## Part D.

## VBGC 1.1 - The extension and generalization of BGC as applied on o-primeths ($^{o}\wp^{*}$)

**VBGC 1.1 (version 1.1, the same with the version of this article) – main statement**:

1. **Defining:** $^{0}P_x = P\left(\underset{0 iterations}{x}\right)$, $^{1}P_x = P\left(\underset{1 iteration}{P(x)}\right)$, $^{2}P_x = P\left(\underset{2 iterations}{P(P(x))}\right) \ldots {}^{o}P_x = P\left(\underset{o iterations}{P(P(\ldots P(x)))}\right)$, with $P(x)$ being the x-th prime in the set of standard primes (usually denoted as $P(x)$ or $P_x$ and equivalent to $^{0}P_x$) and the generic $^{o}P_x$ being named the generic set of o-primeths (with" o" being the number of /"order" of iterations).

2. **For any pair of finite positive integers** $\boxed{(a,b), with\, a \geq b \geq 0}$ defining the (recursive) orders of an a-primeth $\left({}^{a}P\right)$ and a b-primeth respectively $\left({}^{b}P\right)$, **there will always exist a single finite positive integer** $\boxed{\left(n_{a,b} = n_{b,a}\right) \geq 3}$ **so that, for any positive integer** $\boxed{m > n_{a,b}}$ **it will always exist at least one pair of finite *distinct* positive integers** $\boxed{(x,y), with\, x > y > 1}$ (indexes of distinct odd o-primeths) **so that:** $\boxed{{}^{a}P_x + {}^{b}P_y = 2m}$ **AND** $\boxed{{}^{a}P_x > {}^{b}P_y}$ **AND the function** $\boxed{f(a,b) = f(b,a) = \left(n_{a,b} = n_{b,a}\right) \geq 3}$ **has a finite positive integer value for any combination of finite positive integers** $\boxed{(a,b)}$**, without any catastrophic-like infinities for any** $\boxed{(a,b)}$ **pair of finites positive integers.**

    a. **<u>Important note</u>. I have chosen the additional conditions** $\boxed{(a \geq b \geq 0) \wedge (x > y > 1)} \Leftrightarrow \boxed{{}^{a}P_x > {}^{b}P_y}$ **so that to lower the nof. lines per each GM and to simplify the algorithm of searching** $\left({}^{a}P_x, {}^{b}P_y\right)$ **pairs, as the set** ${}^{a}P$ **is much less dense that the set** ${}^{b}P$ **for** $a > b$ **AND the sieve using** ${}^{a}P$ **(which searches an** ${}^{a}P$ **starting from** $2m\ to\ 3$**) finds a** $\left({}^{a}P_x, {}^{b}P_y\right)$ **pair much more quicker than a sieve using** ${}^{b}P$ **(if** $a > b$**).**

    b. $\boxed{f(0,0) = \left(n_{0,0}\right) = 3}$

    c. $\boxed{f(1,0) = f(0,1) = \left(n_{1,0} = n_{0,1}\right) = 3}$

    d. $\boxed{f(2,0) = f(0,2) = \left(n_{2,0} = n_{0,2}\right) = 2564}$

    e. $\boxed{f(1,1) = \left(n_{1,1}\right) = 40\ 306}$

    f. $\boxed{f(2,1) = f(1,2) = \left(n_{2,1} = n_{1,2}\right) = 1\ 765\ 126}$

    g. $\boxed{f(2,2) = \left(n_{2,2}\right) = 161\ 352\ 166}$

    h. $\boxed{f(3,0) = f(0,3) = \left(n_{3,0} = n_{0,3}\right) = ?}$ **[working in progress on this function value]**

i. $\boxed{f(3,1)=f(1,3)=(n_{3,1}=n_{1,3})=?}$ **[working in progress on this function value]**

j. $\boxed{f(3,2)=f(2,3)=(n_{3,2}=n_{2,3})=?}$ **[working in progress on this function value]**

k. $\boxed{f(3,3)=(n_{3,3})=?}$ **[working in progress on this function value]**

l. **…[working progress on other higher indexes function values]**

3. **AND**

   a. **for** $\boxed{(a,b)=(1,0)}$ **AND** $\boxed{m>28}$**, it will always exist at least one pair of finite distinct positive integers** $\boxed{(x,y), with\, x>y>1}$ **AND** $\boxed{{}^1P_x+{}^0P_y=2m}$ **AND x (or y) in the double-open interval** $\boxed{(\ln(2m),2m/\ln(2m))}$**.**

      i. **Important note:** VBGC is much "stronger" and general than BGC and proposes a much more rapid and efficient (at-least-one-GIP)-sieve than the GKRC. The GM of GIPs generated by VBGC has a smaller nof. lines than the GM of GIPs generated by GKRC. VBGC is a useful optimized sieve to push forward the limit $4\cdot10^{18}$ to which BGC was verified to hold **[53]**. When verifying BGC for a very large number $N$, one can use the VBGC(a,b) with a minimal positive value for the difference $N-f(a,b)$.

4. **Important note:** VBGC essentially (and alternatively) states that there is an infinite number of conjectures indexable as VBGC(a,b), all stronger than BGC, EACH of if associated with a pair $\boxed{(a,b), with\, a\geq b>0}$ AND a finite positive integer $\boxed{n_{a,b}=f(a,b)}$.

   a. VBGC(0,0) is in fact ntBGC.

**VBGC 1.1 – secondary statements (also part of VBGC):**

1. **The different special cases of VBGC can be named after the pair (a,b) [VBGC(a,b)] AND:**

   a. **VBGC(0,0)** is in fact ntBGC (defined in the Part B of this article)

   b. **VBGC(1,0)**[1] is a GLC stronger and more elegant than ntBGC, as it acts on a limit( $2n_{1,0}=6$) close to ntBGC inferior limit (l=4) BUT the associated $G_{1,0}$ (n) function (which counts the number of pairs of possible GIPs for any even integer n>6) has significantly smaller values than the $G_{0,0}$(n) function of ntBGC [which is VBGC(0,0)]

   c. **VBGC(2,0)** is obviously a stronger GLC than VBGC(1,0) is.

   d. **VBGC(1,1**) (anticipated by my discovery of **VBGC(1,0) from 2007 and officially registered in 2012 at OSIM** [1]) is an obviously stronger GLC than VBGC(1,0 ) and is equivalent to Bayless-Klyve-Oliveira e Silva Goldbach-like Conjecture (BKOS-GLC) published in Oct. 2013 [**54**] (alias "Conjecture 9.1" [rephrased]: all even integers n>80612[>**2·40306>2·$n_{1,1}$**] can be expressed as the sum of at least one pair of prime-indexed primes [PIPs] [1-primeths] and tested up to n=$10^{10}$). This article of Bayless. Klyve and Oliveira (2012, 2013) was based on a previous article by Barnett and Broughan (published in 2009) **[55]**, but BKOS-GLC was an additional result to this 2009 article. Mr. George Anescu (a friend and collaborator) have also helped me to retest VBGC(1,1) up to $n=10^{10}$, but also helped me testing all VBGC for all pairs (a,b)={(1,0), (2,0), (2,1)} [6].

2. **2.** $G_{a,b}(n_{a,b}+1) \rightarrow 1$, when $(a,b) \rightarrow \infty$ and the "comets" of VBGC(a,b) tend to narrow progressively for each pair $(a_2, b_2)$, with $a_2>a_1$ and $b_2>b_1$

3. **3.** All VBGC($a>0, b\geq 0$) can be used to produce more rapid algorithms for the experimental verification of ntBGC for very large positive integers

   **a.** For VBGC(1,0), the average number of attempts (ANA) to find the first pair (x,y) for each integer m, in the interval [3,2m] tends asymptotically to $\ln\left(\sqrt{n}\right) = \ln(n)/2$ when searching just the 1-primeths subset in descending manner, starting from the largest 1-primeth $\leq 2m$ and verifying if $2m-{}^{1}P_x$ is a 0- primeth)

**Conclusions on VBGC 1.1:**

1. VBGC(a,b) is essentially an extension and generalization of BGC as applied on (the extended and generalized concept of) all ${}^{o}\wp^{*}$ subsets of o-primeths.

2. VBGC distinguishes as a very important (unified) conjecture of primes and a very special self-similar propriety of the primes as the rarefied ${}^{o}\wp^{*}$ is self-similar to the more dense ${}^{o-1}\wp^{*}$ in respect to the ntBGC. In other words, each of the o-primeths sets behaves as a "summary of" the 0-primeths set in respect to the ntBGC: this is a (quasi)fractal-like BGC-related behavior of the infinite number of the o-primeths sets (Batchko R.G. has also reported other quasi-fractal/quasi-self-similar structure in the distribution of the prime-indexed primes **[56]:** Batchko also used a similar general definition for primes with recursive prime indexes, briefly named in my article as "o-primeths"). **Essentially, VBGC conjectures that ntBGC is a common propriety of all the o-primeths sets (for any positive integer order o), differing just by the inferior limit of each VBGC(a,b). I have called VBGC as "vertical" motivated by the fact that VBGC is a "vertical" (recursive) generalization of the ntBGC on the infinite super-set of o-primeths sets.**

   a. The set n(a,b) is a set of critical density thresholds/points of each o-primeths set in respect to the set VBGC(a,b) conjectures.

**Future challenges for VBGC (to be also approached in the next versions of this article):**

1. **To calculate $f(a,b)=n_{a,b}$ and test VBGC(a,b) for large positive integers pairs (a,b), but also for the pairs (a,b) with large (a-b) differences.**

**Potential applications of VBGC (to also be created in the next versions of this article):**

1. VBGC can offer a potential infinite set of Goldbach Comets, one for each sub-VBGC applied on each order of o-primeths

2. **VBGC can be used to optimize the algorithms of finding/verifying very large primes (o-primeths)/potential primes (o-primeths)**

3. **VBGC can be used as a model to also formulate a Vertical (generalization) of the Ternary Goldbach Conjecture/Theorem (VTGC)**

4. **VBGC can be theoretically used to optimize the algorithms of prime/integer factorization[URL2,URL3] (the main tool of cryptography)**

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**[6]** The code-source (written by Mr. George Anescu in Microsoft Visual C++ language/environment using parallel processing) that was used to test BKOS-GLC up to $n=10^{10}$ (using a laptop PC with an Intel[R] Core[TM] processor i7-3630 QM CPU at 2.4 GHz with 4 processors (8 hyper-threads), can be found at this URL: dragoii.com/test_primes.rar

5. **VBGC can offer a rule of decomposition of Euclidean[URL2,URL3,URL4]/non-Euclidean[URL2] spaces/volumes with a finite 2N (positive) integer number of dimensions into pair of spaces, both with a (positive) o-primeth number of dimensions**
6. **VBGC can be used in M-Theory to simulate decompositions of 2N-branes (with a finite 2N [positive] integer number of dimensions) into pair of branes both with a (positive) o-primeth number of dimensions**
7. **VBGC can be also used to predict possible symmetries/asymmetries in crystallography, as based on o-primeths.**

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## Competing interests

Author has declared that no competing interests exist.

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## Addendum

**Method for verifying VBGC**. We have used Microsoft Visual C++. First, we have created (and stored on hard-disk) a set of ".bin" files containing all the standard primes (alias 0-primeths) (a file of ~3.9GigaBytes), the 1-primeths and the 2-primeths respectively, all in the double-open interval $\left(1,10^{10}\right)$. We have then computed each $n_{a,b}$ (with the additional condition ${}^{a}P_{x} \neq {}^{b}P_{y}$ in at least one Goldbach partition for any $m > n_{a,b}$, with ${}^{a}P_{x} + {}^{b}P_{y} = 2m$)

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[7] The CV of Professor Albu T. is also available online (URL)
[8] The CV of Professor Strătilă Ş-V. is also available online (URL)

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