Non-trivial non-Archimedean extension of real numbers.

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Abstract

We propose an extension of real numbers which reveals a surprising algebraic role of Bernoulli numbers, Hurwitz Zeta function, Euler-Mascheroni constant as well as generalized summations of divergent series and integrals. We extend elementary functions to the proposed numerical system and analyze some symmetries of the special elements. This reveals intriguing closed-form relations between trigonometric and inverse trigonometric functions. Besides this we show that the proposed system can be naturally used as a cardinality measure for fine comparison between infinite countable sets in metric space which respects the intuitive notion of the set's size.

Question regarding comparison of infinite sets

Intuitively it is conceivable that the set of even numbers is "smaller" than the set of integer numbers, because the integers are composed of even and odd numbers. Thus we want more densely distributed sets having greater "numerocity" measure than less densely distributed. For uniformly distributed ones (having equal spacing between numbers included) the measure should be inversely proportional to the interval. Similarly it is conceivable that the set of non-negative integers is "smaller" than the set of positive integers because it includes one more number (zero). Thus we also want the numerocities being strictly additive (the numerocity of a union of two sets should be equal to the sum of the numerocities of the both sets).

Cantor's method of comparison between infinite sets (cardinals) does not fit these intuitive notions because it is based on bijection between elements of the sets. Cantorean cardinality of all the mentioned sets is the same.

Here we propose another measure for infinite sets. It is meant for comparison of sets in metric spaces. We base it on extending the real numbers with a set of extended numbers. Extended numbers consist of the extended part and standard part which is a usual real or complex number.

Although this theory has no technical overlap with Robinson's nonstandard analysis, it shares some philosophical goals with it (such as the goal of giving an approach to computing the derivative of a function without recourse to the concept of limits)

If we denote the measure of the set of integers as 2τ, the measure of the set of even numbers, as well as odd numbers will be τ. At the same time we cannot divide the set of integers into two equal left and right parts because there is the zero at the origin. That's why one of the parts will be greater than the other by one. This way we denote the measure of positive integers as $\omega_- = \tau - 1/2$ and the measure of non-negative integers as $\omega_+ = \tau + 1/2$ so that $\omega_- + \omega_+ = 2\tau$. We will call these values "special elements".

We postulate that τ is a purely extended number (having no standard part). If so, the standard part of ω_- is -1/2 and the standard part of ω_+ is 1/2.

It is possible to define some operations on sets that do not change their measure, for instance, moving a finite number of elements along the real line. More details on this below.

Connection with the divergent series.

If the elements of a set are situated at the points corresponding to natural numbers, we can naturally postulate equivalence between the measure of such set and the sum of the series, which has under its sum sign the membership function of the set (1 if the natural number belongs to the set and 0 of it does not). This way we can build correspondence between a series (possibly, divergent) and all subsets of natural numbers.

Particularly,

$$\omega_- = \sum_{k=1}^{\infty} 1$$

$$\omega_+ = \sum_{k=0}^{\infty} 1$$

We thus postulate that the standard part of the measure of such sets is equal to the regularized sum of the corresponding divergent series. In particular, the above-mentioned series have regularized sums of -1/2 and 1/2 correspondingly.

We can extend our definition to consider not just subsets of natural numbers but sets of weighted dots. The standard part of the measure of such sets will be expressed as regularized sum as well.

To derive further results we will use Faulhaber's formula for Ramanujan's summation:

$$\sum_{n\geq 1}^{\Re} f(n) = -\sum_{n=1}^{\infty} \frac{f^{(n-1)}(0)}{n!} B_n(1)$$

In fact, Faulhaber's formula is a Taylor series for standard part of antiderivative of $f(x)$. We arrive at the following relation:

$$\operatorname{st} f'(\omega_- + z) = \Delta f(z)$$

Particularly, we obtain the standard part of the powers of special elements as Bernoulli numbers:

$$\operatorname{st} \omega_-^n = B_n$$

$$\operatorname{st} \omega_+^n = B_n^*$$

where B_n^* are the second Bernoulli numbers (those which have $B_1^* = 1/2$).

This is fascinating relation, that reveals the algebraic role of the Bernoulli numbers: they represent the standard part of the two special elements.

This is especially notable because many series representations use Bernoulli numbers. This way, the sums of these series can be expressed as standard parts. Having two series:

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos z$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k} z^{2k-1}}{(2k)!} = \cot z, |z| < \pi$$

we can obtain a relation:

$$\operatorname{st} \cos(z\omega_-) = \operatorname{st} \cos(z\omega_+) = \frac{z}{2} \cot\left(\frac{z}{2}\right), |z| < \pi$$

From the definition of Bernoulli numbers:

$$\operatorname{st} e^{z\omega_-} = \frac{z}{e^z - 1}$$

In general, this property allows us to find standard part of any function of non-standard numbers.

Particularly, the formula for the standard part of the powers of the special elements can be generalized with use of Hurwitz Zeta function, revealing its algebraic role:

$$\operatorname{st}(\tau + y)^x = -x\zeta(1 - x, 1/2 + y)$$

and, in particular,

$$\operatorname{st} \omega_-^x = -x\zeta(1 - x, 0)$$

$$\operatorname{st} \omega_+^x = -x\zeta(1 - x, 1) = -x\zeta(1 - x)$$

$$\operatorname{st}(\omega_- + z)^n = B_n(z)$$

where $B_n(z)$ are Bernoulli polynomials.

We see that Zeta function is essentially the standard part of exponential function.

Symmetries.

We can observe relations between standard parts of the powers of special elements:

$$\operatorname{st} \omega_-^x = \operatorname{st} \omega_+^x, x > 1$$

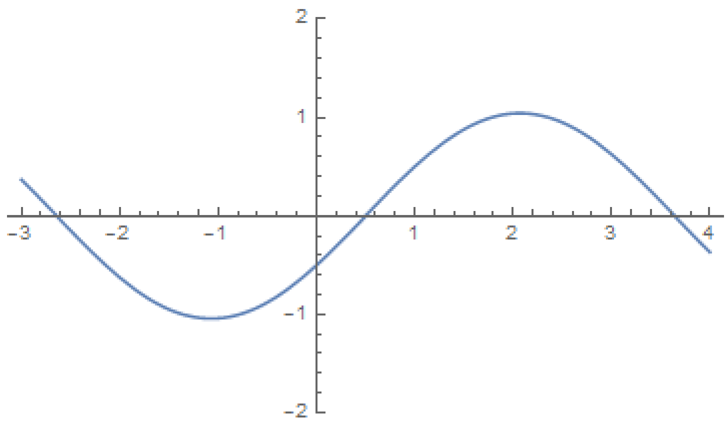
$$\operatorname{st} \tau^x = \operatorname{st} \omega_+^x (2^{1-x} - 1)$$

This way the standard parts of powers of ω_- and ω_+, exceeding 1, coincide. This reflexes the deep symmetry between these special elements.

We also can obtain expressions for standard parts of the elementary functions of the special elements:

$$\operatorname{st}\sin(a\omega_- + x) = \frac{a}{2}\cot\left(\frac{a}{2}\right)\sin x - \frac{a}{2}\cos x$$

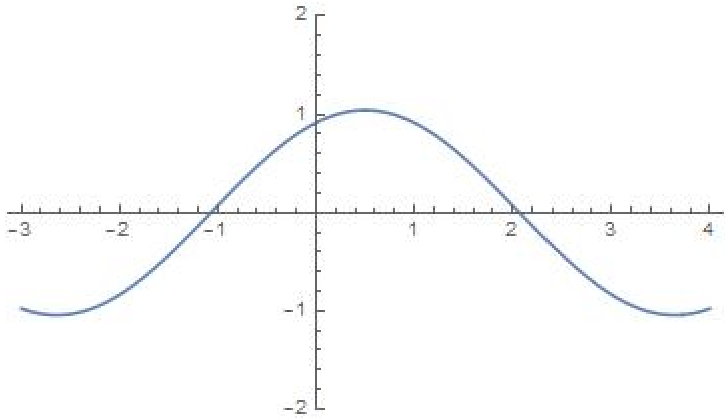
Plot for $\operatorname{st}\sin(\omega_- + x)$:



In particular, we see that $\operatorname{st}\sin\omega_+ = 1/2,\ \operatorname{st}\sin\omega_- = -1/2,\ \operatorname{st}\sin\tau = 0$

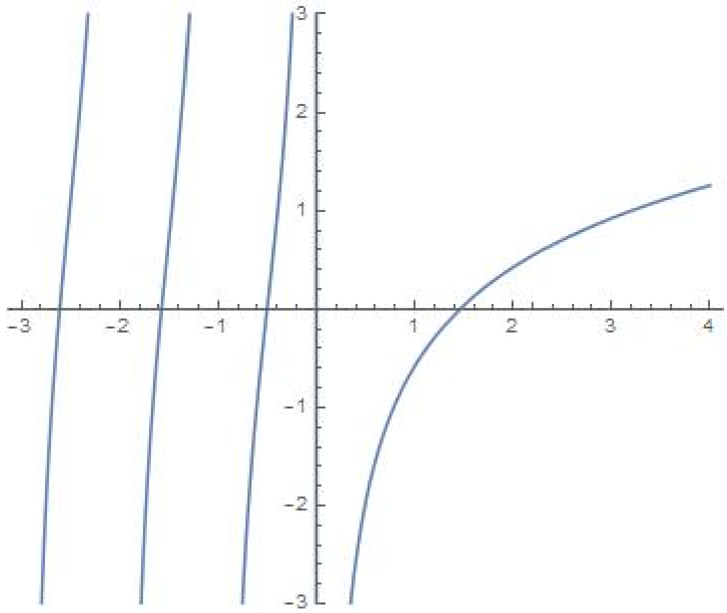
$$\operatorname{st}\cos(a\omega_- + x) = \frac{a}{2}\csc\left(\frac{a}{2}\right)\cos\left(\frac{a}{2} - x\right)$$

Plot for $\operatorname{st}\cos(\omega_- + x)$:



On the other hand,

$$\operatorname{st}\ln(\omega_- + z) = \psi(z)$$



where $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ - is the digamma function. We can see that this function has a pole at $z=0$, thus $\ln(\omega_-)$ is not defined. At the same time, $\operatorname{st}\log(\omega_+)=-\gamma$, where γ is the Euler-Mascheroni constant.

The same way we observe that it is impossible to divide by ω_-. Moreover, all negative powers of this element are undefined. On the other hand, it is completely possible to divide by ω_+. Moreover, thanks to the Riemann functional equation, posiutive and negative powers of ω_+ display symmetry:

$$\operatorname{st}\omega_+^{-x}=\operatorname{st}\frac{-\omega_+^{x+1}2^x\pi^{x+1}}{\sin(\pi x/2)\Gamma(x)(x+1)}$$

Impressive feature of this relation that uncovers the algebraic role of the Riemann functional equation is that when it is applied to a Taylor series term by term, it converts exponential functions to logarithmic ones and vice versa.

This way we can write relations between trigonometric and hyperbolic functions and their inverse ones in closed form:

$$\operatorname{st}\left(1-\cosh\left(2x\omega_+\right)\right)=\operatorname{st}\frac{x}{\pi}\operatorname{artanh}\left(\frac{x}{\pi\omega_+}\right)=\operatorname{st}\frac{x}{\pi}\operatorname{arcoth}\left(\frac{\pi\omega_+}{x}\right)=1-x\coth(x)$$

$$\operatorname{st}\frac{z}{2\pi}\ln\left(\frac{\omega_+-\frac{z}{2\pi}}{\omega_-+\frac{z}{2\pi}}\right)=\operatorname{st}\cos(z\omega_-)=\operatorname{st}\cos(z\omega_+)=\frac{z}{2}\cot\left(\frac{z}{2}\right)$$

It remains unclear whether similar relations can be obtained without the operation of taking the standard part, and thus connect the logarithmic and exponential functions in the same way as Euler’s formula connects trigonometric and hyperbolic functions.

Expression for the derivative without using limits.

Coming back to the Faulhaber's formula and noticing that all positive integer powers of ω_- and ω_+ except the power one have the same standard part, we can separate the coefficient of the power one. This coefficient is the derivative:

$$f'(x) = \operatorname{st}(f(\omega_+ + x) - f(\omega_- + x)) = \operatorname{st}\Delta f(\omega_- + x)$$

There exist other similar formulas:

$$f'(x) = \operatorname{st}(f(\omega_+ + x) - f(-\omega_+ + x))$$

$$f'(x) = \operatorname{st}(f(-\omega_- + x) - f(\omega_- + x))$$

These relations allow to express the derivative of an analytic function without using the idea of limit. This is in a sense similar to the formula for the derivative in Robinson's Non-Standard Analysis.

If the function $f(x)$ is odd, then $\operatorname{st} f(\omega_+) = -\operatorname{st} f(\omega_-) = \frac{1}{2}f'(0)$

Other relations coming from Faulhaber's formula::

$$\operatorname{st}\sum_{k=0}^{\infty} f(k) = -\operatorname{st}\int_0^{\omega_-} f(x)dx$$

$$\operatorname{st}\sum_{k=1}^{\infty} f(k) = -\operatorname{st}\int_0^{\omega_+} f(x)dx$$

In last two relations the integrals should be understood as differences between the antiderivatives evaluated at the integration limits.

Norm.

Using analogy with the complex numbers we can introduce the notion of the norm of an extended number:

$$\|w\| = \exp(\operatorname{st}\ln w)$$

In this case appears an interesting role of the Euler-Mascheroni constant:

$$\|\omega_+\| = e^{-\gamma}$$

Divergent integrals.

Since we can represent the sums of divergent series as non-standard numbers, a question arises whether the same can be done with divergent integrals.

We postulate that any set of weighted dots on the real axis can be represented by a sum of symmetric distributions, cantered at the weighted dots and each having area the same as that of the dot it is centered at.

This way we can establish correspondence between sets of weighted dots and (divergent) integrals.

Particularly, we can see that

$$\int_{-1/2}^{\infty} dx = \sum_{k=0}^{\infty} 1 = \omega_+,$$

$$\int_{1/2}^{\infty} dx = \sum_{k=1}^{\infty} 1 = \omega_-,$$

because these integrals correspond to the sets of weighted dots at integers.

Moreover, we come to the following:

$$\int_0^{\infty} dx = \omega_- + 1/2 = \tau$$

An objection is possible here that the integral at the left will have a different value after variable change. That's why it is very important that x axis has the same scale as the function under integral. We will consider this true for x axis in this paper.

Following our principle, we can obtain exact expressions for other integrals as well:

$$\int_{-\frac{1}{2}}^{\infty} \pi \cos(\pi x)\, dx = \sum_{k=0}^{\infty} (-2)^k = 1$$

$$\int_1^{\infty} \frac{1}{x} = \sum_{k=1}^{\infty} \frac{1}{k} - \gamma$$

Besides this, from our definition follows the general rule: if $f(x)$ is periodic and integral over the period is zero, then

$$\int_{-\infty}^{+\infty} f(x) dx = 0$$

and if also even, then

$$\int_0^{+\infty} f(x)dx = 0$$

In particular,

$$\int_0^{\infty} \cos x \, dx = 0$$

It is important to notice that we are speaking about the precise value here rather than just standard part.

Standard parts of divergent integrals.

Using the definition of integral as a limit of the integral sum, we can derive the following formula for standard part of an improper integral.

$$\operatorname{st} \int_0^{\infty} f(x) \, dx = \lim_{s \to 0} \operatorname{st} s \sum_{k=1}^{\infty} f(sk)$$

Particularly,

$$\operatorname{st} \int_0^{\infty} e^x dx = -1$$

Connection with generalized functions and Dirac Delta function.

According the integral definition of Delta function,

$$\delta(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixz} dx$$

With real $x \neq 0$ the value of the integral is zero, according to our previous result. But at $z = 0$ we arrive

$$\delta(0) = \frac{1}{\pi} \int_0^{\infty} dx$$

This expression is equal to the $\frac{\tau}{\pi}$. We can arrive at the same result, considering Fourier transform of the Dirac comb.

Since Delta-function is a derivative of the step function, we can arrive at a conclusion that a derivative of a step function at the step point can be represents as extended numbers: the step size defines the extended part, and limit of derivative defines the standard part. Particularly,

$$\operatorname{sgn}'(0) = \frac{2\tau}{\pi}$$

Equvality between two extended numbers.

We postulate that two extended numbers are equal if distributional representation of one can be transformed into distributional representation of the other using only operations that preserve the value of Fourier transform of the distributions at the point zero.

Taking under account this as well as strict additivity we can derive expressions in extended numbers for numerocities of some subsets of integers:

$$N(0,2,4,6,...) = \frac{\tau}{2} + \frac{1}{2}$$

$$N(1,3,5,...) = \frac{\tau}{2}$$

Conjecture regarding closed forms of some infinite series.

We hereby conjecture the following property under the above definition of the extended numbers equality.

$$\sum_{k=0}^{\infty} f(k) = \frac{1}{D(e^D - 1)} \frac{f(x)}{x}|_{x=\omega_+} + 1/2 f(0)$$

Particularly, for $n > 0$

$$\sum_{k=0}^{\infty} x^n = \frac{B_{n+1}(\omega_+)}{n(n+1)}$$

In regards to standard part the conjecture can be trivially seen to hold.

Geometric interpretation of Riemann functional equation.

The volume of the unit n-sphere is defined with the following expression:

$$V(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

and the surface area is

$$S(n) = \frac{n\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

Following Riemann's functional equation, this gives us interesting relations between properties of positive-dimentional and hypothetical negetive-dimentional n-sphere of radius ω_+:

$$\operatorname{st} V(n)\omega_+^n = \operatorname{st} V(1-n)\omega_+^{1-n}$$

$$\operatorname{st}(1-n)S(n)\omega_+^n = \operatorname{st} nS(1-n)\omega_+^{1-n}$$

It is unclear whether these relations may have deeper geometric meaning.

Problem of a weighted dot in a hollow cavity in infinite filled space.

Consider an infinite 3-dimentional Euclidean space uniformly filled with dense matter. There is a spherical cavity in the space and a weighted point in that cavity at a distance from the cavity's center.

The question is: will the weighted dot experience any acceleration? It seems that the answer depends on what point one takes as a start for doing the infinite summation. We believe, our theory provides comprehensive answer by means of extended numbers: yes, the point would be accelerated in the direction opposite to the cavity's center. A similar problem often appears in physics of vacuum, such as calculation of Casimir effect.

Comparison to other non-Archimedean extensions.

Compared to other non-Archimedean extensions of real numbers (non-standard analysis, hyperreals, ordinals etc), this extension is non-trivial because produces fruitful relations between elementary functions. Besides this, the non-standard analysis and hyperreals suffer from the problem of definability: there is no way to define a unique infinite (or infinitesimal) element based on its properties. In the majority of non-Archimedean extensions, even if the definability problem is solved, there exists only one infinite special element, usually corresponding to the counting ordinal.

A difference of the proposed here system is in that there are two equally important special elements, corresponding to the cardinality measure of natural numbers (positive integers) and non-negative integers, expressing symmetry with each other (and, in the end, between 0 and 1).

This system unites quite distant areas of mathematics, dealing with infinities: comparing infinite sets, the theory of divergent series, divergent integrals, number theory, finite differences, theory of distributions, etc.

Extended numbers can be generalized for comparing everywhere dense sets, as well as not countable ones, which was not addressed in this paper.

Physical applications.

Regularization of divergent series and integrals for getting rid of infinities is used in quantum field theory, vacuum physics, quantum chromodynamics. Particularly, Paul Dirac obtained the mean energy of quantum harmonic oscillator using the regularization of the divergent series:

$$\epsilon = \frac{h\nu}{2} + \frac{h\nu}{e^{h\nu/kT} - 1}$$

The same value when using extended numbers would look like:

$$\epsilon = h\nu\omega_+ + kTe^{\frac{h\nu\omega_-}{kT}}$$

The first term here corresponds to zero-point energy. That way, a conjecture arises that ω_+ corresponds to the infinite number of filled negative energy levels in the Dirac sea, and the minimum ofservable energy of a quantum harmonic oscillator is $\frac{h\nu}{2}$ exactly because the standard part of ω_+ is 1/2.

Expression for the vacuum energy for calculation of Casimir effect between two parallel plates also can be written using extended numbers:

$$\frac{E}{A} = \frac{\hbar c \pi^2 \omega_+^4}{24 a^3}$$

The infinite term usually thrown away during regularization here is usually interpreted as vacuum zero-point energy.

Considering the above examples, a possibiility arises that extended numbers may play in quantum field theory and vacuum physics the same role as complex numbers play in quantum mechanics.