The "Vertical" (generalization of) the Binary Goldbach's conjecture (VBGC 1.0) as applied on primes with (recursive) prime indexes (o-primeths)[1 2,3]

*

Andrei-Lucian Drăgoi (24 January 2017) (study research interval: 2007-2017 - ?) [4,5]

*

This article version: 1.0

*

Abstract

This article proposes the generalization of the both binary (strong) and ternary (weak) Goldbach's Conjectures (**BGC and TGC)[1,2,3][4,5,6,7]**, briefly called "the Vertical Goldbach's Conjectures" (**VBGC and VTGC**), discovered in 2007[1] and perfected until 2016[3] by using the arrays ($\mathbf{S_p}$ and $\mathbf{S_{o,p}}$) of Matrix of Goldbach index-partitions (**GIPs**) (simple $\mathbf{M_{p,n}}$ and recursive $\mathbf{M_{o,p,n}}$, with order $\mathbf{o \geq 0}$), which are a useful tool in studying BGC by focusing on prime indexes (as the function P_n that numbers the primes is a bijection). Simple M ($\mathbf{M_{p,n}}$) and recursive M ($\mathbf{M_{o,p,n}}$) are related to the ***concept of generalized "primeths"*** (a term first used by Fernandez N. in his "The Exploring Primeness Project" **[8]**), *which is the generalization with order o≥0 of the known "higher-order prime numbers" (alias "superprime numbers", "super-prime numbers", "super-primes", " super-primes" or "prime-indexed primes[PIPs]") as a subset of (simple or recursive) primes with (also) prime indexes ($^{o}P_x$ is the x^{th} o-primeth, with order $\mathbf{o \geq 0}$ as explained later on).*

The author of this article also brings in a **S-M-synthesis** of some Goldbach-like conjectures (**GLC**) (including those which are "stronger" than BGC) and a new class of **GLC**s "stronger" than BGC, from which VBGC (which is essentially a variant of BGC applied on a serial array of subsets of primeths with a general order $\mathbf{o \geq 0}$) distinguishes as a very important conjecture of primes (with great importance in the optimization of the BGC experimental verification and other potential useful theoretical and practical applications in mathematics [including cryptography and fractals] and physics [including crystallography and M-Theory]), and a very special self-similar propriety of the primes subset of $\mathbb{N}$ (noted/abbreviated as $\wp$ or $\wp^*$ as explained later on in this article).

Keywords: Prime (number), primes with prime indexes, the o-primeths (with order o≥0), the Binary Goldbach Conjecture (BGC), the Ternary Goldbach Conjecture (TGC), Goldbach index-partition (GIP), fractal patterns of the number and distribution of Goldbach index-partitions, Goldbach-like conjectures (GLC), the Vertical Binary Goldbach Conjecture (VBGC) and Vertical Ternary Goldbach Conjecture (VTGC) the as applied on o-primeths

---

[1] Online preprint – VBGC version 1.0 (VBGC 1.0): DOI: 10.13140/RG.2.2.27963.62245

[2] Discovered in December 2007 as a special case of Goldbach's Conjecture and registered in 2012 (in this initial variant) in "Plicul cu idei" ("The envelope of ideas") (OSIM, Romania) with number: **300323/22.08.2012** (generic URL)

[3] ORDA (Romania) registration number: **4856/23.06.2016** (URL: orda.ro/cautare_cerere.aspx?mid=1&rid=1&cerere=4856)

[4] Romanian pediatrician specialist (with no additional academic title) undertaking independent research in the field of Prime Numbers

[5] Contact email: dr.dragoi@yahoo.com

Introduction

*Primes (which are considered natural numbers [positive integers] >1 that each has no positive divisors other than 1 and itself by the latest modern conventional definition, as number 1 is a special case***[9,10]** *which is considered neither prime nor composite, but the unit of all integers) are conjectured to have a sufficiently dense and (sufficiently) uniform distribution in* $\mathbb{N}$ *, so that: 1) any natural even number 2n (with n>1) can be splitted in at least one Goldbach partition/pair(GP)***[11]** OR *2) any positive integer >1 can be expressed as the arithmentic average of at least one pair of primes* (GC is specifically reformulated by the author of this article in order to emphasize the importance of studying the Primes Distribution (**PD**) **[12,13,14,15]** defined by a global and local density and uniformity with multiple interesting *fractal patterns* **[16]**: *GC is in fact an auto-recursive fractal propriety of PD in* $\mathbb{N}$ alias the *Goldbach Distribution of Primes* ***(GDP)*** (as the author will try to prove later on in this article), but also a propriety of $\wp$, a propriety which is indirectly expressed as GC, using the subset of even naturals).

***

Part A.
The array (S_p) of the simple Matrix of Goldbach Index-Partitions ($M_{p,n}$)

Definition of $\wp^*$ and $\wp$. We may define the prime subset of $\mathbb{N}$ as $\wp^*=\{P_1(=2), P_2(=3),\ldots,P_x,\ldots,P_y,\ldots P_\infty\}$, with $x, y\in\mathbb{N}^*$ ($0<x<y$), with $P_x(P_y)$ being the x^{th} (y^{th}) primes of $\wp$ and P_∞ marking the already proved fact that $\wp^*$ has an infinite number of (natural) elements (Euclid's 2^{nd} theorem **[17]**). The numbering function of primes (P_n) is a bijection that interconnects $\wp^*$ with $\mathbb{N}^*$ so that each element of $\wp^*$ corresponds to only (just) one element of $\mathbb{N}^*$ and vice versa: $1\leftrightarrow P_1(=2)$, $2\leftrightarrow P_2(=3)$, ..., $x \leftrightarrow P_x$(the x^{th} prime), $y \leftrightarrow P_y$(the y^{th} prime), …, $\infty\leftrightarrow P_\infty$. Originally, Goldbach considered that number 1 was the first prime: although still debated until present, today the mainstream considers that number 1 is neither prime or composite, but the unity of all the other integers.**[9,10]**. However, in respect to the first "ternary" formulation of GC (**TGC**) (which was re-formulated by Euler as the BGC and also demonstrated by the same Euler to be stronger than TGC, as TGC is a consequence of BGC), the author of this article also defines $P_0=1$ (the unity of all integers) and $\wp=\{P_0(=1), P_1(=2), P_2(=3),\ldots,P_x,\ldots,P_y,\ldots P_\infty\}$, with $x, y\in\mathbb{N}^*$ ($0\le x<y$), although only $\wp^*$ shall be used in this work.

***

The 1^{st} formulation of BGC. For any even integer $n>2$, it will always exist at least one pair of (other 2) integers ($x,y\in\mathbb{N}^*$) (with $x\le y$) so that $P_x+P_y=n$, with $P_x(P_y)$ being the x^{th} (y^{th}) primes of $\wp^*$. **Important observation**: The author considers that analyzing those “homogeneous” triplets of 3 naturals (n,x,y) (no matter if primes or composites) is more convenient and has more “analytical” potential than analyzing (relatively) “inhomogeneous” triplets of type (n,P_x,P_y): that’s why the author proposes *Goldbach index partitions (GIP)* as an alternative to the standard *Goldbach partitions (GP)* proposed by Oliveira e Silva**[11]**. The existence of (at least) a triplet (n,x,y) for each even integer n>2 (as BGC claims) may suggest that BGC is profoundly connected to the generic primality (of any P_x and P_y) and, more specifically, argues that *GC is in fact a propriety of PD in $\mathbb{N}$ (and a propriety of $\wp^*$ as composed of indexed/numbered elements)*. The most important propriety of Primes and PD and is that **$P_x\to x\cdot\ln(x)$ or $P_x\sim x\cdot\ln(x)$ for any large x** (which is the alternative [linearithmic] expression of the Prime Number Theorem **[18]**, as if $\wp$ is a result of an apparent random quantized linearithmization of $\mathbb{N}$ so that **$P_n \sim n\cdot\ln(n)$**). **In conclusion**: *For any even integer n>2, at least one GIP exists **(BGC – 1^{st} condensed formulation)***

***

The 2^{nd} formulation of BGC using *the Matrix of Goldbach index-partitions (M-GIP* or *M).*

[**1**] Let us consider an infinite string of matrix $S=\{M_1, M_2, \ldots, M_n,\ldots M_\infty\}$, with each generic M_n being composed of lines made by GIPs (x,y), such as:

$$M_n=\begin{pmatrix} x_{n,1} & y_{n,1} \\ \vdots & \vdots \\ x_{n,i} & y_{n,i} \\ \vdots & \vdots \\ x_{n,m_n} & y_{n,m_n} \end{pmatrix}, \text{ with } P_{x_{n,i}}+P_{y_{n,i}}=n,\ \forall\ i\in[1,m_n]$$

(i is the index of any chosen line of M_n, $i\ge1$ and $i\le m_n$)
(m_n is the total maximum number of i-indexed lines of M_n)
($x_{n,i},y_{n,i}\in\mathbb{N}^*$, $x_{n,i} < x_{n,i+1}$ for $m_n\ge2$, $\forall\ i\in[1,m_n]$)

[**2**] Let us also consider the function that counts the lines of any M_n, such as: **$l(n)=m_n$**. This function (that numbers the lines of a GM) is classically named as **$r(n)=l(n)=m_n$ (“r” is for the nof. “rows”).[11]**

[**3**] An **empty matrix (em)** is defined as a matrix in which the number of rows and/or columns is 0.

Using S, M, em and l(n) as previously defined, BGC has 2 formulations sub-variants:

1. *M_n≠em (S doesn't contain any em) for any even integer n>2 or shortly:* ***even integer n>2=>M_n≠em (the 2nd formulation of BGC – 1st sub-variant).***
2. *For any even integer n>2, l(n)>0 or shortly:* ***even integer n>2 => r(n)>0 (the 2nd formulation of BGC – 2nd sub-variant).***

***

The 3rd formulation of BGC using *the generalization of S (S_p) and the generalization of M ($M_{p,n}$) for GIPs matrix containing more than 2 columns (as based on GIPs with more than 2 elements).*

[1]Let us consider an infinite set OF infinite strings OF matrix:

a) **S_2**={$M_{2,1}$, $M_{2,2}$, …, $M_{2,n}$,… $M_{2,\infty}$} (the generic $M_{2,n}$ of S_2 has 2 columns based on [binary] GIPs with 2 elements)
b) **S_3**={$M_{3,1}$, $M_{3,2}$, …, $M_{3,n}$,… $M_{3,\infty}$} (the generic $M_{3,n}$ of S_3 has 3 columns based on [ternary] GIPs with 3 elements),
c) …,
d) **S_p**={$M_{p,1}$, $M_{p,2}$, …, $M_{p,n}$,… $M_{p,\infty}$} (the generic $M_{p,n}$ of S_p has p columns based on [p-nary] GIPs with p elements and natural p>3),
e) …,
f) **S_∞**={$M_{\infty,1}$, $M_{\infty,2}$, …, $M_{\infty,n}$,… $M_{\infty,\infty}$} (the generic $M_{\infty,n}$ of S_∞ has potentially infinite(∞) number of columns based on [∞-nary] GIPs with a potentially infinite (∞) number of elements)
g) With **e**ach generic $M_{p,n}$ being composed of $m_{p,n}$ lines and p columns made by p-nary GIPs with p elements, such as:

$$M_{p,n}=\begin{pmatrix} x_{n,1} & \cdots & x_{n,j} & \cdots & x_{n,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n,i} & \cdots & x_{n,j+i} & \cdots & x_{n,p+i} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n,m_{p,n}} & \cdots & x_{n,j+m_{p,n}} & \cdots & x_{n,p+m_{p,n}} \end{pmatrix}, \text{ with } P_{x_{n,i}}+\ldots+P_{x_{n,j+i}}+\ldots+P_{x_{n,p+i}}=n,$$
$$\forall\ i\in\left[1,m_{p,n}\right] and\ \forall\ j\in[1,p],$$

(i is the index of any chosen line of $M_{p,n}$, i≥1 and i≤$m_{p,n}$
and $m_{p,n}$ is the total maximum number of i-indexed lines of $M_{p,n}$)
(j is the index of any chosen column of $M_{p,n}$, j≥1 and j≤p
and p is the total number of j-indexed columns of $M_{p,n}$)
($x_{n,j+i}\in\mathbb{N}^*$, $x_{n,i}\le x_{n,i+1}$ for $m_{p,n}\ge 2$, $\forall\ i\in\left[1,m_{p,n}\right] and\ \forall\ j\in[1,p]$)

[**2**] Let us also consider the function that counts the lines of any $M_{p,n}$, such as: **r(p,n)=l(p,n)=$m_{p,n}$**.
[**3**] An **empty matrix** (**em**) is defined as a matrix in which the number of rows and/or columns is 0.

Using S_p, $M_{p,n}$, em and l(p,n) as previously defined, BGC has 2 formulations sub-variants:

1. *$M_{2,n}$≠em (S_2 doesn't contain any em) for any even integer n>2 or shortly:* ***even integer n>2=>$M_{2,n}$≠em (the 3rd formulation of BGC – 1st sub-variant).***
2. *For any even integer n>2, l(2,n)>0 or shortly:* ***even integer n>2 => r(2,n)>0 (the 3rd formulation of BGC –2nd sub-variant).***

***

Part B.
A synthesis and A/B classification of the main GLCs using $M_{p,n}$ concept

The Goldbach-like conjectures (GLCs) category/class.

GLCs definition. A GLC may be defined as any additional special (observed/conjectured) propriety of S_p and its elements $M_{p,n}$ other that GC (with n>2), with possibly other inferior limits a≥2 (with) n>a≥2.

GLCs classification. GLCs may be classified in 2 major classes using a double criterion such as:

1. **Type A GLCs (A-GLCs)** are *those GLCs that claim: [1] Not only that all $M_{p,n}$≠em for a chosen p>1 and for any / any odd / any even integer n>a (with a being any finite natural established by that A-GLC and a<n) BUT ALSO [2] any other non-trivial(**nt**) accessory propriety/proprieties of all $M_{p,n}$(≠em) of S_p. A specific A-GLC is considered authentic if the other non-trivial accessory propriety/proprieties of all $M_{p,n}$(≠em) (claimed by that A-GLC) isn't/aren't a consequence of the 1^{st} claim (of the same A-GLC). Authentic (at least conjectured as such) A-GLCs are (have the potential to be) "stronger" than GC as they claim "more" than GC does.*

2. **Type B GLCs (B-GLCs)** *are those GLCs that claim: no matter if all $M_{p,n}$≠em or just some $M_{p,n}$≠em for a chosen p>1 and for some / some odd / some even integer n>a (with a being any finite natural established by that B-GLC and a<n), all those $M_{p,n}$ that are yet non-em (for n>a) have (an)other non-trivial accessory propriety/proprieties. A specific B-GLC is considered authentic if the other non-trivial accessory propriety/proprieties of all $M_{p,n}$(≠em) (claimed by that B-GLC for n>a) isn't/aren't a consequence of the fact that some $M_{p,n}$≠em for n>a. Authentic (at least conjectured as such) B-GLCs are "neutral" to GC (uncertainly "stronger" or "weaker" conjectures) as they claim "more" but also "less" than GC does (although they may be globally weaker and easier to formally prove than GC).*

***

Other variants[1] of GC and GLCs include the statements that:

1. "*[...] Every [integer] number that is greater than 2 is the sum of three primes*" (Goldbach's original conjecture formulated in 1742, sometimes called the "ternary" Goldbach conjecture, written in a June 7, 1742 letter to Euler)[1] (which is equivalent to: *"every integer >2 is the sum of at least one triad of primes*", *with the specification that number 1 was also considered a prime by the majority of mathematicians contemporary to Goldbach, which is no longer the case now]"*). This (first) variant of GC can be formulated using (ternary) $M_{3,n}$ (based on GIPs with 3 elements) such as:
   a. **Type A formulation variant as applied to** $\wp$(*not just to* $\wp$*) : **"***integer n>2 => $M_{3,n}$≠em (with $x_{n,i,j} \geq 0$ and $P_{x_{n,i,j}} \in \wp$)"*
   b. **Type B (neutral) formulation variant**: not supported.

2. "*Every even integer n>4 is the sum of 2 odd primes.*" (Euler's binary reformulation of the original GC, which was initially expressed by Goldbach in a ternary form as previously explained) [1]. Since BGC (as originally reformulated by Euler) contains the obvious triviality that there are infinite many even positive integers of form 2p=p+p (with p a prime), the non-trivial BGC (**ntBGC**) sub-variant that shall be used in this article (alias "BGC" or "ntBGC") is that**: "*every even integer n>6 is the sum of at least one pair of distinct odd primes.*" [19,20]** (which is equivalent to: ***"every even integer >3 is the arithmetic average of at least one pair of distinct odd primes"***). Please note that ntBGC doesn't support the definition of a GLC, as 2p=p+p is a trivial propriety of some even integers implying the complementary relative triviality that: 2c≠2p≠p+p (with c a composite number and p a prime). NtBGC can be formulated using (binary) $M_{2,n}$ (based on GIPs with 2 elements) such as:
   a. **Type A formulation variant**: *"even integer n>6 => M_n (or $M_{2,n}$) ≠ em AND M_n (or $M_{2,n}$) contains at least one line with both elements (GIPs)≠1 (as P_1=2 is the only even prime)*

AND distinct to each other (as distinct GIPs means distinct primes as based on the bijection of the prime numbering function)"

b. **Type B (neutral) formulation variant**: "*even integer n>6 => all M_n ($M_{2,n}$) that are non-empty (as S_p may also contain empty $M_{2,n}$ for some specific [but still unfound] n values) will contain at least one line with both elements (GIPs)≠1 (as P_1=2 is the only even prime) AND distinct to each other (as distinct GIPs means distinct primes as based on the bijection of the prime numbering function)"*

3. "*Every odd integer n>5 is the sum of 3 (possibly identical) primes.*" **[1,21]** (the [weak] Ternary Goldbach's conjecture [**TGC/TGT – Ternary Goldbach's conjecture/theorem**]; formally proved by Harald Helfgott in 2013 **[22,23,24]**, so that TGC is very probably [*but not surely however*] a proved theorem, and no longer a "conjecture") (which is equivalent to: *"every integer >5 is the sum of at least one triad of [possibly identical] primes"*). TGC can be formulated using (ternary) $M_{3,n}$ (based on GIPs with 3 elements) such as:
   a. **Type A formulation variant**: **"***odd integer n>5 => $M_{3,n}$ ≠ em"*
   b. **Type B (neutral) formulation variant**: not supported.

4. "*Every integer n>17 is the sum of exactly 3 distinct primes.*" **[1,19]** (cited as "Conjecture 3.2" by Pakianathan and Winfree in their article, which is equivalent to: *"every integer >17 is the sum of at least one triad of distinct primes"*) (this is a conjecture stronger than TGC, but weaker than BGC as it is implied by BGC). This stronger version of TGC(**sTGC**) can be formulated using (ternary) $M_{3,n}$ (based on GIPs with 3 elements) such as:
   a. **Type A formulation variant**: **"***integer n>17 => $M_{3,n}$ ≠ em AND $M_{3,n}$ contains at least one line with all 3 elements (GIPs) distinct from each other"*
   b. **Type B (neutral) formulation variant**: "*integer n>17 => all the $M_{3,n}$ ≠ em will contain at least one line with all 3 elements (GIPs) distinct from each other"*

5. "*Every odd integer n>5 is the sum of a prime and a doubled prime [which is twice of any prime].*" (Lemoine's conjecture [**LC**] **[25,26]** which was erroneously attributed by MathWorld to Levy H. who pondered it in 1963 **[26,27**,**28]**). LC is stronger than TGC, but weaker than BGC. LC also has an extension formulated by Kiltinen J. and Young P. (alias the "refined Lemoine conjecture" **[29]**), which is stronger than LC, but weaker than BGC and won't be discussed in this article (as I shall mainly focus on those GLCs stronger than BGC). LC can be formulated using (ternary, not binary) $M_{3,n}$ (based on GIPs with 3 elements) such as:
   a. **Type A formulation variant**: **"***odd integer n >5 => $M_{3,n}$ ≠ em AND $M_{3,n}$ contains at least one line with at least 2 identical elements (GIPs)"*
   b. **Type B (neutral) formulation variant**: "*odd integer n >5 => all the $M_{3,n}$ ≠ em will contain at least one line with at least 2 identical elements (GIPs)"*

6. There are also a few original conjectures**[30]** on partitions of integers as summations of primes published by Smarandache F. that won't be discussed in this article, as these conjectures depart from VBGC (as VBGC presentation is the main purpose of this article).

***

There are also a number of (relative recently discovered) GLCs stronger than BGC (and implicitly stronger than TGC), that can also be synthesized using $M_{p,n}$ concept: **these stronger GLCs (as VBGC also is) are tools that can inspire new strategies in finding a formal proof for BGC, as I shall try to demonstrate next.** Additionally, there are some arguments that Twin Prime Conjecture (**TPC**) **[31]** may be also (indirectly) related to BGC as part of a more extended and profound conjecture [6] **[16,32,33, 34]**, so that any new clue for BGC formal proof may also help in TPC (formal) demonstration. Moreover, TPC may be weaker (and possibly easier to proof) than BGC (at least regarding the efforts toward the final formal proof) as the superior limit of the primes gap was recently "pushed" to be ≤246 **[35**], but the Chen's Theorem I (that *"every sufficiently large even number can be written as the sum of either 2 primes, OR a prime and a semiprime [the product of just 2 primes]*" [**36**,**37**,**38]**) has not been improved since a long time (at least by the set of proofs that are accepted in the present by the mainstream) except Cai's new proved theorem published in 2002 ("*There exists a natural number N such that every even integer n larger than N is a sum of* *a prime* $\leq n^{0.95}$ *and a semi-prime*" **[39,40]**, a theorem which is a similar but a weaker statement than LC that hasn't a formal proof yet).

1. *"For each even integer n>4 there is at least one prime number p [so that] sqrt(n)<p≤n/2 and q=n-p is also prime [with n=p+q implicitly]*" (the Goldbach-Knjzek conjecture [**GKC**] **[41]** which is stronger than BGC) **(**GKC can also be reformulated as: "*every even integer n>4 is the sum of at least one pair of primes with one element in the semi-open interval* $\left(\sqrt{n}, n/2\right]$". GKC can be formulated using (binary) $M_{2,n}$ (based on GIPs with 2 elements) such as:
    a. **Type A formulation variant**: **"***even integer n>4 => M_n ($M_{2,n}$) ≠ em AND M_n ($M_{2,n}$) contains at least one line with one element in the semi-opened interval* $\left(\sqrt{n}, n/2\right]$".
    b. **Type B (neutral) formulation variant**: "*even integer n>4 => all the M_n ($M_{2,n}$) ≠ em will contain at least one line with one element in the semi-opened interval* $\left(\sqrt{n}, n/2\right]$"

2. *"For each even integer n>4 there is at least one prime number p [so that] sqrt(n)<p<4·sqrt(n) and q=N-p is also prime [with n=p+q implicitly]*" (the Goldbach-Knjzek-Rivera conjecture [**GKRC**] **[42]** which is obviously stronger than BGC, but also stronger than GKC for n≥64) **(**GKRC can also be reformulated as: "*every even integer n>4 is the sum of at least one pair of primes with one element in the double-open interval* $\left(\sqrt{n}, 4\sqrt{n}\right)$". GKRC can be formulated using (binary) $M_{2,n}$ (based on GIPs with 2 elements) such as:
    a. **Type A formulation variant**: **"***even integer n>4 => M_n ($M_{2,n}$) ≠ em AND M_n ($M_{2,n}$) contains at least one line with one element in the double-open interval* $\left(\sqrt{n}, 4\sqrt{n}\right)$"
    b. **Type B (neutral) formulation variant**: "*even integer n>4 => all the M_n ($M_{2,n}$) ≠ em will contain at least one line with one element in the double-open interval* $\left(\sqrt{n}, 4\sqrt{n}\right)$"

3. Any other GLC that establishes an additional inferior limit a>0 for l(2,n) so that l(2,n)≥a>0 (like Woon's GLC **[43]**) can also be considered stronger that BGC, as BGC only suggests l(2,n)>0 for any even integer n>6 (which implies a greater average number of GIPs per each n than the more selective Woon's GLC does).

***

There is also a remarkable set of original conjectures (many of them stronger than BGC and/or TPC) originally proposed by Sun Zhi-Wei **[44,45]**, a set from which I shall cite **[46]** (by rephrasing) some of those conjectures that have an important element in common with the first special case of VBGC: the recursive P_{Px} function in which P_x is the x^{th} prime and P_{Px} is the P_x^{th} prime (which is denoted in the next cited conjectures as P_q which is the q^{th} prime, with q also a prime)

1. **Conjecture 3.1 (Unification of GC and TPC, 29 Jan. 2014)**. For any integer n>2 there is at least one triad of primes [(1<)q(<2n-1); (2n-q); (P_{q+2}+2)] (Sun's Conjecture 3.1 [**SC3.1 or U-GC-TPC**], which is obviously stronger than BGC and was tested up to $2\cdot10^8$)

2. **Conjecture 3.2 (Super TPC [SPTC], 5 Feb. 2014)**. For any integer n>2 there is at least one triad [(0<)k(<n); (P_k+2) prime; ($P_{P(n-k)}$+2) prime] (Sun's Conjecture 3.2 [**SC3.2 or SPTC**], which is obviously stronger than TPC and was tested up to 10^9) **[47,48]**

3. **Conjecture 3.3 (28 Jan. 2014)**. For any integer n>2 there is at least one pentad [(0<)k(<n-1); (6k-1) prime; (6k+1) prime; (P_{n-k}) prime; (P_{n-k}+2) prime] (Sun's Conjecture 3.3 [**SC3.3]**, which is obviously stronger than TPC as it implies TPC; SC3.3 was tested up to $2•10^7$)

4. **Conjecture 3.7-i (1 Dec. 2013)**. There are infinite many positive even integers n>3 which are associated with a hexad of primes [(n+1); (n-1); (P_n+n); (P_n-n); (nP_n+1); (nP_n-1)] (Sun's

Conjecture 3.7-1 [**SC3.7-i]**, which is obviously stronger than TPC as it implies TPC; the first n predicted by SC3.7-I is 22110)

5. **Conjecture 3.12-i (5 Dec. 2013)**. All positive integers n>7 have at least one associated pair [k(<n-1); (2^k+P_{n-k}) prime] (Sun's Conjecture 3.12-i [**SC3.12-i]**)

6. **Conjecture 3.12-ii (6 Dec. 2013)**. All positive integers n>3 have at least one associated pair [k(<n-1); ($k!+P_{n-k}$) prime] (Sun's Conjecture 3.12-ii [**SC3.12-ii]**)

7. **Remark 3.19 (which is an implication of the Conjecture 3.19 not cited in this article)**. There is an infinite number of triads of primes [q(>1), r(=P_q-q+1), (P_r-r+1)] (Sun's Remark on Sun's Conjecture 3.19 [**SRC3.19**])

8. **Conjecture 3.21-i (6 Mar. 2014)**. For any integer n>5 there will always exist at least one triad [(0<)k(<n), (2k+1) prime, (P_{kn}+kn) prime]. (Sun's Conjecture 3.21-i [**SC3.21-i**])

9. **Conjecture 3.23-i (1 Feb. 2014)**. For any integer n>13 there is at least one triad of primes [(1<)q(<n), (q+2), (P_{n-q}+q+1)] (the Sun's Conjecture 3.23-i [**SC3.23-i**])

***

Part C.
The 'o-primeths' ($^o\wp^*$) definition

The definition of "o-primeths", which is slightly different from Fernandez's definition[8]

I have chosen to use the term "primeth(s)" (Fernandez N. introduced it for the first time in 1999, in his "The Exploring Primeness Project" [8]) because this is the shortest and also the most suggestive of all the alternatives **[49]** used until now (as the "th" suffix includes by abbreviation the idea of "index of primes").

Primeths were originally defined by Fernandez N. as a subset of primes with (also) prime indexes[8] (the numbering of the elements of $\wp^*$ starts with $P_1=2$). As primes are in fact those positive integers with a prime index [8] (the "prime index" being non-tautological defined as a positive integer >1 that has only 2 distinct divisors: 1 and itself), all the primes may be considered primeths with order o=0 (or shortly: 0-primeths) NOT with o=1 (as Fernandez first considered[8]) (as the order o=0 marks the genesis of $\wp^*$ from the ordinary $\mathbb{N}\supset\wp^*$). This new definition of o-primeths ($^o\wp^*$ containing ^oP_x elements with $o\geq 0$ and $x\in\mathbb{N}^*$) has 3 advantages:

1. the order o is also the number of ("vertical") iterations for producing the o-primeths from the 0-primeths ($\wp^*$) (as in the Fernandez's original primeths definition, the standard primes were considered 1-primeths not 0-primeths, as if they were produced from $\mathbb{N}$ using 1 vertical iteration, but $\mathbb{N}$ doesn't contain just primes);
    a. these iterations numbered by order o are easy to follow when implemented in different algorithms using a programming language on a computer
2. the concept of primes can be generalized as o-primeths that also includes $\wp^*$ as the special case of 0-primeths ($^0\wp^*$);
3. this definition clearly separates $\wp^*$ from the ordinary $\mathbb{N}$ using 0 (not 1) as a starting order (o) for $\wp^*$ ($^0\wp^*$) and considering $\mathbb{N}$ as a $^{-1}\wp^*$ (a "bulky" $^{-1}\wp^*$ contaminated with composite positive integers that can be considered "-1-primeths" convertible to 0-primeths by different sieves of primes, which are another kind of iterations than those producing o-primeths from 0-primeths)
    a. $^0\wp^*$ inevitably "contains" $\mathbb{N}$ by its indexes , in the sense that $^o\wp^*$ contains ^oP_x elements with indexes $x\in\mathbb{N}^*$ (an index x that scrolls all $\mathbb{N}$). The same positive integer may be part of more than one $^o\wp^*$, as x is not necessarily a prime.
    b. This slightly different definition of the o-primeths ($^o\wp^*$ containing ^oP_x elements with $o\geq 0$ and $x>0$) (as explained previously) is not a new "anomaly" and it was also practiced by Smarandache F. as cited by Murthy A.[**50]** and also by Seleacu V. and Bălăcenoiu I. [**51]**

The elements of the group $^o\wp^*$

$^0\wp^*=\{\,^0P_1(=P_1=2),\ ^0P_2(=P_2=3),\ ^0P_3(=P_3(=5),\dots,\ ^0P_x(=P_x),\dots\}$ (alias 0-primeths)

$^1\wp^*=\{^1P_1(=P_{P_1}=P_2=3),\ ^1P_2(=P_{P_2}=P_3=5),\dots,\ ^1P_x(=P_{P_X}),\dots\}$ (alias 1-primeths **[52]**)

$^2\wp^*=\{^2P_1(=P_{P_{P_1}}=P_3=5),\ ^2P_2(=P_{P_{P_2}}=P_5=11),\dots,\ ^2P_x(=P_{P_{P_X}}),\dots\},\ \dots$

$$^o\wp^*=\{^oP_1=P_{\underbrace{P_{\dots P_1}}_{o\ iterations\ of\ P}},\ ^oP_2=P_{\underbrace{P_{\dots P_2}}_{o\ iterations\ of\ P}},\dots,\ ^oP_x=P_{\underbrace{P_{\dots P_X}}_{o\ iterations\ of\ P}},\dots\},\ \text{with } x\in\mathbb{N}^*-\{1,2\}$$

Part D.

VBGC 1.0 - The extension and generalization of BGC as applied on o-primeths ($^{o}\wp^{*}$)

VBGC 1.0 (version 1.0, the same with the version of this article) – main statement:

1. **For any pair of finite positive integers (a,b)** (with a≥b≥0) defining the (recursive) orders of an a-primeth (^{a}P) and a b-primeth respectively(^{b}P), **there will always exist a single finite** positive integer $n_{a,b}$**≥3 so that, for any positive even integer m>2$n_{a,b}$, it will always exist at least one pair of finite *distinct* positive odd integers (x,y)** (with x≠y, x>1, y>1 indexes of distinct odd primes) **so that: $^{a}P_{x}+{}^{b}P_{y}=2m$ AND $^{a}P_{x}>{}^{b}P_{y}$ AND the function f(a,b)=$n_{a,b}$ has a finite positive integer value for any combination of finite positive integers (a,b), without any catastrophic-like infinities for any (a,b) pair of finites positive integers. Important note. I have chosen the additional condition $^{a}P_{x}>{}^{b}P_{y}$ for a≥b≥0 so that to lower the nof. lines per each GM and to simplify the algorithm of searching for ($^{a}P_{x}$,$^{b}P_{y}$) pairs, as the a-primeths are much fewer that b-primeths if a>b AND the sieve using a-primeths (which searches an a-primeth from 2m to 3) finds a ($^{a}P_{x}$,$^{b}P_{y}$) pair much more quicker than a sieve using b-primeths if a>b.**
   a. **f(0,0)=$n_{0,0}$=3, f(1,0)= $n_{1,0}$=701, f(2,0)= $n_{2,0}$=29009, f(1,1)= $n_{1,1}$=40306, f(2,1)= $n_{2,1}$=[working progress on this result] , f(2,2)= $n_{2,2}$=[working progress on this result], …**

2. **AND**
   a. **for (a,b)=(1,0) and m>28, it will always exist at least one pair of finite distinct positive integers (x,y) (with x,y>1 AND $^{1}P_{x}+{}^{0}P_{y}=2m$ AND (x or y) in the double-open interval (ln[2m],2m/ln[2m]).**
      i. **Important note:** VBGC is much stronger and general than BGC and proposes a much more rapid and efficient (at-least-one-GIP)-sieve than the GKRC. The GM of GIPs generated by VBGC has a smaller nof. lines than the GM of GIPs generated by GKRC. VBGC is a useful optimized sieve to push forward the limit $4\cdot10^{18}$ to which BGC was verified to hold **[53]**. When verifying BGC for a very large number L, one can use the VBGC(a,b) with a minimal positive value for the difference L-f(a,b).

3. **Important note:** VBGC essentially states that there is an infinite number of conjectures named VBGC(a,b), all stronger than BGC, each of if associated with a pair (a,b) (with a,b>0) and a finite positive integer $n_{a,b}$=f(a,b).
   a. VBGC(0,0) is in fact ntBGC.

VBGC 1.0 – secondary statements (also part of VBGC):

1. **The different special cases of VBGC can be named after the pair (a,b) [VBGC(a,b)] AND:**
   a. **VBGC(0,0)** is in fact ntBGC (defined in the Part B of this article)
   b. **VBGC(1,0)**[1] is a GLC stronger and more elegant than ntBGC, as it acts on a limit(2$n_{1,0}$=6) close to ntBGC inferior limit (l=4) BUT the associated $G_{1,0}$ (n) function (which counts the number of pairs of possible GIPs for any even integer n>6) has significantly smaller values than the $G_{0,0}$(n) function of ntBGC [which is VBGC(0,0)]
   c. **VBGC(2,0)** is obviously a stronger GLC than VBGC(1,0) is.
   d. **VBGC(1,1**) (anticipated by my discovery of **VBGC(1,0) from 2007 and officially registered in 2012 at OSIM** [1]) is an obviously stronger GLC than VBGC(1,0) and is equivalent to Bayless-Klyve-Oliveira e Silva Goldbach-like Conjecture (BKOS-GLC)

published in Oct. 2013 [**54]** (alias “Conjecture 9.1” [rephrased]: all even integers n>80612[>**2·40306>2·$n_{1,1}$**] can be expressed as the sum of at least one pair of prime-indexed primes [PIPs] [1-primeths] and tested up to $n=10^{10}$). This article of Bayless. Klyve and Oliveira (2012, 2013) was based on a previous article by Barnett and Broughan (published in 2009) **[55]**, but BKOS-GLC was an additional result to this 2009 article. Mr. George Anescu (a friend and collaborator) have also helped me to retest VBGC(1,1) up to $n=10^{10,}$ but also helped me testing all VBGC for all pairs (a,b)={(1,0), (2,0), (2,1)} **[6]**.

2. $G_{a,b}(n_{a,b}+1)\rightarrow 1$, when $(a,b)\rightarrow\infty$ and the “comets” of VBGC(a,b) tend to narrow progressively for each pair (a_2, b_2), with $a_2>a_1$ and $b_2>b_1$

3. All VBGC(a>0,b≥0) can be used to produce more rapid algorithms for the experimental verification of ntBGC for very large positive integers

a. For VBGC(1,0), the average number of attempts (ANA) to find the first pair (x,y) for each integer m, in the interval [3,2m] tends asymptotically to $\ln\left(\sqrt{n}\right) = \ln(n)/2$ when searching just the 1-primeths subset in descending manner, starting from the largest 1-primeth ≤ 2m and verifying if $2m-{}^{1}P_x$ is a 0- primeth)

Conclusions on VBGC 1.0:

1. VBGC(a,b) is essentially an extension and generalization of BGC as applied on (the extended and generalized concept of) all ${}^{o}\wp^{*}$ subsets of o-primeths.

2. VBGC distinguishes as a very important (unified) conjecture of primes and a very special self-similar propriety of the primes as the rarefied ${}^{o}\wp^{*}$ is self-similar to the more dense ${}^{o-1}\wp^{*}$ in respect to the ntBGC. In other words, each of the o-primeths sets behaves as a “summary of” the 0-primeths set in respect to the ntBGC: this is a (quasi)fractal-like BGC-related behavior of the infinite number of the o-primeths sets (Batchko R.G. has also reported other quasi-fractal/quasi-self-similar structure in the distribution of the prime-indexed primes **[56]:** Batchko also used a similar general definition for primes with recursive prime indexes, briefly named in my article as “o-primeths”). **Essentially, VBGC conjectures that ntBGC is a common propriety of all the o-primeths sets (for any positive integer order o), differing just by the inferior limit of each VBGC(a,b). I have called VBGC as “vertical” motivated by the fact that VBGC is a “vertical” (recursive) generalization of the ntBGC on the infinite super-set of o-primeths sets.**

   a. The set n(a,b) is a set of critical density thresholds/points of each o-primeths set in respect to the set VBGC(a,b) conjectures.

Future challenges for VBGC (to be also approached in the next versions of this article):

1. **To calculate $f(a,b)=n_{a,b}$ and test VBGC(a,b) for large positive integers pairs (a,b), but also for the pairs (a,b) with large (a-b) differences.**

Potential applications of VBGC (to also be created in the next versions of this article):

1. VBGC can offer a potential infinite set of Goldbach Comets, one for each sub-VBGC applied on each order of o-primeths

2. **VBGC can be used to optimize the algorithms of finding/verifying very large primes (o-primeths)/potential primes (o-primeths)**

---

[6] The code-source (written by Mr. George Anescu in Microsoft C#.NET language/environment using parallel processing) that was used to test BKOS-GLC up to $n=10^{10}$ (using a laptop PC with an Intel[R] Core[TM] processor i7-3630 QM CPU at 2.4 GHz with 4 processors (8 hyper-threads), can be found at this URL: dragoii.com/test_primes.rar

3. **VBGC can be used as a model to also formulate a Vertical (generalization) of the Ternary Goldbach Conjecture/Theorem (VTGC)**

4. **VBGC can be theoretically used to optimize the algorithms of prime/integer factorization[URL2,URL3] (the main tool of cryptography)**

5. **VBGC can offer a rule of decomposition of Euclidean[URL2,URL3,URL4]/non-Euclidean[URL2] spaces/volumes with a finite 2N (positive) integer number of dimensions into pair of spaces, both with a (positive) o-primeth number of dimensions**

6. **VBGC can be used in M-Theory to simulate decompositions of 2N-branes (with a finite 2N [positive] integer number of dimensions) into pair of branes both with a (positive) o-primeth number of dimensions**

7. **VBGC can be also used to predict possible symmetries/asymmetries in crystallography, as based on o-primeths.**

Acknowledgements

I would like to express all my sincere gratitude and appreciation to all my mathematics, physics, chemistry and medicine teachers for their support and fellowship throughout the years, which provided substantial and profound inner motivation for the redaction and completion of this manuscript. I would also like to emphasize my friendship with George Anescu (physicist and mathematician) who helped me verify the VBGC up to $n=10^{10}$.

My special thanks to professor Toma Albu[7] who had the patience to read and listen my weak voice in mathematics as a hobbyist. Also my sincere gratitude to professor Şerban-Valentin Strătilă[8] that adviced me on the first special case of VBGC discovered in 2007 and he urged me to look for a more general conjecture based on my first observation.

***

Competing interests

Author has declared that no competing interests exist.

[7] The CV of Professor Albu T. is also available online (URL)
[8] The CV of Professor Strătilă Ş-V. is also available online (URL)

ENDNOTE ADDITIONAL REFERENCES (in order of citation in this article)

[1] Weisstein E. W. (1999-2014). "Goldbach Conjecture", web article From MathWorld—A Wolfram Web Resource. (URL1, URL2; URL3)
[2] Caldwell C.K. (1999-2015). "Goldbach's conjecture (another Prime Pages' Glossary entries)", web article. (URL)
[3] Oliveira e Silva T. (2014). "Goldbach conjecture verification", web article. (URL)
[4] Ye J. D., Liu C. (2013). "A Study of Goldbach's conjecture and Polignac's conjecture equivalence issues", IACR Cryptology ePrint Archive, Volume 2013 (URL)
[5] Ye J. D., Liu C. (2014). "A Study of Relationship of RSA with Goldbach's Conjecture and Its Properties" (URL1, URL2)
[6] Liu C. (2015). "A Study of Relationship Among Goldbach Conjecture, Twin Prime and Fibonacci Number" (URL1, URL2)
[7] Liu C., Chang C-C., Wu Z-P., Ye S-L (2015). "A Study of Relationship between RSA Public Key Cryptosystem and Goldbach's Conjecture Properties", International Journal of Network Security, Vol.17, No.4, PP.445-453, July 2015 (URL)
[8] Fernandez N. (1999). "The Exploring Primeness Project", website. (URL1, URL2, URL3-OIES page, URL4-OIES page)
[9] Weisstein E. W. (1999-2015). "Prime Number", web article From MathWorld—A Wolfram Web Resource. (URL)
[10] Wells, D (1986). "The Penguin Dictionary of Curious and Interesting Numbers", Middlesex, England: Penguin Books, 1986, p. 31. (URL)
[11] Weisstein E. W. (1999-2015). "Goldbach Partition", web article (URL); Some JavaScript/ Wolfram Language online calculator of Goldbach partitions can be found at: URL1, URL2, URL3)
[12] Granville A. (1993). "Harald Cramér and the distribution of prime numbers" (based on a lecture presented on 24th September 1993 at the Cramér symposium in Stockholm. (URL)
[13] Granville A. (2009). "Different Approaches to the Distribution of Primes", Milan Journal of Mathematic vol. 78 (2009), p. 1–25 (URL)
[14] Soundararajan K.(2006). "The distribution of prime numbers" (URL)
[15] Diamond H.G.(1982). "Elementary methods in the study of the distribution of prime numbers", Bull. Amer. Math. Soc. (N.S.), Volume 7, Number 3 (1982), p. 553-589. (URL)
[16] Liang W.,Yan H., Zhi-cheng D. (2006). "Fractal in the statistics of Goldbach partition" (URL)
[17] Weisstein E. W. (1999-2014). "Euclid's Theorems", web article From MathWorld—A Wolfram Web Resource. (URL)
[18] Weisstein E. W. (1999-2014). "Prime Number Theorem", web article From MathWorld—A Wolfram Web Resource. (URL)
[19] Pakianathan J. and Winfree T. (2011). "Quota Complexes, Persistent Homology and the Goldbach Conjecture", pages 9-10 (URL)
[20] Zhang S. (2008, 2010). "Goldbach Conjecture and the least prime number in an arithmetic progression", page 2 (URL)
[21] Wikipedia contributors (last update: 2 Aug. 2015). "Goldbach's weak conjecture", Wikipedia, The Free Encyclopedia (URL accessed on 13 December 2015)
[22] Helfgott H.A. (2013). "The ternary Goldbach conjecture is true"* (URL1, URL2, URL3) (*although it still has to go through the formalities of publication, Helfgott's preprint is endorsed and believed to be true by top mathematicians, including the Fields medalist Terence Tao who showed in 2012 that any odd integer is the sum of at most 5 primes, as can be found at: URL1, URL2)
[23] Helfgott H.A. (2014, 2015). "The ternary Goldbach problem", Snapshots of modern mathematics from Oberwolfach, No. 3/2014 (URL1; URL2, URL3)
[24] Platt D.J. (2014). "Proving Goldbach's Weak Conjecture" (URL)
[25] Lemoine E.(1894). "L'intermédiare des mathématiciens", 1 (1894), 179; ibid 3 (1896), page 151
[26] Wikipedia contributors (last update: 25 Nov 2014). "Lemoine's conjecture", Wikipedia, The Free Encyclopedia (URL accessed on 4 Jan 2016)
[27] Levy H.(1963). "On Goldbach's Conjecture", Math. Gaz. 47 (1963): page 274
[28] Weisstein E. W. (1999-2014). "Levy's Conjecture", web article From MathWorld—A Wolfram Web Resource. (URL)
[29] Kiltinen J. and Young P. (September 1984). "Goldbach, Lemoine, and a Know/Don't Know Problem", Mathematics Magazine (Mathematical Association of America) 58 (4): pages 195–203 (URL1, URL2)
[30] Smarandache F.(1999, 2000[republished], 2007[republished]). "Conjectures on partitions of integers as summations of primes", "Math Power", Pima Community College, Tucson, AZ, USA, Vol. 5, No. 9, pp. 2-4, September 1999; (URL)
[31] Weisstein E. W. (1999-2014). "Twin Prime Conjecture", web article From MathWorld—A Wolfram Web Resource. (URL)
[32] Ikorong G.A.N (2007). "Around the twin primes conjecture and the Goldbach conjecture I", Analele Științifice ale Universității "Al. I. Cuza" din Iași (S.N.), Matematică, Tomul LIII, 2007, f.1 (URL)
[33] Ikorong G.A.N (2008). "Playing with the Twin Primes Conjecture and the Goldbach Conjecture", Alabama Journal of Mathematics, Spring/Fall 2008, pages 45-52 (URL)
[34] Gerstein L.J. (1993). "A Reformulation of the Goldbach Conjecture", Mathematics Magazine vol. 66, no.1, February 1993. pages 44-45 (URL)
[35] Polymath D.H.J. (2014). "The "bounded gaps between primes" Polymath project - a retrospective" (URL)
[36] Chen, J.R. (1966). "On the representation of a large even integer as the sum of a prime and the product of at most two primes", Kexue Tongbao 11 (9): pages 385–386
[37] Chen, J.R. (1973). "On the representation of a larger even integer as the sum of a prime and the product of at most two primes", Scientia Sinica 16: pages 157–176
[38] Cheng-Dong P., Xia-Xi D., Yuan W. (1975). "On the representation of every large even integer as a sum of a prime and an almost prime", Scientia Sinica Vol. XVIII No.5 Sept.-Oct. 1975: pages 599–610 (URL)
[39] Cai, Y.C. (2002). "Chen's Theorem with Small Primes", Acta Mathematica Sinica 18 (3): pages 597–604 (URL)
[40] Cai, Y.C. (2008). "On Chen's theorem (II)", Journal of Number Theory, Volume 128, Issue 5, May 2008, pages: 1336–1357 (URL)

[41] Rivera C. (1999-2001). "Conjecture 22. A stronger version of the Goldbach Conjecture (by Mr. Rudolf Knjzek, from Austria)", web article from Prime Problems & Puzzles. (URL)
[42] Rivera C. (1999-2001). "Conjecture 22. A stronger version of the Goldbach Conjecture (by Mr. Rudolf Knjzek, from Austria and narrowed by Rivera C.)", web article from Prime Problems & Puzzles. (URL)
[43] Woon M.S.C. (2000). "On Partitions of Goldbach's Conjecture" (URL)
[44] Sun Z-W. (2013, 2014). Chapter "Problems on combinatorial properties of primes" (19 pages) in "Number Theory: Plowing and Starring Through High Wave Forms: Proceedings of the 7th China-Japan Seminar" (edited by Kaneko M., Kanemitsu S. and Liu J.), pages: 169 – 188 (URL1-book excerpt, URL2- full book)
[45] Sun Z-W. (2014). "Towards the Twin Prime Conjecture", A talk given at: NCTS (Hsinchu, Taiwan, August 6, 2014), Northwest University (Xi'an, October 26, 2014) and at Center for Combinatorics, Nankai University (Tianjin, Nov. 3, 2014) (URL)
[46] See also Sun's Z-W. personal web page on which all conjectures are presented in detail (URL)
[47] See also the first announcement of this conjecture made by Sun Z-W. himself on 6 Feb 2014) (URL)
[48] See also the sequence A218829 on OEIS.org proposed by Sun Z-W. (URL1, URL2)
[49] Alternative terms for "primeths": "higher-order prime numbers", "superprime numbers", "super-prime numbers", "super-primes", " superprimes" or "prime-indexed primes[PIPs]" (URL-OIES page)
[50] Murthy A. (2005). "Generalized Partitions and New Ideas on Number Theory and Smarandache Sequences" (book), page 91 (URL1-book, URL2 – page 181)
[51] Seleacu V. and Bălăcenoiu I. (2000). "Smarandache Notions, Vol. 11" (book), page181 (URL)
[52] Primes subset (3, 5, 11, 17, 31, 41, 59, 67, 83, 109, 127, 157, …) **,** also known as sequence **A006450** in OEIS (URL-OIES page)
[53] Oliveira e Silva T. (30 Dec. 2016). "Goldbach conjecture verification" (web article) (URL)
[54] Bayless J., Klyve D. and Oliveira e Silva T. (2012, 2013). "New bounds and computations on prime-indexed primes" (23 pages article,), Integers: Annual Volume 13 (2013), page 17 (URL1, URL2, URL3)
[55] Broughan K.A., Ross Barnett A. (2009). "On the Subsequence of Primes Having Prime Subscripts" (10 pages), Article 09.2.3 from the Journal of Integer Sequences, Vol. 12 (2009) (URL1, URL2, URL3, URL4)
[56] Batchko R.G. (2014). "A prime fractal and global quasi-self-similar structure in the distribution of prime-indexed primes", ArXiv article, submitted on 10 May 2014 (v1), last revised 17 May 2014 (this version, v2) (URL1, URL2)