AN ALGORITHMIC PROOF TO THE TWIN PRIMES CONJECTURE AND THE GOLDBACH CONJECTURE

JUAN GUILLERMO OROZCO

Abstract. This paper introduces proofs to several open problems in number theory, particularly the Goldbach Conjecture and the Twin Prime Conjecture. These two conjectures are proven by using a greedy elimination algorithm, and incorporating Mertens' third theorem and the twin prime constant. The argument is extended to Germain primes, Cousin Primes, and other prime related conjectures. A generalization is provided for all algorithms that result in a Euler product$\prod 1 - \frac{a}{p}$.

1. Introduction

In 1742, at age 52, Christian Goldbach wrote to Leonard Euler formulating his conjecture, proposing that all numbers can be stated as the sum of three primes. In more modern terms, it can be restated as whether all even numbers can be expressed as the sum of two primes. As of April 2012, Tomas Oliveira e Silva had verified Goldbach's conjecture by "brute force" up to 4×10^{18} and had double checked up to 2×10^{17} [1][2]. In 2014, Harald Helfgott proved the ternary Goldbach conjecture using ideas from Hardy, Littlewood and Vinogradov [3][4].

In 1825, 83 years after Goldbach's conjecture, Sophie Germain used the now called "Germain primes" in an attempt to prove a weaker version of Fermat's Last Theorem, that there are no solutions to the equation $x^n + y^n = z^n$ for $n > 4$. The Germain prime pairs are such that both p and $2p + 1$ are primes, for example {2,5}, {3,7}, and {5,11}. RSA encryption protocol relies on certain properties of Germain primes that make encryption keys hard to factorize using heuristics like Pollard rho.

Although the existence of twin primes may have been known since 300 BC, (Euclid), in 1849 de Polignac proposed a more general conjecture that there are infinitely many primes p such that $p + 2 \cdot k$ is also prime (where k=1 would be the twin prime conjecture). A more extensive historic review of twin primes can be found in [5].

In 2013, Yitang Zhang published a paper where, for the first time, the prime gap is bounded[6]. Although it was not the purpose of his paper to minimize the gap, it is the first "official prime gap" to be estimated, at 70 million. Subsequently, Terrence Tao launched the Polymath Project[7], an online collaborative effort to optimize Zhang's bound. In order to prove the Twin Prime conjecture, the bound would need to be brought down to 2, which according to Tao[8] is not likely using Zhang's method.

In 1874, Franciszek Mertens proposes multiple theorems approximating products and sums related to primes. One of them specifically, Mertens' third theorem on

Date: September 2016.

the approximation to the product of $(1-\frac{1}{p})$, will play a critical role in the resolution of several open prime problems.

As late as 1912, during the Mathematical Congress in Cambridge, UK, Edmund Landau proposes what are now known as "Landau's problems". These were the twin prime conjecture, Golbach conjecture, Legendre conjecture, and the infinity of primes $p = n^2 + 1$.

This paper presents proofs to these problems using a greedy elimination algorithm and Mertens' approximation to $\prod\left(1-\frac{1}{p}\right)$. Results are extended to other open problems such as cousin primes and sexy primes.

2. Proof of Goldbach's Conjecture

The Goldbach Conjecture states that every even number can be expressed as the sum of 2 prime numbers. An extended version of this is the weak Goldbach Conjecture which states that any odd number can be expressed as the sum of 3 primes. Proving the strong Goldbach conjecture immediately implies the weak version by noting that$N_{odd} = 3 + N_{even} = 3 + p_i + p_j$.

2.1. Elimination algorithm.

We present a greedy elimination algorithm, and prove that such algorithm returns all possible Goldbach pairs for a given n.

The algorithm is a greedy elimination algorithm that works as two parallel Eratosthenes sieves[9]. Once the algorithm reaches a critical prime (approximately $\sqrt{n}$, defined below), the algorithm will guarantee the nonexistence of composites in the elimination array. The width of the array decreases as a function of $O\left(\frac{n}{(\ln n)^2}\right)$.

First, we introduce the main parameters of the algorithm.

Symbol	Parameter
n	Number being tested for Goldbach pairs.
S_p	Ordered sequence of primes. $S = 2, 3, 5, 7, 11, 13, ...$
p	Prime in sequence S_p testing for compositeness of elements in array E.
$E(n \times 2)$	Elimination array. $E(i,1) = i$,$E(i,2) = n - i$. So $E(i,1) + E(i,2) = n$.
w_k	Width of elimination array at iteration k ($w_{k=0} = n/2$)
$\|E\|_p(x)$	Number of primes in E when $p = x$.
$\|E\|_c(x)$	Number of composites in E when $p = x$.
	The last three parameters are related to each other as: $\frac{\|E\|_p(p_k)+\|E\|_c(p_k)}{2} = w_k$

The elimination algorithm is as follows.

Lines 1-4 populate the elimination array. The second part of the algorithm (lines 5-12) go through the ordered sequence of primes S_p and test each of the individual cells in the array E for compositeness. If one of the values in a column is divisible by the prime p_i, then the column pair is eliminated from E.

Definition 1. Critical prime: We define a critical prime p_* as the prime in the sequence S_p that guarantees that no composite numbers are left in the array. That is, $|E|_c(p_*) = 0$. The critical prime is a prime smaller than the square root of the maximum value in the array.

Algorithm 1 Goldbach Greedy Elimination Algorithm

```
for x = 1 to n/2 do
    E(x,1) = x
    E(x,2) = n − x
end for
while p_i ≤ √n do
    for x = 1 to n/2 do
        if p_i|E(x,1) or p_i|E(x,2) then
            Remove E(x) column
        end if
    end for
    p_i ← p_{i+1} ∈ S_p
end while
```

Theorem 2. *For algorithm 1, there is a critical $p_* \simeq \sqrt{n}$ such that $|E|_c(p_*) = 0$.*

Proof. Proof is done by contradiction. Let's assume that there is a column x_i that is not eliminated and contains an invalid Goldbach pair combination, i.e., one of the numbers is composite. Without loss of generality, let's say that $E(x_i, 1) = p \cdot q$. Since the cell was not eliminated, it means that 2 of its prime factors must be larger than the critical prime. But, if $p > \sqrt{n}$ and $q > \sqrt{n}$ then $p \cdot q > n$, which results in a contradiction. □

Theorem 3. *The width of array E after iteration k is:*

$$w_k = \frac{n}{2} \cdot \prod_{p|n}^{p_s} \left(1 - \frac{1}{p_k}\right) \cdot \prod_{p\nmid n}^{p_s} \left(1 - \frac{2}{p_k}\right) \tag{2.1}$$

Proof. This is obtained from the following iteration:

$$w_k = \left\{ w_{k-1} \cdot \left(1 - \tfrac{1}{p}\right) \text{ if p|n}, \quad w_{k-1} \cdot \left(1 - \tfrac{2}{p}\right) \text{ if p}\nmid\text{n} \right. \tag{2.2}$$

That is, on every iteration, a factor of $(1 - \frac{1}{p})$ columns is removed if p divides n, or $(1 - \frac{2}{p})$ if p does not divide n. □

Corollary. The lower bound for the width w_k is given by:

$$w_k > \frac{n}{4} \cdot \prod_{p>2}^{p_*} \left(1 - \frac{2}{p}\right) \tag{2.3}$$

In order to approximate the *Euler product*, we use Mertens' third theorem and the Twin Prime constant definition.

Theorem 4. *Mertens' third theorem*

$$\frac{1}{2} \prod_{p>2}^{x} \left(1 - \frac{1}{p}\right) \to \frac{e^{-\lambda}}{\ln x} \tag{2.4}$$

Proof. Proof can be obtained in [10][11]. □

By squaring and rearranging, we have:

$$\prod_{p>2}^{x}\left(1-\frac{1}{p}\right)^2 \to 4\cdot\frac{e^{-2\gamma}}{(\ln x)^2} \tag{2.5}$$

Definition 5. The Twin Prime constant, π_2, is defined as[1]:

$$\pi_2 = \frac{\prod_{p>2}^{\infty}\left(1-\frac{2}{x}\right)}{\prod_{p>2}^{\infty}\left(1-\frac{1}{x}\right)^2} \tag{2.6}$$

Rearranging, we can state that the following two products converge as:

$$\prod_{p>2}^{\infty}\left(1-\frac{2}{x}\right) = \prod_{p>2}^{\infty}\left(1-\frac{1}{x}\right)^2\cdot\pi_2 \tag{2.7}$$

Replacing in Mertens' third theorem, we have:

$$\prod_{p>2}^{x}\left(1-\frac{2}{p}\right) \to 4\cdot\pi_2\cdot\frac{e^{-2\gamma}}{(\ln x)^2} \tag{2.8}$$

Theorem 6. *The lower bound of Goldbach pairs is given by:*

$$w_k > \frac{4\cdot n\cdot\pi_2\cdot e^{-2\gamma}}{(\ln n)^2} \tag{2.9}$$

Proof. Replacing the product in 2.3, and evaluating at the critical prime $\sqrt{n}$ we have:

$$w_k > \frac{n}{4}\cdot 4\cdot\pi_2\cdot\left.\frac{e^{-2\gamma}}{(\ln x)^2}\right|_{x=\sqrt{n}} = \frac{4\cdot n\cdot\pi_2\cdot e^{-2\gamma}}{(\ln n)^2} \tag{2.10}$$

□

Theorem 7. *Goldbach Conjecture: All even numbers are the sum of 2 primes.*

Proof. Theorem 2 guarantees that there are no composites left in the array once the algorithm reaches the critical prime $p_* = \sqrt{n}$. Theorem 3 gives a lower bound on the with of the array of $O(\frac{n}{(\ln n)^2})$ at the critical prime. Then all columns remaining at the critical prime are valid Goldbach pairs for n. □

Note that the actual number of Goldbach pairs depends on the prime factors of n. When $p|n$, the adjusting factor for the array's width is $\left(1-\frac{1}{p}\right)$, instead of $\left(1-\frac{2}{p}\right)$. In order to obtain a more accurate count of Goldbach pairs, the lower bound w_k must be multiplied by the product:

[1] Although the definition is sometimes presented as $\prod\frac{p(p-2)}{(p-1)^2}$, it can be easily converted to the version above by dividing numerator and denominator by p^2.

$$\prod_{p|n}\left(\frac{1-\frac{1}{p}}{1-\frac{2}{p}}\right)=\prod_{p|n}\left(1+\frac{1}{p-2}\right) \tag{2.11}$$

It is also important to note that Mertens' theorem is an approximation to the Eulerian product. For smaller numbers, the error is large, so the product should be evaluated explicitly instead of using Mertens' approximation. The critical primes up to 290 would be given by:

Up to n	p*	Factor
6	2	$\frac{1}{2}$
24	3	$\frac{1}{6}$
48	5	$\frac{1}{10}$
120	7	$\frac{1}{14}$
168	11	$\frac{9}{154}$
290	13	$\frac{9}{182}$
${p_{n+1}}^2-1$	p_n	$\frac{1}{2}\prod\left(1-\frac{2}{p}\right)$

For example, if $n = 20$, then $w_0 = 10$. And the factor would be given by $\frac{1}{2}(1-\frac{2}{3}) = \frac{1}{6}$. Then, the lower bound for Goldbach pairs would be $w_k > \frac{10}{6} > 1$. The Goldbach pairs for 20 are (3,17), and (7,13).

3. Proof of the Twin Prime Conjecture

For the Twin Prime conjecture, the only change needed in the algorithm is the way the elimination array is populated. Row 1 of the array contains all natural numbers up to a number n and row 2, contains $n+2$.

Algorithm 2 Twin Primes Greedy Elimination Algorithm

```
for x = 1 to n do
    E(x, 1) = x
    E(x, 2) = x + 2
end for
while p_i ≤ √(n + 2) do
    for x = 1 to n do
        if p_i|E(x, 1) or p_i|E(x, 2) then
            Remove E(x) column
        end if
    end for
    p_i ← p_{i+1}
end while
```

Theorem 8. *For algorithm 2, there is a critical* $p_* = \sqrt{n+2}$ *such that* $|E|_c = 0$.

Proof. The proof is the same as for Goldbach's algorithm. Let's assume that there is a column x_i such that $E(x_i, 2)$ is composite after the algorithm reaches p_*. That is, $E(x_i, 2) = p \cdot q$. Since the number was not eliminated by the algorithm, both of its prime factors must be larger than $\sqrt{n+2}$. But, if $p > \sqrt{n+2}$ and $q > \sqrt{n+2}$, then $p \cdot q > n+2$, which is a contradiction. □

Theorem 9. *The width of array E is given by:*

$$w_k = w_0 \cdot \frac{1}{2} \prod_{p>2}^{\sqrt{n+2}} \left(1 - \frac{2}{p}\right) \tag{3.1}$$

Using equation 2.8, we have:

$$w_k = w_0 \cdot \frac{1}{2} \frac{4\pi_2 e^{-2\gamma}}{(\ln x)^2} \tag{3.2}$$

And evaluating at the critical $p_c = \sqrt{n+2}$, *it results in:*

$$w_k = n \cdot \frac{8\pi_2 e^{-2\gamma}}{(\ln(n+2))^2} \tag{3.3}$$

Proof. When p=2, all even numbers in the array are removed. This explains the $\frac{1}{2}$ fraction in the formula. Thereafter, all composites multiples of a given p are offset by 2 from each other. Therefore, the reducing factor is given by $\left(1 - \frac{2}{p}\right)$. □

Theorem 10. *Twin prime conjecture. There are infinite pairs of primes such that* $p_{i+1} = p_i + 2$.

Proof. From theorem 8, we have that $|E|_c(p_*) = 0$. But we have a lower bound w_k of $O\left(\frac{n}{(\ln n)^2}\right)$ on the width of the array at the critical p_*. Then $|E_p|(p_*) = 2 \cdot w_k$. That is, all the pairs in the array at the critical p_* are valid twin prime pairs. □

4. A generalization

For any greedy composite elimination algorithm, the lower bound of prime elements in the array can be stated as:

$$w_k = w_0 \cdot \prod_{\{p \in A\}}^{p_*} \left(1 - \frac{h}{p}\right) \tag{4.1}$$

Parameter	Description	Examples	
p_*	Critical prime as defined in definition 1		
h	Height of the elimination array	For Goldbach, h=2; twin primes h=2; prime triplets, h=3; arithmetic progressions, h=number of terms in progression.	
A	Group of primes applied in the elimination algorithm	For Goldbach, Twin, triplets, A=all primes; For $p = n^2 + 1$, $A = p \equiv 1 \mod 4$	

The product can then be approximated by using the following:

$$\prod_{p \in A}^{x} \left(1 - \frac{h}{p}\right) \sim L \left(\frac{e^{-\lambda}}{\ln(x)}\right)^{h} \tag{4.2}$$

Here, L is a constant that "absorbs" the product of those primes not belonging to A.

Recalling the critical prime p_* as a power function of the original size of the array that guarantees $|E|_c(p_*) = 0$, it can reformulated more generically as:

$$p_* = (f_m(x))^{\frac{1}{u}} \tag{4.3}$$

Then, the lower bound for the array size is as follows:

$$w_k = w_0 \cdot \prod_{p \in A}^{x} \left(1 - \frac{h}{p}\right) \sim w_0 \cdot L \left(\frac{u \cdot e^{-\lambda}}{\ln(f_m(x))}\right)^{h} \tag{4.4}$$

This resolves the following conjectures, affirmatively on all counts. Using the generalized formula, and a greedy elimination algorithm, we can guarantee that there are no composite numbers in the array once the algorithm reaches the critical prime p_*.

Problem	Critical prime (p_*)	$A \in S_p$	w_0	u	h	Result
Goldbach conjecture	$\sqrt{n}$	All primes	$\frac{n}{2}$	2	2	Infinite
Twin primes	$\sqrt{n+2}$	All primes	n	2	2	Infinite
Cousin primes	$\sqrt{n+4}$	All primes	n	2	2	Infinite
Sexy primes	$\sqrt{n+6}$	All primes	n	2	2	Infinite
Prime pairs $(n,\ an+b)$	$\sqrt{an+b}$	All primes	n	2	2	Infinite
De Polignac $(p,\ p+2\cdot k)$	$\sqrt{n+2\cdot k}$	All primes	n	2	2	Infinite
Prime triplets $(n,\ n+2,\ n+6)$	$\sqrt{n+6}$	All primes	n	2	3	Infinite
Prime quadruples	$\sqrt{n+10}$	All primes	n	2	4	Infinite
Sophie Germain primes	$\sqrt{2n+1}$	All primes	n	2	2	Infinite
Primes n^2+1	$\sqrt{n^2+1} \sim n$	$\{p \equiv 1 \mod 4\}$	n	2	2	Infinite
k-tuples	$\sqrt{\max(n_{k,h})}$	All primes	n	2	k	Infinite

5. Conclusion

This paper introduces a greedy elimination algorithm to prove various open conjectures related to primes. Then, it is proven that such algorithm guarantees that no composites remain in the array once the *critical prime* is reached. The width of the array is estimated using Mertens' third theorem and the twin prime constant.

By generalizing, multiple open conjectures related to primes are proven affirmatively.

References

[1] T. Oliveira e Silva, S. Herzog, and S. Pardi, "Empirical verification of the even goldbach conjecture and computation of prime gaps up to $4{\cdot}10^{18}$," *Mathematics of Computation*, vol. 83, no. 288, pp. 2033–2060, 2014.

[2] J. Richstein, "Verifying the goldbach conjecture up to $4 \cdot 10^{14}$," *Mathematics of computation*, vol. 70, no. 236, pp. 1745–1749, 2001.

[3] H. Helfgott, "Minor arcs for goldbach's problem," *arXiv preprint arXiv:1205.5252*, 2012.

[4] H. A. Helfgott, "Major arcs for goldbach's theorem," *arXiv preprint arXiv:1305.2897*, 2013.

[5] S. Nazardonyavi, "Some history about twin prime conjecture," *arXiv preprint arXiv:1205.0774*, 2012.

[6] Y. Zhang, "Bounded gaps between primes," *Annals of Mathematics*, vol. 179, no. 3, pp. 1121–1174, 2014.

[7] D. H. J. Polymath, "The "bounded gaps between primes" Polymath project - a retrospective," *ArXiv e-prints*, Sept. 2014.

[8] T. Tao, "Small and large gaps between primes," 10 2007.

[9] Wikipedia, "Sieve of eratosthenes — wikipedia, the free encyclopedia," 2016. [Online; accessed 28-July-2016].

[10] M. B. Villarino, "Mertens' proof of mertens' theorem," *arXiv preprint math/0504289*, 2005.

[11] A. J. Hildebrand, *Introduction to Analytic Number Theory, Math 531 Lecture Notes*. Fall 2005.

[12] H. A. Helfgott, "The ternary Goldbach conjecture is true," *ArXiv e-prints*, Dec. 2013.

[13] A. Berke, "An introduction to the twin prime conjecture," 2006.

[14] A. Granville, "Primes in intervals of bounded length," *Bulletin of the American Mathematical Society*, vol. 52, no. 2, pp. 171–222, 2015.

[15] H. Halberstam and H. E. Richert, *Sieve methods*. Courier Corporation, 2013.

[16] Y. Wang, *The Goldbach Conjecture*. World scientific, 2002.

[17] Wikipedia, "Goldbach's conjecture — wikipedia, the free encyclopedia," 2016. [Online; accessed 25-June-2016].

[18] Wikipedia, "Twin prime — wikipedia, the free encyclopedia," 2015. [Online; accessed 25-June-2016].

E-mail address: `jgorozco@hotmail.com`