b),with\,a\geq b\geq 0}$ **<u>defining the (recursive) orders of each of those i-primeths AND the pair of distinct positive integers</u>** $\boxed{(x,y),with\,x>y>1}$ **<u>defining the indexes of each of those i-primeths</u>.".**

3. **<u>Alternative formulation for the inductive variant of VBGC, using the standard notation</u>** $P^{1}(x)=P(x)={}^{0}P_{x}$, $P^{2}(x)=P(P(x))={}^{1}P_{x}$ and $P^{a}(x)={}^{(a-1)}P_{x}$: **"<u>Any even positive integer</u>** $\boxed{2m>2\cdot 2^{a(b+1)(a+b-1)}}$ **<u>can be written as the sum of at least one pair of distinct i-primeths</u>** $\boxed{P^{a}(x)>P^{b}(y)}$ **<u>, with the positive integers pair</u>** $\boxed{(a,b),with\,a\geq b\geq 0}$ **<u>defining the (recursive)</u>**

**<u>orders of each of those i-primeths</u>** $P^a(x)$ **<u>and</u>** $P^b(y)$ **<u>AND the pair of distinct positive integers</u>** $(x,y), with\, x > y > 1$ **<u>defining the indexes of each of those i-primeths</u>".**

4. **<u>The analytic variant of VBGC (from which the inductive VBGC can be intuitively inducted) states that</u>: "For any pair of finite positive integers** $(a,b), with\, a \ge b \ge 0$ defining the (recursive) orders of an a-primeth $\left({}^aP\right)$ and a b-primeth respectively $\left({}^bP\right)$, **there will always exist a single finite positive integer** $\left(n_{a,b} = n_{b,a}\right) \ge 3$ **so that, for any positive integer** $m > n_{a,b}$ **it will always exist at least one pair of finite *distinct* positive integers** $(x,y), with\, x > y > 1$ (indexes of distinct odd i-primeths) **so that:** ${}^aP_x + {}^bP_y = 2m$ **AND** ${}^aP_x > {}^bP_y$ **AND the function** $f(a,b) = f(b,a) = \left(n_{a,b} = n_{b,a}\right) \ge 3$ **has a finite positive integer value for any combination of finite positive integers** $(a,b)$**, without any catastrophic-like infinities for any** $(a,b)$ **pair of finites positive integers.**

   a. **<u>Important note</u>. I have chosen the additional conditions** $(a \ge b \ge 0) \wedge (x > y > 1) \Leftrightarrow$ ${}^aP_x > {}^bP_y$ **so that to lower the nof. lines per each GM and to simplify the algorithm of searching** $\left({}^aP_x, {}^bP_y\right)$ **pairs, as the set** ${}^aP$ **is much less dense that the set** ${}^bP$ for $a > b$ **AND the sieve using** ${}^aP$ **(which searches an** ${}^aP$ **starting from** $2m\ to\ 3$**) finds a** $\left({}^aP_x, {}^bP_y\right)$ **pair much more quicker than a sieve using** ${}^bP$ **(if** $a > b$**).**

   b. $f(0,0) = \left(n_{0,0}\right) = 3$

   c. $f(1,0) = f(0,1) = \left(n_{1,0} = n_{0,1}\right) = 3$

   d. $f(2,0) = f(0,2) = \left(n_{2,0} = n_{0,2}\right) = 2564$

   e. $f(1,1) = \left(n_{1,1}\right) = 40\ 306$

   f. $f(2,1) = f(1,2) = \left(n_{2,1} = n_{1,2}\right) = 1\ 765\ 126$

   g. $f(2,2) = \left(n_{2,2}\right) = 161\ 352\ 166$

   h. $f(3,0) = f(0,3) = \left(n_{3,0} = n_{0,3}\right) = ?$ **[working in progress on this function value]**

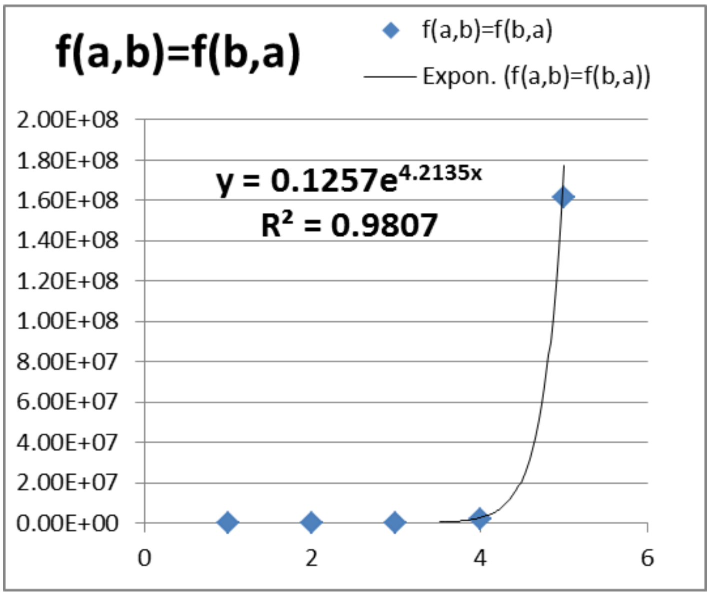
   i. $f(3,1) = f(1,3) = \left(n_{3,1} = n_{1,3}\right) = ?$ **[working in progress on this function value]**

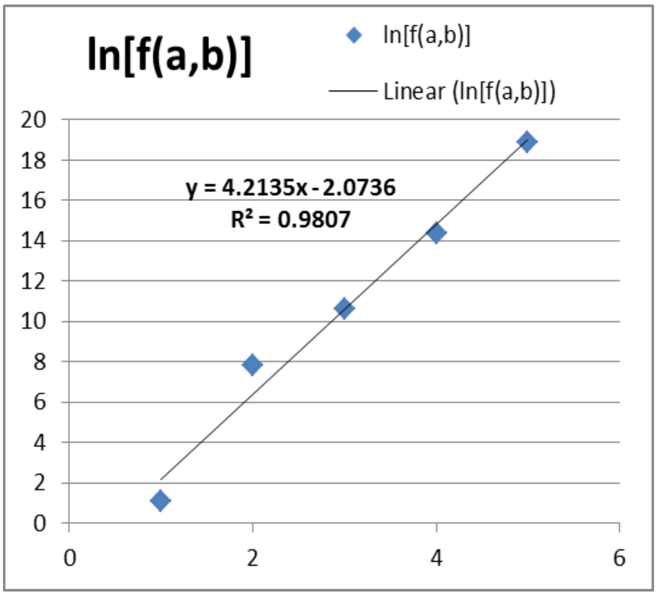
   j. $f(3,2) = f(2,3) = \left(n_{3,2} = n_{2,3}\right) = ?$ **[working in progress on this function value]**

k. $\boxed{f(3,3) = (n_{3,3}) = ?}$ **[working in progress on this function value]**

l. **…[working progress on other higher indexes function values]**

m. **Interestingly,** $\boxed{f(a,b)}$ **applied on** $a \in \{0,1,2\}$ **and** $b \in \{0,1,2\}$ **has its value in the set** $\boxed{F = \{3,\ 3,\ 2564,\ 40306,\ 1\,765\,126,\ 161\,352\,166\}}$ **which has an exponential pattern such as:** $\boxed{E_F = \{{\cong}1.1,\ {\cong}1.1,\ {\cong}7.8,\ {\cong}10.6,\ {\cong}14.4,\ {\cong}18.9\}}$**, with a relatively constant geometric progression between its last 4 elements so that** $\boxed{({\cong}18.9 / {\cong}14.4) \cong \overset{\cong}{}(14.4 / {\cong}10.6) \cong ({\cong}10.6/{\cong}7.8) \cong 1.32}$**. The gap between the exponents** ${\cong}1.1$ **and** ${\cong}7.8$ **may be possibly filled by** $\ln[f(3,0) = f(0,3)]$**,** $\ln[f(4,0) = f(0,4)]$ **… which are still in work to compute in the near future (see the next figures).**



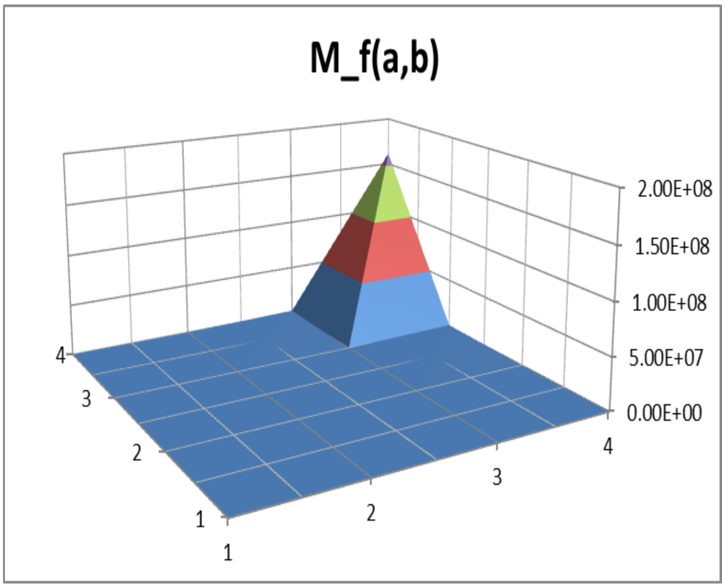



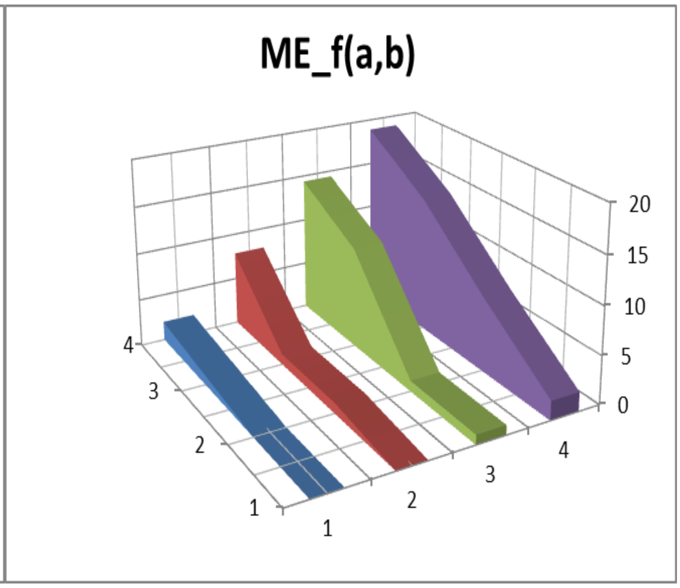

n. $F = \{3,\ 3,\ 2564,\ 40306,\ 1\,765\,126,\ 161\,352\,166\}$ **also has a correspondent matrix**

$$\boxed{M_{f(a,b)} = \begin{pmatrix} n_{0,0} & n_{1,0} = n_{0,1} & n_{2,0} = n_{0,2} \\ n_{0,1} = n_{1,0} & n_{1,1} & n_{2,1} = n_{1,2} \\ n_{0,2} = n_{2,0} & n_{1,2} = n_{2,1} & n_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{3} & \mathbf{3} & \mathbf{2564} \\ \mathbf{3} & \mathbf{40306} & \mathbf{1\,765\,126} \\ \mathbf{2564} & \mathbf{1\,765\,126} & \mathbf{161\,352\,166} \end{pmatrix}}$$ **and**

**a matrix of exponents** $$\boxed{ME_{f(a,b)} = \begin{pmatrix} \ln(n_{0,0}) & \ln(n_{1,0} = n_{0,1}) & \ln(n_{2,0} = n_{0,2}) \\ \ln(n_{0,1} = n_{1,0}) & \ln(n_{1,1}) & \ln(n_{2,1} = n_{1,2}) \\ \ln(n_{0,2} = n_{2,0}) & \ln(n_{1,2} = n_{2,1}) & \ln(n_{2,2}) \end{pmatrix}},$$

$$ME_{f(a,b)} = \begin{pmatrix} \mathbf{1.1} & \mathbf{1.1} & \mathbf{7.85} \\ \mathbf{1.1} & \mathbf{10.6} & \mathbf{14.38} \\ \mathbf{7.85} & \mathbf{14.38} & \mathbf{18.9} \end{pmatrix}$$ **which can both be graphed as a surfaces (see the next figure).**




o. **More interestingly, the function** $\boxed{fx(a,b)=2^{(a+1)(b+2)(a+b+1)}}$ **generates positive integer values that are relatively close BUT strictly larger than the values of** $f(a,b)$ **for** $a\in\{0,1,2\}$ **and** $b\in\{0,1,2\}$**, so that the author proposes a variant of inductive VBGC stating that:**

**"<u>Any even positive integer</u>** $\boxed{2m>2\cdot 2^{(a+1)(b+2)(a+b+1)}}$ **<u>can be written as the sum of at least one pair of distinct i-primeths</u>** $\boxed{{}^{a}P_{x}>{}^{b}P_{y}}$ **<u>, with the positive integers pair</u>** $\boxed{(a,b), with\, a\geq b\geq 0}$ **<u>defining the (recursive) orders of each of those i-primeths AND the pair of distinct positive integers</u>** $\boxed{(x,y), with\, x>y>1}$ **<u>defining the indexes of each of those i-primeths</u>."**

p. **The function** $\boxed{fx(a,b)=2^{(a+1)(b+2)(a+b+1)}}$ **has its values in the matrix**

$$\boxed{M_{fx(a,b)}=\begin{pmatrix}\mathbf{4} & \mathbf{64} & \mathbf{4096}\\ \mathbf{256} & \cong\mathbf{2.621\times10^{5}} & \cong\mathbf{4.295\times10^{9}}\\ \cong\mathbf{2.621\times10^{5}} & \cong\mathbf{6.872\times10^{10}} & \cong\mathbf{1.153\times10^{18}}\end{pmatrix}}$$

in which each element is larger than its correspondent element from

$$\boxed{M_{f(a,b)}=\begin{pmatrix}\mathbf{3} & \mathbf{3} & \mathbf{2564}\\ \mathbf{3} & \mathbf{40306} & \mathbf{1\,765\,126}\\ \mathbf{2564} & \mathbf{1\,765\,126} & \mathbf{161\,352\,166}\end{pmatrix}}$$

5. **AND**

a. **for** $\boxed{(a,b)=(1,0)}$ **AND** $\boxed{m>28}$**, it will always exist at least one pair of finite distinct positive integers** $\boxed{(x,y), with\, x>y>1}$ **AND** $\boxed{{}^{1}P_{x}+{}^{0}P_{y}=2m}$ **AND x (or y) in the double-open interval** $\boxed{(\ln(2m),2m/\ln(2m))}$**.**

i. **Important note:** VBGC is much "stronger" and general than BGC and proposes a much more rapid and efficient (at-least-one-GIP)-sieve than the GKRC. The GM of GIPs generated by VBGC has a smaller nof. lines than the GM of GIPs generated by GKRC. VBGC is a useful optimized sieve to push forward the limit

$4 \cdot 10^{18}$ to which BGC was verified to hold **[53]**. When verifying BGC for a very large number $N$, one can use the VBGC(a,b) with a minimal positive value for the difference $\left[N - f(a,b)\right]$.

6. **Important note:** VBGC essentially (and alternatively) states that there is an infinite number of conjectures indexable as VBGC(a,b), all stronger than BGC, EACH of if associated with a pair $\boxed{(a,b), with\, a \geq b > 0}$ AND a finite positive integer $\boxed{n_{a,b} = f(a,b)}$.

    **a.** VBGC(0,0) is in fact ntBGC.

**VBGC 1.2 – secondary statements (also part of VBGC):**

1. **The different special cases of VBGC can be named after the pair (a,b) [VBGC(a,b)] AND:**

    **a.** **VBGC(0,0)** is in fact ntBGC (defined in the Part B of this article)

    **b.** **VBGC(1,0)[1]** is a GLC stronger and more elegant than ntBGC, as it acts on a limit $\boxed{2f(1,0)=6}$ identical to ntBGC inferior limit (which is $\boxed{2f(0,0)=6}$) BUT the associated $\boxed{G_{1,0}(m)}$ (which counts the number of pairs of possible GIPs for any even integer $m > 3$) has significantly smaller values than the function $\boxed{G_{0,0}(m)}$ of ntBGC [which is VBGC(0,0)]

    **c.** **VBGC(2,0)** is obviously a stronger GLC than VBGC(1,0) is AND ALSO $\boxed{G_{2,0}(m)}$ has smaller non-0 values than $\boxed{G_{1,1}(m)}$ for $m \in \left(f(2,0),\infty\right)$

    **d.** **VBGC(1,1**) (anticipated by my discovery of **VBGC(1,0) from 2007 and officially registered in 2012 at OSIM [1]**) is an obviously stronger GLC than VBGC(1,0 ) and is equivalent to Bayless-Klyve-Oliveira e Silva Goldbach-like Conjecture (BKOS-GLC) published in Oct. 2013 [**54]** alias "Conjecture 9.1" (rephrased) (tested by these authors up to $2m = 10^9$): all even integers $2m > \left[2 \cdot 40306 (= 2f(1,1))\right]$ can be expressed as the sum of at least one pair of prime-indexed primes [PIPs] (1-primeths ${}^1P_x$ and ${}^1P_y$ ). This article of Bayless. Klyve and Oliveira (2012, 2013) was based on a previous article by Barnett and Broughan (published in 2009) **[55]**, but BKOS-GLC was an additional result to this 2009 article. Mr. George Anescu (a friend and collaborator) have also helped me to retest VBGC(1,1) up to $2m = 10^{10}$, but also helped me verifying all VBGC(a,b) for all pairs $\boxed{(a,b) \in \{(1,0),(1,1),(2,0),(2,1),(2,2)\}}$ **[6]**.

2. When $a \to \infty, b \to \infty\, and\, m \to \infty$, $\boxed{G_{a,b}\left(f(a,b)+1\right) \underset{\geq}{\to} 1}$ and the "comets" of VBGC(a,b) tend to narrow progressively for each pair of positive integers $(a_2,b_2), with\, a_2 > a_1\, and\, b_2 > b_1$.

3. All VBGC(a>0,b≥0) can be used to produce more rapid algorithms for the experimental verification of ntBGC for very large positive integers

    **a.** For VBGC(1,0), the average number of attempts (ANA) to find the first pair (x,y) for each integer m, in the interval [3,2m] tends asymptotically to $\boxed{\ln\left(\sqrt{n}\right) = \ln(n)/2}$ when searching

**[6]** The code-source (written by Mr. George Anescu in Microsoft Studio 2015 - Visual C++ language/environment using parallel processing) that was used to test BKOS-GLC up to $n=10^{10}$ (using a laptop PC with an Intel^R Core^TM processor i7-3630 QM CPU at 2.4 GHz with 4 processors (8 hyper-threads), can be found at this URL (the old variant can be found at this URL-old) just the 1-primeths subset in descending manner, starting from the largest 1-primeth ≤2m-1 and verifying if $\left(2m-{}^{1}P_x\right)$ is a 0- primeth)

**Conclusions on VBGC 1.2:**

1. VBGC(a,b) is essentially an extension and generalization of BGC as applied on (the extended and generalized concept of) all subsets of i-primeths.

2. VBGC distinguishes as a very important (unified) conjecture of primes and a very special self-similar property of the primes as the rarefied ${}^{i}P$ is self-similar to the more dense ${}^{(i-1)}P$ in respect to the ntBGC. In other words, each of the i-primeths sets behaves as a "summary of" the 0-primeths set in respect to the ntBGC: this is a (quasi)fractal-like BGC-related behavior of the infinite number of the i-primeths sets. **Essentially, VBGC conjectures that ntBGC is a common property of all the i-primeths sets (for any positive integer order i), differing just by the inferior limit of each VBGC(a,b) defined by the function $f(a,b)$). I have called VBGC as "vertical" motivated by the fact that VBGC is a "vertical" (recursive) generalization of the ntBGC on the infinite super-set of i-primeths sets.**

    a. The set of values of $f(a,b)$ is a set of critical density thresholds/points of each i-primeths set in respect to the set VBGC(a,b) conjectures.

    b. Batchko R.G. has also reported other quasi-fractal/quasi-self-similar structure in the distribution of the prime-indexed primes **[56]:** Batchko also used a similar general definition for primes with (recursive) prime indexes (PIPs), briefly named in my article as "i-primeths".

    c. Carlo Cattani and Armando Ciancio also reported a quasi-fractal distribution of primes (including i-primeths) similar to a Cantor set (Cantor dust) by mapping primes and i-primeths into a binary image which visualizes the distribution of i-primeths **[57]**. VBGC may be an intrinsic property of all sets of i-primeths that can also explain OR be explained by this Cantor dust-like distribution of these i-primeths.

3. All sets ${}^{(i>0)}P$ are subsets of ${}^{0}P=\wp^{*}$ and come in an infinite number: this family of subsets is governed/defined by the Prime number theorem. There is a potential infinite number of rules/criterions/theorems to extract an infinite number of subsets from ${}^{0}P$ (grouped in a family of subsets defined by that specific rule/criterion/theorem), like the Dirichlet's theorem on arithmetic progressions for example[URL2] OR other prime formulas[URL2, URL3] that generate infinite subsets of primes. It would be an interesting research subfield of BGC to test what are those families (of subsets of primes) that respect ntBGC and generate functions with finite values similar to $f(a,b)=n_{a,b}=n_{b,a}$. This potential future research subfield may also help in optimizing the algorithms used in the present for ntBGC verification on large numbers. However, one special property of the family ${}^{(i>0)}P$ is that each subset of this family is a commutative monoid[URL2].

4. It is an interesting fact per se that all ${}^{(i>0)}P$ subsets have very low densities (when compared to ${}^{0}P$ and $\mathbb{N}^{*}$) BUT NOT sufficiently low densities to NOT generate a function $f(a,b)$ with finite values for any pair of finites $(a,b)$.

**Future challenges for VBGC (to be also approached in the next versions of this article):**

1. **To calculate the values of the function $f(a,b)=n_{a,b}=n_{b,a}$ and test/verify VBGC(a,b) for large positive integers pairs $(a>2,b>2)$ (a,b), but also for the pairs $(a,b)$ with large $(a-b)$ differences.**

**Potential applications of VBGC (to also be created in the next versions of this article):**

1. **VBGC can offer a potential infinite set of Goldbach Comets, one for each sub-VBGC applied on each order of i-primeths**
2. **VBGC can be used to optimize the algorithms of finding/verifying very large primes (i-primeths)/potential primes (i-primeths)**
3. **VBGC can be used as a model to also formulate a Vertical (generalization) of the Ternary Goldbach Conjecture/Theorem (VTGC)**
4. **VBGC can be theoretically used to optimize the algorithms of prime/integer factorization[URL2,URL3] (the main tool of cryptography)**
5. **VBGC can offer a rule of decomposition of Euclidean[URL2,URL3,URL4]/non-Euclidean[URL2] spaces/volumes with a finite 2N (positive) integer number of dimensions into pair of spaces, both with a (positive) i-primeth number of dimensions**
6. **VBGC can be used in M-Theory to simulate decompositions of 2N-branes (with a finite 2N [positive] integer number of dimensions) into pair of branes both with a (positive) i-primeth number of dimensions**
7. **VBGC can be also used to predict possible symmetries/asymmetries in crystallography, as based on i-primeths.**

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My special thanks to professor Toma Albu[7] who had the patience to read and listen my weak voice in mathematics as a hobbyist. Also my sincere gratitude to professor Şerban-Valentin Strătilă[8] that adviced me on the first special case of VBGC discovered in 2007 and he urged me to look for a more general conjecture based on my first observation.

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## Competing interests

Author has declared that no competing interests exist.

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## Addendum

**Method for verifying VBGC**. We have used Microsoft Visual C++. First, we have created (and stored on hard-disk) a set of ".bin" files containing all the standard primes (alias 0-primeths) (a file of ~3.6GigaBytes), the 1-primeths and the 2-primeths respectively, all in the double-open interval $\left(1,10^{10}\right)$.

For every $(a,b)$ pair with $a \geq b$, we have verified each ${}^{a}P_{x}\left(> {}^{b}P_{x}\right)$ from the (less) dense subset of ${}^{a}P$ superposing the double-open interval $(2, 2m \geq 6)$ (starting from that ${}^{a}P_{x}$ which was the closest to $2m-1$ in descending order): we have then verified if the difference $\left(2m - {}^{a}P_{x}\right)$ is an element in the (more) dense set ${}^{b}P$ by using binary section method.

We have then computed each value of $f(a,b)$ (with the additional condition ${}^{a}P_{x} \neq {}^{b}P_{y} \Leftrightarrow {}^{a}P_{x} > {}^{b}P_{y}$ in at least one Goldbach partition for any $m > f(a,b)$, with ${}^{a}P_{x} + {}^{b}P_{y} = 2m$). The computing time for determining and verifying $\boxed{f(2,1) = f(1,2) = \left(n_{2,1} = n_{1,2}\right) = 1\ 765\ 126}$ and $\boxed{f(2,2) = \left(n_{2,2}\right) = 161\ 352\ 166}$ was about 30 hours.

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[7] The CV of Professor Albu T. is also available online (URL)
[8] The CV of Professor Strătilă Ş-V. is also available online (URL)

## ENDNOTE ADDITIONAL REFERENCES (in order of citation in this article)


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