I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 1/6.”

which has been published under the ‘VIXRA’ Mathematics subheading: ‘Functions and Analysis’.


For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.
we inevitably continuous, being a characteristic property of all such real
variable polynomial functions, and hence the quaternion hyper-complex function,
g(t), is also continuous.

Evidently, the continuity of g(t) is dependant upon that of q(t), in conjunction
with the proviso that q(t) ≠ 0, ∀t ∈ (a, b), so much so that it indeed
epitomizes the smoothness of each one and hence, for this reason, we shall
explicitly refer to a smooth arc or C, whenever we talk about the derivatives of
quaternion hyper-complex functions for the remainder of this section. Finally, we
remark that the meaning of Definition DIV-6 should also be self-explan-
atory insofar as it purports to be nothing else other than a logical extension
of its analogue from complex variable analysis.

3. Differentiation Formulas pertaining to Sums, Products
and Quotients of Quaternion Hypercomplex Functions
restricted to Smooth Arcs embedded in q-Space.

Various differentiation formulas pertaining to the sums, products and
quotients of both real and complex variable functions have already been
well established and this in turn motivates us to subsequently derive appropri-
ate analogues thereof with respect to our ensuing analysis of the properties of
quaternion hypercomplex functions. In so doing, we are naturally mindful of the
previous originating from the preceding Parts 1 and 2 of this section, as is
evident from our next theorem which has been explicitly stated and likewise
modified for that very purpose.
Let there exist two quaternion hyper-complex functions, \( \phi_1(q) \) and \( \phi_2(q) \), which are restricted to a smooth arc, \( C \), thus defined by the equation,

\[
q(t) = x(t) + iy(t) + jx(t) + kh(t), \quad \forall t \in [a, b].
\]

Hence, it may be shown that, if \( \phi_1(q) \) and \( \phi_2(q) \) respectively are both differentiable in \( t \), \( \forall t \in (a, b) \), then the following differentiation formulae shall apply, \( \forall t \in (a, b) \):

\[
(i) \quad \frac{d}{dt} \left[ \phi_1(q) + \phi_2(q) \right] = \left\{ \begin{array}{l}
\frac{\hat{z} [\phi_1(q(t))] + \hat{z} [\phi_2(q(t))]}{\hat{z} [q(t)]} \frac{\hat{z} [q(t)]}{\hat{z} [q(t)]^2} \\
\frac{\hat{z} [q(t)] (\hat{z} [\phi_1(q(t))] + \hat{z} [\phi_2(q(t))])}{\hat{z} [q(t)]^2}
\end{array} \right.
\]

\[
(ii) \quad \frac{d}{dt} \left[ \phi_1(q) \phi_2(q) \right] = \left\{ \begin{array}{l}
\frac{\hat{z} [\phi_1(q(t))] \hat{z} [q(t)]}{\hat{z} [q(t)]} \frac{\hat{z} [q(t)]}{\hat{z} [q(t)]^2} \\
\frac{\hat{z} [q(t)] \hat{z} [\phi_2(q(t))]}{\hat{z} [q(t)]^2}
\end{array} \right. ,
\]

where \( \hat{z} [f(q(t))] = \phi_2(q(t)) \hat{z} [\phi_2(q(t))] + \hat{z} [\phi_1(q(t))] \phi_2(q(t)) \).
\[
\left[ \frac{\partial}{\partial \mathbf{q}} \right] C (\phi_2 (q), \phi_1 (q)) = \left\{ \begin{array}{c}
\frac{\delta [\phi_1 (q)]^{\phi_2 (q)}}{\delta [\phi_1 (q)]^{\phi_2 (q)}} \frac{\delta [\phi_2 (q)]^{\phi_1 (q)}}{\delta [\phi_1 (q)]^{\phi_2 (q)}} \frac{\delta [\phi_1 (q)]^{\phi_2 (q)}}{\delta [\phi_1 (q)]^{\phi_2 (q)}} \\
\frac{\delta [\phi_2 (q)]^{\phi_1 (q)}}{\delta [\phi_2 (q)]^{\phi_1 (q)}} \frac{\delta [\phi_1 (q)]^{\phi_2 (q)}}{\delta [\phi_2 (q)]^{\phi_1 (q)}} \frac{\delta [\phi_2 (q)]^{\phi_1 (q)}}{\delta [\phi_2 (q)]^{\phi_1 (q)}} \\
\frac{\delta [\phi_1 (q)]^{\phi_2 (q)}}{\delta [\phi_1 (q)]^{\phi_2 (q)}} \frac{\delta [\phi_2 (q)]^{\phi_1 (q)}}{\delta [\phi_2 (q)]^{\phi_1 (q)}} \frac{\delta [\phi_1 (q)]^{\phi_2 (q)}}{\delta [\phi_1 (q)]^{\phi_2 (q)}} \\
\frac{\delta [\phi_2 (q)]^{\phi_1 (q)}}{\delta [\phi_2 (q)]^{\phi_1 (q)}} \frac{\delta [\phi_1 (q)]^{\phi_2 (q)}}{\delta [\phi_2 (q)]^{\phi_1 (q)}} \frac{\delta [\phi_2 (q)]^{\phi_1 (q)}}{\delta [\phi_2 (q)]^{\phi_1 (q)}} \\
\frac{\delta [\phi_1 (q)]^{\phi_2 (q)}}{\delta [\phi_1 (q)]^{\phi_2 (q)}} \frac{\delta [\phi_2 (q)]^{\phi_1 (q)}}{\delta [\phi_2 (q)]^{\phi_1 (q)}} \frac{\delta [\phi_1 (q)]^{\phi_2 (q)}}{\delta [\phi_1 (q)]^{\phi_2 (q)}} \\
\end{array} \right.
\]

where \( \frac{\delta}{\delta \mathbf{q}} [\mathbf{q} (t)] = \mathbf{q}_1 (t) \frac{\delta}{\delta \mathbf{q}_1 (t)} \mathbf{q}_2 (t) + \mathbf{q}_2 (t) \frac{\delta}{\delta \mathbf{q}_2 (t)} \mathbf{q}_1 (t) \),

\[
\left[ \frac{\partial}{\partial \mathbf{q}} \right] C (\phi_2 (q), \phi_1 (q)) = \left\{ \begin{array}{c}
\frac{\delta [\psi_1 (q(t))]^{\phi_2 (q(t))}}{\delta [\psi_1 (q(t))]^{\phi_2 (q(t))}} \frac{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}}{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}} \frac{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}}{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}} \\
\frac{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}}{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}} \frac{\delta [\psi_1 (q(t))]^{\phi_2 (q(t))}}{\delta [\psi_1 (q(t))]^{\phi_2 (q(t))}} \frac{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}}{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}} \\
\frac{\delta [\psi_2 (q(t))]^{\phi_2 (q(t))}}{\delta [\psi_2 (q(t))]^{\phi_2 (q(t))}} \frac{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}}{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}} \frac{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}}{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}} \\
\frac{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}}{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}} \frac{\delta [\psi_2 (q(t))]^{\phi_2 (q(t))}}{\delta [\psi_2 (q(t))]^{\phi_2 (q(t))}} \frac{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}}{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}} \\
\frac{\delta [\psi_2 (q(t))]^{\phi_2 (q(t))}}{\delta [\psi_2 (q(t))]^{\phi_2 (q(t))}} \frac{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}}{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}} \frac{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}}{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}} \\
\frac{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}}{\delta [\phi_2 (q(t))]^{\phi_1 (q(t))}} \frac{\delta [\psi_2 (q(t))]^{\phi_2 (q(t))}}{\delta [\psi_2 (q(t))]^{\phi_2 (q(t))}} \frac{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}}{\delta [\phi_1 (q(t))]^{\phi_2 (q(t))}} \\
\end{array} \right.
\]

where the derivatives,

\[
\frac{\delta}{\delta \mathbf{q}} [\psi_1 (q(t))] = \psi_1 (q(t)) \frac{\delta}{\delta \psi_1 (q(t))} \left[ \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right] + \frac{\delta}{\delta \psi_1 (q(t))} \left[ \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right] \frac{\delta}{\delta \psi_1 (q(t))} \left[ \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right],
\]

\[
\frac{\delta}{\delta \mathbf{q}} [\psi_2 (q(t))] = \left( \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right)^2 \frac{\delta}{\delta \psi_2 (q(t))} \left[ \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right] + \frac{\delta}{\delta \psi_2 (q(t))} \left[ \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right] \frac{\delta}{\delta \psi_2 (q(t))} \left[ \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right],
\]

\[
\frac{\delta}{\delta \mathbf{q}} \left[ \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right] = \left( \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right)^2 \frac{\delta}{\delta \phi_2 (q(t))} \left[ \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right] - \frac{\delta}{\delta \phi_2 (q(t))} \left[ \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right] \frac{\delta}{\delta \phi_2 (q(t))} \left[ \frac{\phi_2 (q(t))}{\phi_2 (q(t))} \right].
\]
and \( \phi_2(q(t)) \neq 0, \forall t \in (a, b) \).

\[
\frac{dy}{dt} \bigg|_{C(x)} (f(q)) = \begin{cases} 
\frac{\frac{d}{dt}[f(q(t))] \frac{d}{dt}[g(t)]}{\frac{d}{dt}[g(t)]^2}, & \forall t \in (a, b). \\
\frac{\frac{d}{dt}[g(t)] \frac{d}{dt}[f(q(t))]}{\frac{d}{dt}[g(t)]^2} 
\end{cases}
\]

Consequently, we are now in a position to directly apply this formula in those cases where \( f \) is expressed successively as a sum, product and quotient of its component functions, \( \phi_1 \) and \( \phi_2 \).

\( \star \) Let us initially define a function,

\[ f(q(t)) = \phi_1(q(t)) + \phi_2(q(t)) \iff f(q) = \phi_1(q) + \phi_2(q). \]

We now construct a difference quotient for such a function which is accordingly expressed as:

\[
\frac{f(q(T)) - f(q(t))}{T - t} = \frac{\phi_1(q(T)) + \phi_2(q(T)) - (\phi_1(q(t)) + \phi_2(q(t)))}{T - t}
\]
\[
\begin{aligned}
\frac{\phi_1(q(T)) - \phi_1(q(t))}{T-t} + \frac{\phi_2(q(T)) - \phi_2(q(t))}{T-t}, \quad \forall t, T \in \mathbb{R}.
\end{aligned}
\]

Furthermore, if \(\phi_1(q)\) and \(\phi_2(q)\) are differentiable functions, with respect to the real parameter, \(t\), and we also set

\[
T = t + \delta t,
\]

then clearly,

\[
\begin{aligned}
\lim_{T \to t} \left[ \frac{\phi_1(q(T)) - \phi_1(q(t))}{T-t} \right] &= \frac{d}{dt} \left[ \phi_1(q(t)) \right], \\
\lim_{T \to t} \left[ \frac{\phi_2(q(T)) - \phi_2(q(t))}{T-t} \right] &= \frac{d}{dt} \left[ \phi_2(q(t)) \right],
\end{aligned}
\]

and hence, in accordance with Theorem III - 6, it naturally follows that

\[
\begin{aligned}
\lim_{T \to t} \left[ \frac{\phi_1(q(T)) - \phi_1(q(t))}{T-t} + \frac{\phi_2(q(T)) - \phi_2(q(t))}{T-t} \right] &= \frac{d}{dt} \left[ \phi_1(q(t)) \right] + \frac{d}{dt} \left[ \phi_2(q(t)) \right] \\
&= \frac{d}{dt} \left[ \phi_1(q(t)) \right] + \frac{d}{dt} \left[ \phi_2(q(t)) \right] \\
&= \frac{d}{dt} \left[ \phi_1(q(t)) \right] + \frac{d}{dt} \left[ \phi_2(q(t)) \right].
\end{aligned}
\]

AND

\[
\lim_{T \to t} \left[ \frac{\phi_1(q(T)) - \phi_1(q(t))}{T-t} \right] = \frac{d}{dt} \left[ f(q(t)) \right],
\]

\[
\therefore \frac{d}{dt} \left[ f(q(t)) \right] = \frac{d}{dt} \left[ \phi_1(q(t)) \right] + \frac{d}{dt} \left[ \phi_2(q(t)) \right].
\]

Finally, we deduce that
\[
\left[ \frac{\partial}{\partial t} \right] \big(e^{\frac{\phi_1(q)}{2}} \phi_2(q) + \phi_2(q) \big) = \left\{ \begin{aligned}
\left( \frac{\partial}{\partial t} \left[ \phi_1(q(T)) \right] + \frac{\partial}{\partial t} \left[ \phi_2(q(t)) \right] \right) \frac{\partial}{\partial t} \left[ q(t) \right] \\
\left[ \frac{\partial}{\partial t} \left[ q(t) \right] \right]^2 \left( \frac{\partial}{\partial t} \left[ \phi_1(q(t)) \right] + \frac{\partial}{\partial t} \left[ \phi_2(q(t)) \right] \right) \frac{\partial}{\partial t} \left[ q(t) \right] \end{aligned} \right. \\
\frac{\partial}{\partial t} \left[ q(t) \right]^2 \\
\right.
\] . Q.E.D.

(ii) Let us initially define a function,

\[ f(q(t)) = \phi_1(q(t)) \phi_2(q(t)) \iff f(q) = \phi_1(q) \phi_2(q) . \]

We now construct a difference quotient for such a function, which is accordingly expressed as

\[
\frac{f(q(T)) - f(q(t))}{T - t} = \frac{\phi_1(q(T)) \phi_2(q(T)) - \phi_1(q(t)) \phi_2(q(t))}{T - t}
\]

\[
= \left[ \frac{\phi_1(q(T)) \phi_2(q(T)) - \phi_1(q(T)) \phi_2(q(t)) + \phi_1(q(t)) \phi_2(q(T)) - \phi_1(q(t)) \phi_2(q(t))}{T - t} \right]
\]

\[
= \frac{\phi_1(q(T)) \left[ \phi_2(q(T)) - \phi_2(q(t)) \right] + \left[ \phi_1(q(T)) - \phi_1(q(t)) \right] \phi_2(q(T))}{T - t}
\]

\[
= \phi_1(q(T)) \left[ \frac{\phi_2(q(T)) - \phi_2(q(t))}{T - t} \right] + \left[ \frac{\phi_1(q(T)) - \phi_1(q(t))}{T - t} \right] \phi_2(q(T))
\]

\[ \forall t, T \in \mathbb{R} . \]
Furthermore, if \( \phi_1(q) \) and \( \phi_2(q) \) are differentiable functions, with respect to the real parameter \( t \), and we also set

\[ T = t + h, \]

then clearly

\[
\lim_{T \to t} \left[ \phi_1(q(T)) \right] = \phi_1(q(t)),
\]

\[
\lim_{T \to t} \left[ \phi_2(q(T)) \right] = \phi_2(q(t)),
\]

\[
\lim_{T \to t} \left[ \frac{\phi_2(q(T)) - \phi_2(q(t))}{T - t} \right] = \frac{d}{dt} \left[ \phi_2(q(t)) \right],
\]

and hence, in accordance with Theorem \( \text{III-6} \), it naturally follows that

\[
\lim_{T \to t} \left[ \frac{f(q(T)) - f(q(t))}{T - t} \right] = \lim_{T \to t} \left[ \phi_1(q(T)) \left[ \frac{\phi_2(q(T)) - \phi_2(q(t))}{T - t} \right] + \left[ \frac{\phi_2(q(T)) - \phi_2(q(t))}{T - t} \right] \phi_2(q(t)) \right] = \phi_2(q(t)) \frac{d}{dt} \left[ f(q(t)) \right] + \frac{df}{dt} \left[ f(q(t)) \right] \phi_2(q(t))
\]

AND

\[
\lim_{T \to t} \left[ \frac{f(q(T)) - f(q(t))}{T - t} \right] = \frac{d}{dt} \left[ f(q(t)) \right],
\]
as previously indicated in part (i) of this theorem.

Finally, we deduce that

$$\left[ \frac{\partial}{\partial q} \right]_{C} (\phi_1(q), \phi_2(q)) = \left\{ \frac{\partial}{\partial C} \left[ \frac{\phi_1(q(q))}{\phi_2(q(q))} \right] \frac{\partial}{\partial q} \left[ \frac{\phi_2(q(q))}{\phi_1(q(q))} \right] \right\} \right. $$

$$\left. \left\{ \frac{\partial}{\partial q} \left[ \frac{\phi_1(q(q))}{\phi_2(q(q))} \right] \frac{\partial}{\partial q} \left[ \frac{\phi_2(q(q))}{\phi_1(q(q))} \right] \right\} \right. $$

where

$$\frac{\partial}{\partial q} \left[ f(q(q)) \right] = \phi_1(q(q)) \frac{\partial}{\partial q} \left[ \phi_2(q(q)) \right] + \phi_2(q(q)) \frac{\partial}{\partial q} \left[ \phi_1(q(q)) \right] \phi_2(q(q)). \quad \text{Q.E.D.}$$

---

(iii) The proof required for this part of the theorem is completely analogous with part (i) thereof, insofar as the functions, $\phi_1(q)$ and $\phi_2(q)$, may be merely transposed with respect to the product function, $\phi_1(q) \phi_2(q)$ and $\phi_2(q) \phi_1(q)$.

Q.E.D.

(iv) Let us initially define a function,

$$f(q(q)) = \frac{\phi_1(q(q))}{\phi_2(q(q))} \iff f(q) = \phi_1(q)/\phi_2(q).$$
\[ \phi_3(q(t)) = \left( \frac{\phi_1(q(t))}{\phi_2(q(t))} \right)^2 \]
\[ \begin{pmatrix} \phi_1(q(t)) \\ \phi_2(q(t)) \end{pmatrix} \]
\[ = \begin{pmatrix} \psi_1(q(t)) \\ \psi_2(q(t)) \end{pmatrix} \]

Hence, by virtue of the preceding parts (ii) and (iii) of the theorem, we perceive that

\[
\left[ \frac{\partial}{\partial q} \right] \left( \frac{\phi_1(q)}{\phi_2(q)} \right) = \begin{pmatrix} \frac{\partial \psi_1(q(t))}{\partial q} \frac{\partial q(t)}{\partial q} - \frac{\partial q(t)}{\partial q} \frac{\partial \psi_1(q(t))}{\partial q} & \frac{\partial \psi_2(q(t))}{\partial q} \frac{\partial q(t)}{\partial q} - \frac{\partial q(t)}{\partial q} \frac{\partial \psi_2(q(t))}{\partial q} \\
\frac{\partial \psi_1(q(t))}{\partial q} \frac{\partial q(t)}{\partial q} & \frac{\partial \psi_2(q(t))}{\partial q} \frac{\partial q(t)}{\partial q} \end{pmatrix}
\]

where the derivatives,
\[ \begin{align*}
\frac{d}{dt} \left[ \phi_1(q(t)) \right] &= \frac{d}{dt} \left[ \phi_1(q(t)) \left( \frac{\phi_2(q(t))}{|\phi_2(q(t))|^2} \right) \right] \\
&= \phi_1(q(t)) \frac{d}{dt} \left[ \frac{\phi_2(q(t))}{|\phi_2(q(t))|^2} \right] + \frac{d}{dt} \left[ \phi_1(q(t)) \right] \left( \frac{\phi_2(q(t))}{|\phi_2(q(t))|^2} \right) , \\
\frac{d}{dt} \left[ \phi_2(q(t)) \right] &= \frac{d}{dt} \left[ \left( \frac{\phi_2(q(t))}{|\phi_2(q(t))|^2} \right)^2 \phi_1(q(t)) \right] \\
&= \left( \frac{\phi_2(q(t))}{|\phi_2(q(t))|^2} \right) \frac{d}{dt} \left[ \phi_1(q(t)) \right] + \frac{d}{dt} \left[ \left( \frac{\phi_2(q(t))}{|\phi_2(q(t))|^2} \right)^2 \phi_1(q(t)) \right] .
\end{align*} \]

We now construct a difference quotient for the function, \( |\phi_2(q(T))|^2 \), which is accordingly expressed as

\[
\frac{(\phi_2(q(T)) - \phi_2(q(t)))}{|\phi_2(q(T))|^2 (T-t)} = \frac{|\phi_2(q(T))|^2 \phi_2(q(T)) - |\phi_2(q(T))|^2 \phi_2(q(t))}{|\phi_2(q(T))|^2 |\phi_2(q(t))|^2 (T-t)}
\]

\[
= \left[ \frac{|\phi_2(q(T))|^2 \phi_2(q(T)) - |\phi_2(q(T))|^2 \phi_2(q(t))}{|\phi_2(q(T))|^2 |\phi_2(q(t))|^2 (T-t)} \right]
\]

\[
= \frac{1}{|\phi_2(q(T))|^2 (T-t)} \left[ |\phi_2(q(T))|^2 \left( \frac{\phi_2(q(T)) - \phi_2(q(t))}{T-t} \right) + \left( |\phi_2(q(T))|^2 - |\phi_2(q(T))|^2 \right) \phi_2(q(t)) \right]
\]

\[
= \frac{1}{|\phi_2(q(T))|^2 (T-t)} \left[ \phi_2(q(t)) \left( \frac{\phi_2(q(T)) - \phi_2(q(t))}{T-t} \right) + \left( |\phi_2(q(T))|^2 - |\phi_2(q(T))|^2 \right) \phi_2(q(t)) \right].
\]

\[
\forall t, T \in \mathbb{R}.
\]

Furthermore, if \( \phi_1(q) \) and \( \phi_2(q) \) are differentiable functions, with respect to
the real parameter, \( t \), and we also set

\[
T = x + k,
\]

then clearly,

\[
\lim_{t \to t} \left[ \phi_2(q(T)) \right] = \phi_2(q(t)) \quad \Rightarrow \quad \left\{ \begin{array}{l}
\lim_{T \to t} \left[ \left| \phi_2(q(T)) \right| \right] = \left| \phi_2(q(t)) \right| \\
\lim_{T \to t} \left[ \overline{\phi_2(q(T))} \right] = \overline{\phi_2(q(t))}^*,
\end{array} \right.
\]

\[
\lim_{t \to t} \left[ \phi_2(q(t)) \right] = \phi_2(q(t)) \quad \Rightarrow \quad \left\{ \begin{array}{l}
\lim_{T \to t} \left[ \left| \phi_2(q(T)) \right| \right] = \left| \phi_2(q(t)) \right| \\
\lim_{T \to t} \left[ \overline{\phi_2(q(T))} \right] = \overline{\phi_2(q(t))},
\end{array} \right.
\]

\[
\lim_{t \to t} \left[ \frac{\phi_2(q(T)) - \phi_2(q(t))}{T - t} \right] = \frac{d}{dt} \left[ \phi_2(q(t)) \right],
\]

\[
\lim_{t \to t} \left[ \frac{|\phi_2(q(T))|^2 - |\phi_2(q(t))|^2}{T - t} \right] = \frac{d}{dt} \left[ |\phi_2(q(t))|^2 \right],
\]

\[
\lim_{t \to t} \left[ \frac{\left( \frac{\phi_2(q(T))}{|\phi_2(q(T))|} - \frac{\phi_2(q(t))}{|\phi_2(q(t))|} \right)}{(T - t)} \right] = \frac{d}{dt} \left[ \frac{\phi_2(q(t))}{|\phi_2(q(t))|^2} \right],
\]

and hence, in accordance with Theorem III-6, it naturally follows that

\[
\lim_{t \to t} \left[ \frac{\left( \frac{\phi_2(q(T))}{|\phi_2(q(T))|} - \frac{\phi_2(q(t))}{|\phi_2(q(t))|} \right)}{(T - t)} \right]
\]
\[\begin{align*}
&= \frac{|\psi_2(q(t))|^2 \frac{\text{d}^2}{\text{d}t^2} [\text{Im} [\phi_2(q(t))]]^2 - \frac{\text{d}}{\text{d}t} [\text{Im} [\phi_2(q(t))]]^2}{|\phi_2(q(t))|^4} \\
\end{align*}\]

\[\begin{align*}
&\Rightarrow \frac{\text{d}}{\text{d}t} \left[ \frac{\psi_2(q(t))}{|\phi_2(q(t))|^2} \right] = \frac{|\psi_2(q(t))|^2 \frac{\text{d}^2}{\text{d}t^2} [\text{Im} [\phi_2(q(t))]]^2 - \frac{\text{d}}{\text{d}t} [\text{Im} [\phi_2(q(t))]]^2}{|\phi_2(q(t))|^4} \\
\end{align*}\]

\[\text{†} \quad \text{A formal justification for this statement is provided in Appendix A.3.}\]

Finally, though with the appropriate substitutions, we deduce that

\[\begin{align*}
\left[ \frac{\text{d}}{\text{d}t} \right] \left[ \frac{\phi_2(q(t))}{\phi_2(q(t))} \right] &= \left\{ \begin{array}{c}
\frac{\text{d}}{\text{d}t} [\psi_2(q(t))] \frac{\text{d}^2}{\text{d}t^2} [\text{Im} [\phi_2(q(t))]]^2 \\
\frac{\text{d}}{\text{d}t} [\phi_2(q(t))] \frac{\text{d}^2}{\text{d}t^2} [\phi_2(q(t))] \\
\frac{\text{d}}{\text{d}t} [\phi_2(q(t))] \frac{\text{d}^2}{\text{d}t^2} [\text{Im} [\phi_2(q(t))]]^2 \\
\frac{\text{d}}{\text{d}t} [\phi_2(q(t))] \frac{\text{d}^2}{\text{d}t^2} [\phi_2(q(t))] \\
\end{array} \right\} \\
\end{align*}\]

where the derivatives,

\[\begin{align*}
\frac{\text{d}}{\text{d}t} [\psi_2(q(t))] &= \phi_2(q(t)) \frac{\text{d}}{\text{d}t} \left[ \frac{\text{d}^2}{\text{d}t^2} [\text{Im} [\phi_2(q(t))]]^2 \right] + \frac{\text{d}}{\text{d}t} \left[ \phi_2(q(t)) \right] \left( \frac{\text{d}^2}{\text{d}t^2} [\text{Im} [\phi_2(q(t))]]^2 \right) \\
\frac{\text{d}}{\text{d}t} [\phi_2(q(t))] &= \left( \frac{\text{d}^2}{\text{d}t^2} [\text{Im} [\phi_2(q(t))]]^2 \right) \frac{\text{d}}{\text{d}t} \left[ \phi_2(q(t)) \right] + \frac{\text{d}}{\text{d}t} \left[ \frac{\text{d}^2}{\text{d}t^2} [\phi_2(q(t))]^2 \right] \phi_2(q(t)) \\
\end{align*}\]
\[
\frac{\phi_2(q(t))}{|\phi_2(q(t))|^2} = \frac{1}{|\phi_2(q(t))|^4} \left( \frac{|\phi_2(q(t))|^2}{|\phi_2(q(t))|^2} - \frac{\phi_1(q(t)) \phi_2(q(t))}{|\phi_2(q(t))|^2} \right),
\]

and the function,

\[\phi_2(q(t)) \neq 0, \quad \forall t \in (a, b). \quad \text{Q.E.D.}\]

From the remarks made in the preceding Parts 1 and 2 of this section, we

\[ q(t) = \begin{cases} x(t) + iy(t), & (4-24), \\ x(t) - iy(t), \\ x(t) - iy(t), \\ x(t) + iy(t) \end{cases} \]

then the differentiation formulas for the sums, products and quotients of the

functions, \( \phi_1(q) \) and \( \phi_2(q) \), restricted to such arcs, are reduced to

\[ \frac{\partial}{\partial t} \left[ \frac{\phi_1(q(t)) + \phi_2(q(t))}{\phi_1(q(t))} \right] = \frac{\frac{\partial}{\partial t} \phi_1(q(t)) + \frac{\partial}{\partial t} \phi_2(q(t))}{\phi_1(q(t))} \]

\[ = \frac{\left( \frac{\partial}{\partial t} \phi_1(q(t)) + \frac{\partial}{\partial t} \phi_2(q(t)) \right) \phi_1(q(t))}{\phi_1(q(t))^2} \]

\[ = \frac{\phi_1(q(t)) \left( \frac{\partial}{\partial t} \phi_1(q(t)) + \frac{\partial}{\partial t} \phi_2(q(t)) \right)}{\phi_1(q(t))^2} \]
\[ \begin{align*}
\left[ \frac{\partial}{\partial q} \right]_C (\phi_1(q), \phi_2(q)) &= \left[ \frac{\partial}{\partial q} \right]_C (\phi_1(q), \phi_1(q)) \\
&= \frac{\partial}{\partial q} \left[ \phi_1(q(t)), \phi_1(q(t)) \right] \\
&= \frac{\partial}{\partial q(t)} \left[ \phi_1(q(t)), \phi_1(q(t)) \right] \frac{\partial q(t)}{\partial q(t)} \\
&= \frac{\partial}{\partial q(t)} \left[ \phi_1(q(t)), \phi_2(q(t)) \right] \\
&= \frac{\partial}{\partial q(t)} \left[ \phi_1(q(t)), \phi_1(q(t)) \right] \\
\text{where } \frac{\partial}{\partial q(t)} \left[ \phi_1(q(t)), \phi_2(q(t)) \right] &= \frac{\partial}{\partial q(t)} \left[ \phi_2(q(t)), \phi_1(q(t)) \right] \\
&= \frac{\partial}{\partial q(t)} \left[ \phi_1(q(t)), \phi_1(q(t)) \right] \\
&= \phi_1(q(t)) \frac{\partial}{\partial q(t)} \left[ \phi_2(q(t)), \phi_1(q(t)) \right] \\
&= \phi_1(q(t)) \frac{\partial}{\partial q(t)} \left[ \phi_2(q(t)), \phi_1(q(t)) \right],
\end{align*} \]

such that

\[ \phi_1(q(t)) \frac{\partial}{\partial q(t)} \left[ \phi_2(q(t)), \phi_1(q(t)) \right] = \frac{\partial}{\partial q(t)} \left[ \phi_2(q(t)) \right] \phi_1(q(t)), \]

\[ \frac{\partial}{\partial q(t)} \left[ \phi_1(q(t)), \phi_2(q(t)) \right] = \phi_2(q(t)) \frac{\partial}{\partial q(t)} \left[ \phi_2(q(t)) \right]. \]
\[
\frac{\frac{d}{dt} \left[ \phi_1(g(t)) \right]}{\phi_2(g(t))} = \frac{\frac{d}{dt} \left[ \phi_1(g(t)) \phi_2(g(t)) \right]}{\phi_2(g(t))},
\]

where the function,

\[
\frac{\phi_2(g(t))}{\phi_2(g(t))} = \frac{\phi_1(g(t)) \phi_2(g(t))}{\phi_2(g(t))} = \frac{\phi_2(g(t)) \phi_1(g(t))}{\phi_2(g(t))},
\]

and the derivatives,

\[
\frac{d}{dt} \left[ \frac{\phi_1(g(t))}{\phi_2(g(t))} \right] = \phi_1(g(t)) \frac{d}{dt} \left[ \frac{\phi_2(g(t))}{\phi_2(g(t))} \right] + \frac{d}{dt} \left[ \phi_1(g(t)) \right] \frac{\phi_2(g(t))}{\phi_2(g(t))},
\]

such that

\[
\phi_1(g(t)) \frac{d}{dt} \left[ \frac{\phi_2(g(t))}{\phi_2(g(t))} \right] = \frac{d}{dt} \left[ \phi_1(g(t)) \phi_2(g(t)) \right],
\]

\[
\frac{d}{dt} \left[ \phi_1(g(t)) \right] \frac{\phi_2(g(t))}{\phi_2(g(t))} = \left( \frac{\phi_2(g(t))}{\phi_2(g(t))} \right)^2 \frac{d}{dt} \left[ \phi_1(g(t)) \right],
\]

\[
\left| \phi_2(g(t)) \right|^2 \frac{d}{dt} \left[ \phi_2(g(t)) \right] = \frac{d}{dt} \left[ \phi_2(g(t)) \right] \left| \phi_2(g(t)) \right|^2,
\]

\[
\frac{d}{dt} \left[ \phi_2(g(t)) \right] \phi_2(g(t)) = \phi_2(g(t)) \frac{d}{dt} \left[ \phi_2(g(t)) \right] \left| \phi_2(g(t)) \right|^2 \quad (4-25),
\]

all of which are in accordance with Eq. (4-14).

Furthermore, if the quaternion hypercomplex functions, \( f(x+iy) \), \( f(x+jy) \), \( f(x+kx) \), \( f(x+iy) \), are also analytic in terms of Eqs. (4-17), (4-18), (4-19) and (4-20), we similarly deduce that the above-mentioned formulas...
may now be rewritten respectively as

\( \left[ \frac{d}{dq} \right] \phi_1(q) + \phi_2(q) \) = \( \frac{d}{dq} \phi_1(q) + \phi_2(q) \)

= \( \frac{d}{dq} \phi_1(q) \) + \( \frac{d}{dq} \phi_2(q) \),

\( \left[ \frac{d}{dq} \right] \phi_1(q) \phi_2(q) \) = \( \phi_1(q) \phi_2(q) \)

= \( \phi_1(q) \phi_2(q) \) + \( \phi_1(q) \phi_2(q) \phi_2(q) \),

\( \left[ \frac{d}{dq} \right] \phi_1(q) \phi_2(q) \) = \( \phi_1(q) \phi_2(q) \)

= \( \frac{\phi_1(q) \phi_2(q) \phi_2(q) - \phi_1(q) \phi_2(q)}{\phi_2(q)} \)  

(4-26),

being completely analogous with their counterparts from both real and complex variable analysis.

4. **Differentiation of the Composites of Quaternion Hyper-complex Functions restricted to Smooth arcs embedded in q-space.**

From the calculus of real variable functions, we recall that any composite function, \((f \circ g)(x)\), which is differentiable in \(x\), will invariably give rise to the derivative,
\[
\frac{d}{dx} [f(g(x))] = \frac{d}{dx} [f(g)]
\]
\[= f'(g(x)) \cdot g'(x) \quad (4-27).
\]

Similarly, we obtain the differential formula,
\[
\frac{d}{dy} [f(g(x))] = \frac{d}{dy} [f(g)]
\]
\[= f'(g) \cdot g'(y) \quad (4-28),
\]

thus corresponding to a composite complex function, \((f \circ g)(y) \in C\), which is likewise differentiable over some arbitrary region of the \(y\)-plane.

However, leaving in mind the criteria previously enumerated in Parts 1 and 2 of this section, we immediately recognize that the formula,
\[
\frac{d}{dy} [f(g(x))] = \frac{d}{dy} [f(g)]
\]
\[= f'(g(y)) \cdot g'(y) \quad (4-29),
\]
cannot, as a general rule, suffice to be the first derivative of every quaternion hypercomplex function, \((f \circ g)(y)\). Consequently, we must once again invoke the notion of a smooth arc, \(C\), embedded in \(y\)-space, before we proceed any further in deriving an appropriate differential formula which clearly pertains to this particular class of functions. Indeed, our first step towards achieving this objective is to formally define the first derivative of any quaternion hypercomplex composite function, \((f \circ g)(y)\), thus constrained to such arcs, as follows:
Definition DIV - 7

Let there exist a quaternion hypercomplex composite function, \((\phi; \phi_2)(q)\), which is restricted to a smooth arc, \(C\), thus defined by the equation,

\[ q(t) = x(t) + iy(t) + jz(t) + kz(t), \forall t \in [a, b]. \]

Clearly, by virtue of Theorem DIV - 1, if \((\phi; \phi_2)(q)\) is differentiable in \(t\), then the following differential formula pertaining to all such quaternion hypercomplex composite functions shall likewise apply, \(\forall t \in (a, b):\)

\[
\left[ \frac{\partial}{\partial t} \right] \hat{c} \left[ (\phi_1 \phi_2)(q(t)) \right] = \frac{\hat{c} \left[ (\phi_1 \phi_2)(q(t)) \right] - \hat{c} \left[ (\phi_2)(q(t)) \right]}{\left[ \hat{c} [q(t)] \right]^2} \cdot \left( \frac{\hat{c} [q(t)]}{\hat{c} [q(t)]} \right) \cdot \hat{c} \left[ \frac{\partial}{\partial t} \left[ (\phi_1 \phi_2)(q(t)) \right] \right]
\]

Although the existence of the above stated definition is certainly a necessary condition for the purposes of our analysis, it is by no means sufficient, since we have not explicitly stated a concise method for evaluating the parametric first derivative, \(\frac{\partial}{\partial t} \left[ (\phi_1 \phi_2)(q(t)) \right]\). Quite conceivably, one might even be tempted to regard Definition DIV - 7 as being little more than a trivial extension of Theorem DIV - 1, in which case we will need to examine the properties of \(\frac{\partial}{\partial t} \left[ (\phi_1 \phi_2)(q(t)) \right]\) more closely in order to overcome the apparent shortcomings of this definition.

Our chosen strategy to that effect thus consists of
ii. Resolving the function, $(\phi, \theta)(q)$, into its constituent real and imaginary parts.

AND

(iii) Differentiating each of these components, in turn, with respect to the real parameter, $t$.

as the reader will no doubt ascertain from our next theorem, which we shall henceforth enunciate and verify accordingly:

**Theorem IV-3**

Let there exist a quaternion hypercomplex composite function, $(\phi, \theta)(q)$, which is restricted to a smooth arc, $C$, thus defined by the equation,

$$q(t) = x(t) + iy(t) + jz(t) + k\omega(t), \quad \forall t \in [a, b].$$

If $(\phi, \theta)(q)$ is differentiable in $t$, such that the functions:

$$\phi_1(q) = \sigma_1(x, y, z) + i\tau_1(x, y, z) + j\sigma_2(x, y, z) + k\tau_2(x, y, z),$$

$$\phi_2(q) = \sigma_1(x, y, z) + i\tau_1(x, y, z) + j\sigma_2(x, y, z) + k\tau_2(x, y, z),$$

**************
\[
\begin{align*}
= L_1(\sigma(x, y, \hat{\gamma}), T_1(x, y, \hat{\gamma}), \sigma(x, y, \hat{\gamma}), T_2(x, y, \hat{\gamma})) + \\
i \beta_1(\sigma(x, y, \hat{\gamma}), T_1(x, y, \hat{\gamma}), \sigma(x, y, \hat{\gamma}), T_1(x, y, \hat{\gamma})) + \\
j \epsilon_1(\sigma(x, y, \hat{\gamma}), T_1(x, y, \hat{\gamma}), \sigma(x, y, \hat{\gamma}), T_2(x, y, \hat{\gamma})) + \\
k \epsilon_2(\sigma(x, y, \hat{\gamma}), T_1(x, y, \hat{\gamma}), \sigma(x, y, \hat{\gamma}), T_2(x, y, \hat{\gamma})),
\end{align*}
\]

are also differentiable in it, then for the composite function \( (g \circ \phi)(x) \), we obtain the corresponding differential formula, which we accordingly express as

\[
\frac{d}{dx} [(g \circ \phi)(x(t))] = \frac{d}{dx} [A_1(x(t), y(t), \hat{\gamma}(t))] + i \phi_{g(t)} B_1(x(t), y(t), \hat{\gamma}(t)) + j \phi_{g(t)} A_2(x(t), y(t), \hat{\gamma}(t)) + k \phi_{g(t)} B_2(x(t), y(t), \hat{\gamma}(t)),
\]

\forall t \in (a, b),

where the functions

\[
\begin{align*}
A_1(x(t), y(t), \hat{\gamma}(t)) &= A_1(x, y, \hat{\gamma}) \\
&= \sigma_1(\sigma(x, y, \hat{\gamma}), T_1(x, y, \hat{\gamma}), \sigma(x, y, \hat{\gamma}), T_2(x, y, \hat{\gamma})),
\end{align*}
\]

\[
\begin{align*}
B_1(x(t), y(t), \hat{\gamma}(t)) &= B_1(x, y, \hat{\gamma}) \\
&= \beta_1(\sigma(x, y, \hat{\gamma}), T_1(x, y, \hat{\gamma}), \sigma(x, y, \hat{\gamma}), T_1(x, y, \hat{\gamma})),
\end{align*}
\]

\[
\begin{align*}
A_2(x(t), y(t), \hat{\gamma}(t)) &= A_2(x, y, \hat{\gamma}) \\
&= \sigma_2(\sigma(x, y, \hat{\gamma}), T_2(x, y, \hat{\gamma}), \sigma(x, y, \hat{\gamma}), T_2(x, y, \hat{\gamma})),
\end{align*}
\]

\[
\begin{align*}
B_2(x(t), y(t), \hat{\gamma}(t)) &= B_2(x, y, \hat{\gamma}) \\
&= \beta_2(\sigma(x, y, \hat{\gamma}), T_2(x, y, \hat{\gamma}), \sigma(x, y, \hat{\gamma}), T_2(x, y, \hat{\gamma})).
\end{align*}
\]
and hence the derivatives,

\[ \frac{\partial}{\partial \theta_k} \left[ A_j (x(t), y(t), z(t), j(t)) \right] = \sum_{i=1}^{4} \frac{\partial}{\partial \theta_k} (\alpha_i (x_i, x_2, x_3, x_4)) \frac{\partial x_i}{\partial \theta_k} \],

\[ \frac{\partial}{\partial \theta_k} \left[ B_j (x(t), y(t), z(t), j(t)) \right] = \sum_{i=1}^{4} \frac{\partial}{\partial \theta_k} (\beta_i (x_i, x_2, x_3, x_4)) \frac{\partial x_i}{\partial \theta_k} \],

\[ \frac{\partial}{\partial \theta_k} \left[ A_j (x(t), y(t), z(t), j(t)) \right] = \sum_{i=1}^{4} \frac{\partial}{\partial \theta_k} (\sigma_i (x_i, x_2, x_3, x_4)) \frac{\partial x_i}{\partial \theta_k} \],

\[ \frac{\partial}{\partial \theta_k} \left[ B_j (x(t), y(t), z(t), j(t)) \right] = \sum_{i=1}^{4} \frac{\partial}{\partial \theta_k} (\tau_i (x_i, x_2, x_3, x_4)) \frac{\partial x_i}{\partial \theta_k} \].

\[ \frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ \sigma_1 (x, y, z, j) \right] = \frac{\partial}{\partial \theta} (\sigma_1 (x, y, z, j)) \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial \theta} (\sigma_2 (x, y, z, j)) \frac{\partial y}{\partial \theta} + \frac{\partial}{\partial \theta} (\sigma_3 (x, y, z, j)) \frac{\partial z}{\partial \theta} + \frac{\partial}{\partial \theta} (\sigma_4 (x, y, z, j)) \frac{\partial j}{\partial \theta} \],

\[ \frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ \tau_1 (x, y, z, j) \right] = \frac{\partial}{\partial \theta} (\tau_1 (x, y, z, j)) \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial \theta} (\tau_2 (x, y, z, j)) \frac{\partial y}{\partial \theta} + \frac{\partial}{\partial \theta} (\tau_3 (x, y, z, j)) \frac{\partial z}{\partial \theta} + \frac{\partial}{\partial \theta} (\tau_4 (x, y, z, j)) \frac{\partial j}{\partial \theta} \],

\[ \frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ \sigma_2 (x, y, z, j) \right] = \frac{\partial}{\partial \theta} (\sigma_2 (x, y, z, j)) \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial \theta} (\sigma_3 (x, y, z, j)) \frac{\partial y}{\partial \theta} + \frac{\partial}{\partial \theta} (\sigma_4 (x, y, z, j)) \frac{\partial z}{\partial \theta} + \frac{\partial}{\partial \theta} (\sigma_4 (x, y, z, j)) \frac{\partial j}{\partial \theta} \],

\[ \frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ \tau_2 (x, y, z, j) \right] = \frac{\partial}{\partial \theta} (\tau_2 (x, y, z, j)) \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial \theta} (\tau_3 (x, y, z, j)) \frac{\partial y}{\partial \theta} + \frac{\partial}{\partial \theta} (\tau_4 (x, y, z, j)) \frac{\partial z}{\partial \theta} + \frac{\partial}{\partial \theta} (\tau_4 (x, y, z, j)) \frac{\partial j}{\partial \theta} \].
\[ \frac{\partial}{\partial x} (r_2(x, y, z)) \frac{\partial}{\partial y} + \frac{\partial}{\partial y} (r_2(x, y, z)) \frac{\partial}{\partial x} + \frac{\partial}{\partial z} (r_2(x, y, z)) \frac{\partial}{\partial z} \]

\[
\begin{align*}
\text{PROOF:} \quad &- \\
\text{We commence our proof by initially defining a composite function,} \\
\end{align*}
\]

\[
(\varphi \circ \varphi)(t) = A_t(x_t, y_t, z_t) + iB_t(x_t, y_t, z_t) + jA_t(x_t, y_t, z_t) + kB_t(x_t, y_t, z_t).
\]

Moreover, by restricting this function to a smooth arc, \( c \), thus defined by the equation,

\[
g(t) = x(t) + iy(t) + j\alpha(t) + k\beta(t), \quad \forall t \in [a, b],
\]

we further obtain

\[
(\varphi \circ \varphi)(g(t)) = A_t(x(t), y(t), z(t), \alpha(t)) + iB_t(x(t), y(t), z(t), \alpha(t)) + jA_t(x(t), y(t), z(t), \alpha(t)) + kB_t(x(t), y(t), z(t), \alpha(t)).
\]

whereupon, in accordance with Theorem IV-1, it subsequently follows that, if \((\varphi \circ \varphi)(g(t))\) is differentiable in \( t \), then

\[
\frac{d}{dt} [(\varphi \circ \varphi)(g(t))] = \frac{d}{dt} [A_t(x(t), y(t), z(t), \alpha(t))] + i\frac{d}{dt} [B_t(x(t), y(t), z(t), \alpha(t))] + j\frac{d}{dt} [A_t(x(t), y(t), z(t), \alpha(t))] + k\frac{d}{dt} [B_t(x(t), y(t), z(t), \alpha(t))], \quad \forall t \in (a, b).
\]
But since we had previously stated in the preambles to this proof that

\[ A_1(x,y,z) = \alpha_1(\alpha_1(x,y,z), \alpha_2(x,y,z), \alpha_3(x,y,z), \alpha_4(x,y,z)) \, , \]
\[ B_1(x,y,z) = \beta_1(\beta_1(x,y,z), \beta_2(x,y,z), \beta_3(x,y,z), \beta_4(x,y,z)) \, , \]
\[ A_2(x,y,z) = \alpha_2(\alpha_1(x,y,z), \alpha_2(x,y,z), \alpha_3(x,y,z), \alpha_4(x,y,z)) \, , \]
\[ B_2(x,y,z) = \beta_2(\beta_1(x,y,z), \beta_2(x,y,z), \beta_3(x,y,z), \beta_4(x,y,z)) \, , \]

we are now in a position to evaluate the derivatives \( \frac{\partial A_i}{\partial x} \), \( \frac{\partial B_i}{\partial x} \), \( \frac{\partial A_i}{\partial y} \), \( \frac{\partial B_i}{\partial y} \), in terms of the first-mentioned component functions.

In order to do this, we shall henceforth utilize a particular differentiation

---

formulas from the calculus of functions of several real variables (cf. Appendix A4) which thus pertains to the derivatives of composite functions of four such variables and is accordingly expressed as

\[ \frac{\partial F}{\partial x} = \frac{\partial F}{\partial x_1} \cdot \frac{\partial x_1}{\partial x} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial x} + \frac{\partial F}{\partial x_3} \cdot \frac{\partial x_3}{\partial x} + \frac{\partial F}{\partial x_4} \cdot \frac{\partial x_4}{\partial x} \, , \]

where \( F = F(x_1, x_2, x_3, x_4) \) and \( x_1, x_2, x_3, x_4 \) are all differentiable functions in \( t \).

Finally, by setting

\[ X_1 = \alpha_1(x,y,z) \, , \quad x = x(t) \, , \]
\[ X_2 = \beta_1(x,y,z) \, , \quad y = y(t) \, , \]

---
Furthermore, we deduce, in an analogous manner to eq. (i) above, that

\[
\frac{\partial}{\partial y} [x, y, \hat{y}(t)] = \frac{\partial}{\partial y} [A, (x(t), y(t), \hat{y}(t))]
\]

\[
= \frac{\partial}{\partial y} (\alpha_1(x, x_2, x_3, x_4)) \frac{\partial x_1}{\partial y} + \frac{\partial}{\partial y} (\alpha_2(x, x_2, x_3, x_4)) \frac{\partial x_2}{\partial y} + \frac{\partial}{\partial y} (\alpha_3(x, x_2, x_3, x_4)) \frac{\partial x_3}{\partial y} + \frac{\partial}{\partial y} (\alpha_4(x, x_2, x_3, x_4)) \frac{\partial x_4}{\partial y}
\]

\[
= \sum_{k=1}^{4} \frac{\partial}{\partial y} (\alpha_k(x, x_2, x_3, x_4)) \frac{\partial x_k}{\partial y},
\]

\[
\frac{\partial}{\partial y} [B, (x, y, \hat{y}(t))] = \frac{\partial}{\partial y} [B, (x(t), y(t), \hat{y}(t))]
\]

\[
= \frac{\partial}{\partial y} (\beta_1(x, x_2, x_3, x_4)) \frac{\partial x_1}{\partial y} + \frac{\partial}{\partial y} (\beta_2(x, x_2, x_3, x_4)) \frac{\partial x_2}{\partial y} + \frac{\partial}{\partial y} (\beta_3(x, x_2, x_3, x_4)) \frac{\partial x_3}{\partial y} + \frac{\partial}{\partial y} (\beta_4(x, x_2, x_3, x_4)) \frac{\partial x_4}{\partial y}
\]

\[
= \sum_{k=1}^{4} \frac{\partial}{\partial y} (\beta_k(x, x_2, x_3, x_4)) \frac{\partial x_k}{\partial y},
\]

\[
\frac{\partial}{\partial y} [A, (x, y, \hat{y}(t))] = \frac{\partial}{\partial y} [A, (x(t), y(t), \hat{y}(t))]
\]

\[
= \frac{\partial}{\partial y} (\alpha_1(x, x_2, x_3, x_4)) \frac{\partial x_1}{\partial y} + \frac{\partial}{\partial y} (\alpha_2(x, x_2, x_3, x_4)) \frac{\partial x_2}{\partial y} + \frac{\partial}{\partial y} (\alpha_3(x, x_2, x_3, x_4)) \frac{\partial x_3}{\partial y} + \frac{\partial}{\partial y} (\alpha_4(x, x_2, x_3, x_4)) \frac{\partial x_4}{\partial y}
\]

\[
= \sum_{k=1}^{4} \frac{\partial}{\partial y} (\alpha_k(x, x_2, x_3, x_4)) \frac{\partial x_k}{\partial y},
\]

\[
\frac{\partial}{\partial y} [B, (x, y, \hat{y}(t))] = \frac{\partial}{\partial y} [B, (x(t), y(t), \hat{y}(t))]
\]
\[
\frac{\partial}{\partial x} \left( \beta_2(x_1, x_2, x_3, x_4) \right) = \frac{\partial}{\partial x} \left( \beta_2(x_1, x_2, x_3, x_4) \right) + \frac{\partial}{\partial x} \left( \beta_2(x_1, x_2, x_3, x_4) \right) + \frac{\partial}{\partial x} \left( \beta_2(x_1, x_2, x_3, x_4) \right) + \frac{\partial}{\partial x} \left( \beta_2(x_1, x_2, x_3, x_4) \right)
\]
\[
= \sum_{k=1}^{4} \frac{\partial}{\partial x} \left( \beta_2(x_1, x_2, x_3, x_4) \right)
\]
and similarly,
\[
\frac{\partial}{\partial x} \left[ \sigma_1(x, y, z, j) \right]
= \frac{\partial}{\partial x} \left[ \sigma_1(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \sigma_1(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \sigma_1(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \sigma_1(x, y, z, j) \right]
\]
\[
\frac{\partial}{\partial x} \left[ \tau_1(x, y, z, j) \right]
= \frac{\partial}{\partial x} \left[ \tau_1(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \tau_1(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \tau_1(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \tau_1(x, y, z, j) \right]
\]
\[
\frac{\partial}{\partial x} \left[ \sigma_2(x, y, z, j) \right]
= \frac{\partial}{\partial x} \left[ \sigma_2(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \sigma_2(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \sigma_2(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \sigma_2(x, y, z, j) \right]
\]
\[
\frac{\partial}{\partial x} \left[ \tau_2(x, y, z, j) \right]
= \frac{\partial}{\partial x} \left[ \tau_2(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \tau_2(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \tau_2(x, y, z, j) \right] + \frac{\partial}{\partial x} \left[ \tau_2(x, y, z, j) \right]
\]
Q.E.D.
The inclusion of the last theorem as an adjunct to Definition DIV-7 therefore not only serves to enhance the latter but also, more importantly, provides us with a rigorous method of evaluating any parametric derivative, \( \frac{\partial}{\partial t} [\Phi(\Phi_0 \Phi_2)(q(t))] \), in terms of its constituent real and imaginary parts.

In finalizing our analysis of the properties of the first derivatives of quaternion hypercomplex composite functions, \((\Phi_0 \Phi_2)(q)\), we perceive that, in the light of what was discussed in the preceding Parts 1 and 2 of this section, if a smooth arc \( C \), is alternatively expressed as

\[
q(t) = \begin{cases} 
\dot{x}(t) + i\dot{y}(t), \\
\dot{x}(t) + j\dot{z}(t), \\
\dot{x}(t) + k\dot{w}(t),
\end{cases}
\quad (4-30),
\]

then the differentiation formula, specified in Definition DIV-7, is subsequently reduced to

\[
\frac{d}{dt} \left[ \frac{\partial}{\partial t} [\Phi(\Phi_0 \Phi_2)(q(t))] \right] = \frac{\frac{d}{dt} [\Phi(\Phi_0 \Phi_2)(q(t))]\overline{\frac{d}{dt} [q(t)]}}{\left| \frac{d}{dt} [q(t)] \right|^2}
\]

\[
= \frac{\frac{d}{dt} [q(t)] \cdot \overline{\frac{d}{dt} [\Phi(\Phi_0 \Phi_2)(q(t))]} \cdot \overline{\left[ \frac{d}{dt} [q(t)] \right]}}{\left| \frac{d}{dt} [q(t)] \right|^2}
\quad (4-31).
\]

Furthermore, if the quaternion hypercomplex composite functions, \((\Phi_0 \Phi_2)(\dot{x}(t) + i\dot{y}(t)), (\Phi_0 \Phi_2)(\dot{x}(t) + j\dot{z}(t)), (\Phi_0 \Phi_2)(\dot{x}(t) + k\dot{w}(t))\), are also analytic in terms of Eqs. (4-17), (4-18), (4-19) and (4-20), we immediately recognize that Eq. (4-31) may now be rewritten as
\[
[\dot{\phi}_2]_{\psi}(\phi_1 \circ \phi_2)(q)) = \frac{\partial}{\partial q} \left[ (\phi_1 \circ \phi_2)(q) \right] \\
= \frac{\partial}{\partial q} \left[ \phi_1(\phi_2(q)) \right] \\
= \phi_1'(\phi_2(q)) \cdot \phi_2'(q) \quad (4-32),
\]

being completely analogous with its more familiar counterparts from both

real and complex variable analysis.

5. **The Second Derivative of Quaternion Hypercomplex Functions restricted to Smooth Arcs embedded in q-space.**

The concept of a second derivative, needless to say, is well understood with respect to the analysis of both real and complex variable functions. We are naturally encouraged to extend this same notion into the realm of quaternion hypercomplex functions, notwithstanding any restrictions such as we previously encountered via the intermediary of a smooth arc, C, embedded in q-space.

Indeed, to facilitate our discussion along these lines, we shall firstly enunciate the following definition:
Definition DIV-8

Let there exist a quaternion-hypercomplex function, \( f(t) \), which is restricted to a smooth arc, \( C_2 \) thus defined by the equation,

\[
q(t) = x(t) + iy(t) + jz(t) + kw(t), \quad \forall t \in [a, b].
\]

Now, if \( f(t) \) is differentiable in \( t \), we may accordingly define the second derivative thereof by the formula,

\[
\left[ \frac{d^2}{dt^2} \right]_c f(t) = \left[ \frac{d}{dt} \right]_c \left( \left[ \frac{d}{dt} \right]_c f(t) \right)
\]

\[
= \left( \left[ \frac{d}{dt} \right]_c \left[ \frac{d}{dt} \right]_c f(t) \right)
\]

\[
= \left[ \frac{d}{dt} \right]_c \left( \left[ \frac{d}{dt} \right]_c f(t) \right), \quad \forall t \in (a, b).
\]

We now take the opportunity to briefly comment on our intended usage of the notation, \( \left[ \frac{d^2}{dt^2} \right]_c \). From the context of Definition DIV-4, the reader may recall that we firstly introduced the notation \( \left[ \frac{d}{dt} \right]_c \), which we then denoted as the first order differential operator restricted to an arc, \( C_1 \), embedded in \( q \)-space. It seems only logical that we should accordingly denote \( \left[ \frac{d^2}{dt^2} \right]_c \) as the second order differential operator restricted to an arc, \( C_2 \), embedded in \( q \)-space, thus having the property

\[
\left[ \frac{d^2}{dt^2} \right]_c = \left[ \frac{d}{dt} \right]_c \left[ \frac{d}{dt} \right]_c
\]
as prescribed above. Apart from yielding a formula which incorporates this particular property, Definition DIV-8 does not, on the other hand, provide a concise method for evaluating any second derivative, \[ \frac{d^2}{dq^2} (f(q)), \]
by means of manipulating the differential operator, \[ \frac{d}{dq}, \] and therefore requires further elucidation. Our next theorem, however, will succinctly fulfill this additional requirement.

**Theorem IV-4**

Let there exist a quaternion hypersurface function, \( f(q) \), which is restricted to a smooth arc, \( C \), thus defined by the equation,

\[
g(t) = x(t) + i y(t) + j z(t) + k w(t), \quad \forall t \in [a, b].
\]

Hence it may be shown that, if \( f(q) \) is differentiable in \( t \), \( \forall t \in (a, b) \), then the following differentiation formula for the second derivative thereof is likewise valid, \( \forall t \in (a, b) \):

\[
\frac{d^2}{dq^2} \left( \frac{d}{dq} f(q) \right) = \frac{d}{dq} \left( \frac{d}{dq} f(q) \right).
\]
\[ \left[ \frac{d^2}{dq^2} \right]_{c}(f(q)) = \left\{ \begin{array}{l}
\frac{\partial}{\partial q} \left( \frac{f'(q(t)) \partial f(q(t))}{\partial [q(t)]^2} \right) \frac{\partial [q(t)]}{\partial [q(t)]^2} \\
\frac{\partial [q(t)]}{\partial [q(t)]^2} 
\end{array} \right\}, \]

where the derivatives,

\[ \frac{\partial}{\partial q} \left( \frac{f'(q(t)) \partial f(q(t))}{\partial [q(t)]^2} \right) = \frac{\partial f(q(t))}{\partial [q(t)]^2} - \frac{f'(q(t)) \partial [q(t)]}{\partial [q(t)]^2} \]

\[ \frac{\partial^2}{\partial q^2} [f(q(t))] \left( \frac{\partial [q(t)]}{\partial [q(t)]^2} \right) \]

To be continued.
\[
\frac{d}{dt} \left( \frac{\frac{d}{dx} [f(q(t))] \frac{d}{dx} [f(g(t))]}{\frac{d}{dx} [q(t)]^3} \right) = \left( \frac{\frac{d}{dx} [q(t)]}{\frac{d}{dx} [g(t)]} \right)^2 \frac{d}{dx} [f(g(t))] + \left( \frac{\frac{d}{dx} [q(t)]}{\frac{d}{dx} [g(t)]} \right) \frac{d}{dx} [f(q(t))] \\
\left( \frac{\frac{d}{dx} [q(t)]}{\frac{d}{dx} [g(t)]} \right)^2 \frac{d}{dx} [f(q(t))] \\
\left( \frac{\frac{d}{dx} [q(t)]}{\frac{d}{dx} [g(t)]} \right) \right) \\
\]
\[
\frac{\delta c(f(t))}{\delta c(g(t))} = \left\{ \begin{array}{l}
\frac{\delta [f(g(t))] \delta [g(t)]}{\delta [g(t)]}^2, \quad \forall t \in (a,b), \\
\frac{\delta [g(t)] \delta [f(g(t))]}{\delta [g(t)]^2}
\end{array} \right.
\]

then, though making the appropriate substitutions, it naturally follows that

\[
\frac{\delta c(f(t))}{\delta c(g(t))} = \left\{ \begin{array}{l}
\frac{\delta [f(g(t))] \delta [g(t)]}{\delta [g(t)]^2} \frac{\delta [g(t)]}{\delta [g(t)]^2} + \frac{\delta [f(g(t))] \delta [g(t)]}{\delta [g(t)]^2} \frac{\delta [g(t)]}{\delta [g(t)]^2} \\
\frac{\delta [g(t)] \delta [f(g(t))]}{\delta [g(t)]^2} \frac{\delta [g(t)]}{\delta [g(t)]^2} \frac{\delta [g(t)]}{\delta [g(t)]^2} \frac{\delta [g(t)]}{\delta [g(t)]^2}
\end{array} \right.
\]

+ For multi-valued functions, it should be noted that the order in which the component functions are listed, is not important, since all such permutations thereof are equivalent, over the same domain.

-181-

whence, by virtue of the preceding parts (ii), (iii) and (iv) of Theorem IV.2, we likewise deduce that

\[
\frac{\delta [f(g(t))] \delta [g(t)]}{\delta [g(t)]^2} = \frac{\delta [f(g(t))] \delta [g(t)]}{\delta [g(t)]^2}
\]
\[
\begin{align*}
\frac{d}{dx} \left[ f(g(x)) \right] & = \frac{df}{dg} \left[ \frac{dg}{dx} \right] + \\
\frac{d^2}{dx^2} \left[ f(g(x)) \right] & = \frac{d^2}{dg^2} \left[ \frac{dg}{dx} \right] \left( \frac{df}{dg} \right) \\
\frac{d}{dx} \left( \frac{\frac{df}{dg}}{\left( \frac{dg}{dx} \right)^2} \right) & = \frac{d}{dg} \left[ \frac{df}{dg} \right] \frac{dg}{dx} \\
& = \frac{df}{dg} \left( \frac{dg}{dx} \right)^2 + \\
& \quad \frac{d}{dx} \left[ \frac{df}{dg} \right] \frac{dg}{dx} \\
\frac{df}{dx} \left[ \frac{dg}{dx} \right] & = \frac{df}{dg} \left( \frac{dg}{dx} \right)^2 - \frac{df}{dg} \frac{d^2g}{dx^2} \frac{dg}{dx} \\
& = \frac{df}{dg} \left( \frac{dg}{dx} \right)^2 - \frac{d^2}{dx^2} \left[ f(g(x)) \right]
\end{align*}
\]

and

\[
\frac{df}{dx} \left[ \frac{dg}{dx} \right]^2 = \frac{df}{dg} \left( \frac{dg}{dx} \right)^2 \frac{df}{dx} \left[ \frac{dg}{dx} \right] - \frac{df}{dg} \frac{d^2g}{dx^2} \frac{dg}{dx}
\]

Q.E.D.

The above-stated theorem, in conjunction with Definition DIV-8, clearly demonstrates that any second derivative, \( \frac{d^2}{dx^2} (f(x)) \), may be obtained simply by applying the first-order differential operator, \( \frac{d}{dx} \), twice upon the function, \( f(x) \), in accordance with Theorem DIV-1. As a matter of fact, we can generalize this whole procedure to include any "nth-order" derivative, \( \frac{d^n}{dx^n} (f(x)) \), in which case we would utilize the operator, \( \frac{d}{dx} \), \( n \) consecutive times with respect to
the same function.

Once again, we examine that, on those occasions where we may choose to denote as \( z, C \), by the equations,

\[
\begin{align*}
q(t) &= x(t) + iy(t) \\
x(t) &= e^{i\gamma(t)} \\
y(t) &= e^{-i\gamma(t)}
\end{align*}
\]

(4 - 34),

the formulae resulting from Theorem TIV-4 must therefore be rewritten as

\[
\frac{d}{dt}C(q(t)) = \frac{\frac{d}{dt}C(q(t)) \frac{d}{dt}C(q(t))}{\left(\frac{d}{dt}C(q(t))\right)^2}
\]

\[
= \frac{\frac{d}{dt}C(q(t)) \frac{d}{dt}C(q(t))}{\left(\frac{d}{dt}C(q(t))\right)^2}
\]

\[
= \frac{\frac{d}{dt}C(q(t)) \frac{d}{dt}C(q(t))}{\left(\frac{d}{dt}C(q(t))\right)^2}
\]

\[
= \frac{\frac{d}{dt}C(q(t)) \frac{d}{dt}C(q(t))}{\left(\frac{d}{dt}C(q(t))\right)^2}
\]

where the derivatives,
\[
\frac{\partial}{\partial q_{(c)}} \left( \frac{\partial f(q_{(c)}), \partial f(q_{(c)})}{\partial x_{(c)}}, \partial x_{(c)} \right)^{2} = \frac{\partial f(q_{(c)}), \partial f(q_{(c)})}{\partial x_{(c)}}, \partial x_{(c)} \right)^{2} + \frac{\partial^{2} f(q_{(c)}), \partial f(q_{(c)})}{\partial x_{(c)}^{2}}, \partial x_{(c)}^{2}, \partial f(q_{(c)}) \right) \frac{\partial^{2} f(q_{(c)})}{\partial x_{(c)}^{2}}, \partial x_{(c)}^{2}, \partial f(q_{(c)}) \right), \]

\[
\frac{\partial^{2} f(q_{(c)}), \partial f(q_{(c)})}{\partial x_{(c)}^{2}}, \partial x_{(c)}^{2}, \partial f(q_{(c)}) \right) = \left( \frac{\partial^{2} f(q_{(c)}), \partial f(q_{(c)})}{\partial x_{(c)}^{2}}, \partial x_{(c)}^{2}, \partial f(q_{(c)}) \right) \frac{\partial^{2} f(q_{(c)}), \partial f(q_{(c)})}{\partial x_{(c)}^{2}}, \partial x_{(c)}^{2}, \partial f(q_{(c)}) \right), \]

\[
\frac{\partial^{2} f(q_{(c)}), \partial f(q_{(c)})}{\partial x_{(c)}^{2}}, \partial x_{(c)}^{2}, \partial f(q_{(c)}) \right) = \left( \frac{\partial^{2} f(q_{(c)}), \partial f(q_{(c)})}{\partial x_{(c)}^{2}}, \partial x_{(c)}^{2}, \partial f(q_{(c)}) \right) \frac{\partial^{2} f(q_{(c)}), \partial f(q_{(c)})}{\partial x_{(c)}^{2}}, \partial x_{(c)}^{2}, \partial f(q_{(c)}) \right), \]

(4-35).

Finally, if the quaternion hypercomplex functions, \( f(x_{(c)} + iy_{(c)}) \), \( f(x_{(c)} + iy_{(c)}) \), \( f(x_{(c)} + iy_{(c)}) \), are also analytic in terms of Eqs. (4-17), (4-18), (4-19) and (4-20), it likewise follows that the second derivative,

\[
\frac{\partial^{2} f(q_{(c)})}{\partial x_{(c)}^{2}}, \partial f(q_{(c)}) \right) = \frac{\partial^{2} f(q_{(c)})}{\partial x_{(c)}^{2}}, \partial f(q_{(c)}) \right)
which is completely analogous with its more familiar counterparts from the calculus of both real and complex variable functions.

The Indefinite and Definite Integrals of Quaternion Hypercomplex Functions restricted to Smooth Arcs embedded in q-space.

We conclude our discussion of the basic principles underlying the calculus of quaternion hypercomplex functions with some elementary definitions and theorems pertaining to integration. Indeed, we had previously remarked in Part 5 of Section III that Churchill et al. (Ref. Reference B2, Section VII) invoked the concept of an arc, C, defined on the complex z-plane, as a basis for analyzing the behavioral properties of any complex function, f(z), under integration. Presumably, this particular approach has enabled mathematicians to apply various techniques of integration upon complex functions which were originally formulated in relation to real variable analysis.

After taking all of these factors into consideration, we will proceed to derive similar techniques for quaternion hypercomplex functions, whereupon our final definition to that effect is accordingly stated as follows:

Definition DIV-9
Let there exist a quaternion hypercomplex function, \( f(t) \), which is restricted to a smooth arc, \( C \), thus defined by the equation,

\[
g(t) = x(t) + iy(t) + jz(t) + k\lambda(t), \quad \forall t \in [a, b].
\]

Consequently, \( \forall t \in [a, b] \), the indefinite integral or anti-derivative of \( f \), with respect to the real parameter, \( t \), is defined by the formula,

\[
F(g(t)) = F^*(t) = \int f(g(t)) \, dt,
\]

such that the derivative,

\[
\frac{\partial}{\partial t} [F(g(t))] = \frac{\partial}{\partial t} [F^*(t)] = \frac{\partial}{\partial t} \left[ \int f(g(t)) \, dt \right] = f(g(t)).
\]

Given the context of Definition DIX-9, we now wish to evaluate the indefinite integral mentioned therein in terms of its constituent real and imaginary parts. It is therefore not surprising that our next theorem should be concerned with just such an objective!
Furthermore, if we define the corresponding real and imaginary parts of $f$, such that

$$f(q(t)) = u_1^*(t) + iv_1^*(t) + ju_2^*(t) + kv_2^*(t),$$

then it may be established that the indefinite integrals,

$$\int f(q(t)) \, dt = \int u_1^*(t) \, dt + i \int v_1^*(t) \, dt + j \int u_2^*(t) \, dt + k \int v_2^*(t) \, dt,$$

exist, \( \forall t \in [a, b] \), provided that the indefinite integrals,

$$\int u_1^*(t) \, dt, \int v_1^*(t) \, dt, \int u_2^*(t) \, dt, \int v_2^*(t) \, dt,$$

are similarly defined with respect to the above mentioned interval.

\[ \boxed{\text{** PROOF:**}} \]

We firstly construct an integral function,

$$F(q(t)) = U_1^*(t) + iV_1^*(t) + ju_2^*(t) + kv_2^*(t),$$

which is accordingly defined, \( \forall t \in [a, b] \). Now, by virtue of Theorem IV-1, we deduce that

$$\frac{d}{dt} \left[ F(q(t)) \right] = u_1^*(t) + iv_1^*(t) + ju_2^*(t) + kv_2^*(t)$$
\[ f(g(t)) = \begin{cases} f(t), & \forall t \in (a, b), \\
\text{if and only if the derivatives,} \\
\frac{d}{dt}[U_1^*(t)] = u_1^*(t), & \frac{d}{dt}[U_2^*(t)] = u_2^*(t), \\
\frac{d}{dt}[V_1^*(t)] = v_1^*(t), & \frac{d}{dt}[V_2^*(t)] = v_2^*(t), \\
\text{also exist on the same interval.} \\
\end{cases} \]

Finally, it follows from Definition D4-9 that

\[ \frac{d}{dt}[F(g(t))] = f(g(t)) \iff \int f(g(t)) dt = F(g(t)) \]

\[ \int f(g(t)) dt = U_1^*(b) + iV_1^*(b) + jU_2^*(b) + kV_2^*(b) \]

\[ = \int u_1^*(t) dt + i \int v_1^*(t) dt + j \int u_2^*(t) dt + k \int v_2^*(t) dt, \]

\[ \forall t \in [a, b], \text{ since the Fundamental Theorem of Calculus, pertaining to functions of a single real variable, automatically gives rise to the equivalent statements.} \]
To be continued via the author’s next submission, namely -

“An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 6/6.”