Novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs

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Abstract
In this research article, we present certain notions of bipolar neutrosophic graphs. We study the dominating and independent sets of bipolar neutrosophic graphs. We describe novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. We develop an algorithm for computing domination in bipolar neutrosophic graphs. We also show that there are some flaws in Broumi et al. [11]’s definition.

Keywords: Bipolar neutrosophic sets, Bipolar neutrosophic graphs, Domination number, Independent number, Decision making, Algorithm.

Mathematics Subject Classification 2010: 03E72, 68R10, 68R05

1 Introduction
A fuzzy set [27] is an important mathematical structure to represent a collection of objects whose boundary is vague. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. In 1994, Zhang [29] introduced the notion of bipolar fuzzy sets and relations. Bipolar fuzzy sets are extension of fuzzy sets whose membership degree ranges $[-1,1]$. The membership degree $(0,1]$ indicates that the object satisfies a certain property whereas the membership degree $[-1,0)$ indicates that the element satisfies the implicit counter property. Positive information represent what is considered to be possible and negative information represent what is granted to be impossible. Actually, a variety of decision making problems are based on two-sided bipolar judgements on a positive side and a negative side. Nowadays bipolar fuzzy sets are playing a substantial role in chemistry, economics, computer science, engineering, medicine and decision making problems. Samarandache [22] introduced the idea of neutrosophic probability, sets and logic. Peng et al. [19], in 2014, described some operational properties and studied a new approach for multi-criteria decision making problems using neutrosophic sets. Ye [25, 26] discussed trapezoidal neutrosophic sets and simplified neutrosophic sets with applications in multi-criteria decision making problems.
The other terminologies and applications of neutrosophic sets can be seen in [13, 23, 25, 26]. In a neutrosophic set, the membership value is associated with truth, false and indeterminacy degrees but there is no restriction on their sum. Deli et al. [12] extended the ideas of bipolar fuzzy sets and neutrosophic sets to bipolar neutrosophic sets and studied its operations and applications in decision making problems.

Graph theory has numerous applications in science and engineering. However, in some cases, some aspects of graph theoretic concepts may be uncertain. In such cases, it is important to deal with uncertainty using the methods of fuzzy sets and logics. Based on Zadeh’s fuzzy relations [28], Kaufmann [14] defined a fuzzy graph. The fuzzy relations between fuzzy sets were also considered by Rosenfeld [20] and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Later on, Bhattacharya [8] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [16]. The complement of a fuzzy graph was defined by Mordeson [17]. Bhutani and Rosenfeld introduced the concept of $M$-strong fuzzy graphs in [9] and studied some of their properties.

The concept of strong arcs in fuzzy graphs was discussed in [10]. The theory of fuzzy graphs has extended widely by many researchers as it can be seen in [15, 17]. The idea of domination was first arose in chessboard problem in 1862. Somasundaram and Somasundaram [24] introduced domination and independent domination in fuzzy graphs. Gani and Chandrasekaran [18] studied the notion of fuzzy domination and independent domination using strong arcs. Akram [1, 2] introduced bipolar fuzzy graphs and discuss its various properties. Akram and Dudek [3] studied regular bipolar fuzzy graphs. Several new concepts on bipolar neutrosophic graphs and bipolar neutrosophic hypergraphs have been studied in [4-7]. In this research article, we present certain notions of bipolar neutrosophic graphs. We study the dominating and independent sets of bipolar neutrosophic graphs. We describe novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. We develop an algorithm for computing domination in bipolar neutrosophic graphs. We also show that there are some flaws in Broumi et al. [11]’s definition.

## 2 Preliminaries

Let $Y$ be a non-empty universe and $\widetilde{Y}^2$ is the collection of all 2-element subsets of $Y$. A pair $G^* = (Y, E)$, where $E \subseteq \widetilde{Y}^2$ is a graph. The cardinality of any subset $D \subseteq Y$ is the number of vertices in $D$, it is denoted by $|D|$.

**Definition 2.1.** [27, 28] A fuzzy set $\mu$ in a universe $Y$ is a mapping $\mu : Y \rightarrow [0, 1]$. A fuzzy relation on $Y$ is a fuzzy set $\nu$ in $Y \times Y$.

**Definition 2.2.** [28] If $\mu$ is a fuzzy set on $Y$ and $\nu$ a fuzzy relation in $Y$. We can say $\nu$ is a fuzzy relation on $\mu$ if $\nu(y, z) \leq \min\{\mu(y), \mu(z)\}$ for all $y, z \in Y$.

**Definition 2.3.** [14] A fuzzy graph on a non-empty universe $Y$ is a pair $G = (\mu, \lambda)$, where $\mu$ is a fuzzy set on $Y$ and $\lambda$ is a fuzzy relation in $Y$ such that $\lambda(yz) \leq \min\{\mu(y), \mu(z)\}$ for all $y, z \in Y$. Note that $\lambda$ is a fuzzy relation on $\mu$, and $\lambda(yz) = 0$ for all $yz \in \widetilde{Y}^2 - E$.

**Definition 2.4.** [29] A bipolar fuzzy set on a non-empty set $Y$ has the form $C = \{\{(y, \mu^p(y), \mu^n(y)) : y \in Y\}$ where, $\mu^p : Y \rightarrow [0, 1]$ and $\mu^n : Y \rightarrow [-1, 0]$ are mappings.
The positive membership value $\mu^p(y)$ represents the strength of truth or satisfaction of an element $y$ to a certain property corresponding to bipolar fuzzy set $C$ and $\mu^n(x)$ denotes the strength of satisfaction of an element $y$ to some counter property of bipolar fuzzy set $C$. If $\mu^p(y) \neq 0$ and $\mu^n(y) = 0$, it is the situation when $y$ has only truth satisfaction degree for property $C$. If $\mu^n(y) \neq 0$ and $\mu^p(y) = 0$, it is the case that $y$ is not satisfying the property of $C$ but satisfying the counter property to $C$. It is possible for $y$ that $\mu^p(x) \neq 0$ and $\mu^n(x) \neq 0$ when $y$ satisfies the property of $C$ as well as its counter property in some part of $Y$.

**Definition 2.5.** [1] Let $Y$ be a nonempty set. A mapping $D = (\mu^p, \mu^n) : Y \times Y \to [0, 1] \times [-1, 0]$ is a bipolar fuzzy relation on $Y$ such that $\mu^p(xy) \in [0, 1]$ and $\mu^n(xy) \in [-1, 0]$ for $y, z \in Y$.

**Definition 2.6.** [1] A bipolar fuzzy graph on $Y$ is a pair $G = (C, D)$ where $C = (\mu^p_C, \mu^n_C)$ is a bipolar fuzzy set on $Y$ and $D = (\mu^p_D, \mu^n_D)$ is a bipolar fuzzy relation in $Y$ such that

$$\mu^p_D(yz) \leq \mu^p_C(y) \wedge \mu^p_C(z) \text{ and } \mu^n_D(yz) \geq \mu^n_C(y) \vee \mu^n_C(z) \text{ for all } y, z \in Y.$$

Note that $D$ is a bipolar fuzzy relation on $C$, and $\mu^p_D(yz) > 0, \mu^n_D(yz) < 0$ for $yz \in \bar{Y}^2$, $\mu^p_D(yz) = \mu^n_D(yz) = 0$ for $yz \in \bar{Y}^2 - E$.

**Definition 2.7.** [23] A neutrosophic set $C$ on a non-empty set $Y$ is characterized by a truth membership function $t_C : Y \to [0, 1]$, indeterminacy membership function $I_C : Y \to [0, 1]$ and a falsity membership function $f_C : Y \to [0, 1]$. There is no restriction on the sum of $t_C(x)$, $I_C(x)$ and $f_C(x)$ for all $x \in Y$.

**Definition 2.8.** [12] A bipolar neutrosophic set on a empty set $Y$ is an object of the form

$$C = \{(y, t^p_C(y), I^p_C(y), f^p_C(y), t^C_C(y), I^C_C(y), f^C_C(y)) : y \in Y\}$$

where, $t^p_C, I^p_C, f^p_C : Y \to [0, 1]$ and $t^C_C, I^C_C, f^C_C : Y \to [-1, 0]$. The positive values $t^p_C(y), I^p_C(y), f^p_C(y)$ denote respectively the truth, indeterminacy and false membership degrees of an element $y \in Y$ whereas $t^C_C(y), I^C_C(y), f^C_C(y)$ denote the implicit counter property of the truth, indeterminacy and false membership degrees of the element $y \in Y$ corresponding to the bipolar neutrosophic set $C$.

### 3 Bipolar neutrosophic graphs

In this section, we discuss the concept of a bipolar neutrosophic graph and its various properties.

**Definition 3.1.** A bipolar neutrosophic relation on a non-empty set $Y$ is a bipolar neutrosophic subset of $Y \times Y$ of the form $D = \{(yz, t^p_D(yz), I^p_D(yz), f^p_D(yz), t^C_D(yz), I^C_D(yz), f^C_D(yz)) : yz \in Y \times Y\}$ where, $t^p_D, I^p_D, f^p_D, t^C_D, I^C_D, f^C_D$ are defined by the mappings $t^p_D, I^p_D, f^p_D : Y \times Y \to [0, 1]$ and $t^C_D, I^C_D, f^C_D : Y \times Y \to [-1, 0]$.

**Definition 3.2.** A bipolar neutrosophic graph on a non-empty set $Y$ is a pair $G = (C, D)$, where $C$ is a bipolar neutrosophic set on $Y$ and $D$ is a bipolar neutrosophic relation in $Y$ such that

$$t^p_D(yz) \leq t^p_C(y) \wedge t^p_C(z), \quad I^p_D(yz) \leq I^p_C(y) \wedge I^p_C(z), \quad f^p_D(yz) \leq f^p_C(y) \vee f^p_C(z),$$
$$t^C_D(yz) \geq t^C_C(y) \vee t^C_C(z), \quad I^C_D(yz) \geq I^C_C(y) \vee I^C_C(z), \quad f^C_D(yz) \geq f^C_C(y) \wedge f^C_C(z),$$

for all $y, z \in Y$. Note that $D(yz) = (0, 0, 0, 0, 0)$ for all $yz \in Y \times Y \setminus E$. 

3
Example 3.1. Here we discuss an example of a bipolar neutrosophic graph such that $Y = \{x, y, z\}$. Let $C$ be a bipolar neutrosophic set on $Y$ given in Table 1 and $D$ be a bipolar neutrosophic relation in $Y$ given in Table 2.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{C}^{p}$</td>
<td>0.3</td>
<td>0.5</td>
</tr>
<tr>
<td>$I_{C}^{p}$</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>$f_{C}^{p}$</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>$t_{C}^{-p}$</td>
<td>-0.6</td>
<td>-0.1</td>
</tr>
<tr>
<td>$I_{C}^{-p}$</td>
<td>-0.5</td>
<td>-0.8</td>
</tr>
<tr>
<td>$f_{C}^{-p}$</td>
<td>-0.2</td>
<td>-0.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$xy$</th>
<th>$yz$</th>
<th>$xz$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{D}^{p}$</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>$I_{D}^{p}$</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>$f_{D}^{p}$</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>$t_{D}^{-p}$</td>
<td>-0.1</td>
<td>-0.1</td>
</tr>
<tr>
<td>$I_{D}^{-p}$</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>$f_{D}^{-p}$</td>
<td>-0.2</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

Routine calculations show that $G = (C, D)$ is a bipolar neutrosophic graph. The bipolar neutrosophic graph $G$ is shown in Fig. 3.1.

![Figure 3.1: Bipolar neutrosophic graph G](image)

**Definition 3.3.** Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two bipolar neutrosophic graphs where, $C_1$ and $C_2$ are bipolar neutrosophic sets on $Y_1$ and $Y_2$, $D_1$ and $D_2$ are bipolar neutrosophic relations in $Y_1$ and $Y_2$, respectively. The union of $G_1$ and $G_2$ is a pair $G_1 \cup G_2 = (C_1 \cup C_2, D_1 \cup D_2)$ such that for all $y, z \in Y$,

1. If $y \in Y_1, y \notin Y_2$ then, $(C_1 \cup C_2)(y) = C_1(y)$.
2. If $y \in Y_2, y \notin Y_1$ then, $(C_1 \cup C_2)(y) = C_2(y)$.
3. If $y \in Y_1 \cap Y_2$ then,

\[
(C_1 \cup C_2)(y) = (t_{C_1}^{p}(y) \lor t_{C_2}^{p}(y), \frac{I_{C_1}^{p}(y) + I_{C_2}^{p}(y)}{2}, f_{C_1}^{p}(y) \land f_{C_2}^{p}(y), t_{C_1}^{-p}(y) \land t_{C_2}^{-p}(y), \frac{I_{C_1}^{-p}(y) + I_{C_2}^{-p}(y)}{2}, f_{C_1}^{-p}(y) \lor f_{C_2}^{-p}(y)),
\]
If $E_1$ and $E_2$ are the sets of edges in $G_1$ and $G_2$ then, $D_1 \cup D_2$ can be defined as:

1. If $yz \in E_1, yz \notin E_2$ then, $(D_1 \cup D_2)(yz) = D_1(yz)$.
2. If $yz \in E_2, yz \notin E_1$ then, $(D_1 \cup D_2)(yz) = D_2(yz)$.
3. If $yz \in E_1 \cap E_2$ then,

   $$(D_1 \cup D_2)(yz) = \left( t_{D_1}^p(yz) \cup t_{D_2}^p(yz) \right), f_{D_1}^p(yz) \wedge f_{D_2}^p(yz), t_{D_1}^n(yz) \wedge t_{D_2}^n(yz),$$

\[f_{D_1}^n(yz) + f_{D_2}^n(yz) \over 2].

**Definition 3.4.** The intersection of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is a pair $G_1 \cap G_2 = (C_1 \cap C_2, D_1 \cap D_2)$ where, $C_1, C_2, D_1$ and $D_2$ are given in Definition 3.3. The membership values of vertices and edges in $G_1 \cap G_2$ can be defined as,

\[(C_1 \cap C_2)(y) = (t_{C_1}^p(y) \wedge t_{C_2}^p(y), f_{C_1}^p(y) \wedge f_{C_2}^p(y), t_{C_1}^n(y) \wedge t_{C_2}^n(y), f_{C_1}^n(y) \wedge f_{C_2}^n(y)), \text{ for all } y \in Y_1 \cap Y_2.\]

\[(D_1 \cap D_2)(yz) = (t_{D_1}^p(yz) \wedge t_{D_2}^p(yz), f_{D_1}^p(yz) \wedge f_{D_2}^p(yz), t_{D_1}^n(yz) \wedge t_{D_2}^n(yz), f_{D_1}^n(yz) \wedge f_{D_2}^n(yz)), \text{ for all } yz \in E_1 \cap E_2.\]

**Definition 3.5.** The join of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is defined by the pair $G_1 + G_2 = (C_1 + C_2, D_1 + D_2)$ such that, $C_1 + C_2 = C_1 \cup C_2$, for all $y \in Y_1 \cup Y_2$, and the membership values of the edges in $G_1 + G_2$ are defined as,

1. $D_1 + D_2 = D_1 \cup D_2$, for all $yz \in E_1 \cup E_2$.
2. Let $E'$ be the set of all edges joining the vertices of $G_1$ and $G_2$ then for all $yz \in E'$, where $y \in Y_1$ and $z \in Y_2$,

\[(D_1 + D_2)(yz) = (t_{D_1}^p(yz) \wedge t_{D_2}^p(yz), f_{D_1}^p(yz) \wedge f_{D_2}^p(yz), t_{D_1}^n(yz) \wedge t_{D_2}^n(yz), f_{D_1}^n(yz) \wedge f_{D_2}^n(yz)).\]

**Definition 3.6.** The Cartesian product of two bipolar neutrosophic graphs $G_1$ and $G_2$ is denoted by the pair $G_1 \square G_2 = (C_1 \square C_2, D_1 \square D_2)$ and defined as,

\[t_{C_1 \square C_2}^p(y) = t_{C_1}^p(y) \wedge t_{C_2}^p(y),\]

\[f_{C_1 \square C_2}^p(y) = f_{C_1}^p(y) \wedge f_{C_2}^p(y),\]

\[t_{C_1 \square C_2}^n(y) = t_{C_1}^n(y) \wedge t_{C_2}^n(y),\]

\[f_{C_1 \square C_2}^n(y) = f_{C_1}^n(y) \wedge f_{C_2}^n(y),\]

for all $y \in Y_1 \times Y_2$. The membership values of the edges in $G_1 \square G_2$ can be calculated as,

1. $t_{D_1 \square D_2}^p((y_1, y_2)(y_1, z_2)) = t_{C_1}^p(y_1) \wedge t_{D_2}^p(y_2 z_2), t_{D_1 \square D_2}^n((y_1, y_2)(y_1, z_2)) = t_{C_1}^n(y_1) \wedge t_{D_2}^n(y_2 z_2)$, for all $y_1 \in Y_1, y_2 z_2 \in E_2$. 

2. $t^p_{D_1 \Box D_2}((y_1, y_2)(z_1, z_2)) = t^p_{D_1}(y_1 z_1) \land t^p_{C_2}(y_2)$, \hspace{1em} $t^n_{D_1 \Box D_2}((y_1, y_2)(z_1, z_2)) = t^n_{D_1}(y_1 z_1) \lor t^n_{C_2}(y_2)$, for all $y_1 z_1 \in E_1, y_2 \in Y_2$.

3. $t^p_{D_1 \Box D_2}((y_1, y_2)(y_1, z_2)) = t^p_{D_1}(y_1 z_1) \land t^p_{D_2}(y_2 z_2)$, \hspace{1em} $I^n_{D_1 \Box D_2}((y_1, y_2)(y_1, z_2)) = I^n_{D_1}(y_1 z_1) \lor I^n_{D_2}(y_2 z_2)$, for all $y_1 \in Y_1, y_2 z_2 \in E_2$.

4. $I^n_{D_1 \Box D_2}((y_1, y_2)(z_1, z_2)) = I^n_{D_1}(y_1 z_1) \lor I^n_{C_2}(y_2)$, \hspace{1em} $f^n_{D_1 \Box D_2}((y_1, y_2)(z_1, z_2)) = f^n_{D_1}(y_1 z_1) \land f^n_{C_2}(y_2)$, for all $y_1 z_1 \in E_1, y_2 \in Y_2$.

5. $f^p_{D_1 \Box D_2}((y_1, y_2)(y_1, z_2)) = f^p_{D_1}(y_1 z_1) \lor f^p_{C_2}(y_2)$, \hspace{1em} $f^n_{D_1 \Box D_2}((y_1, y_2)(z_1, z_2)) = f^n_{D_1}(y_1 z_1) \lor f^n_{C_2}(y_2)$, for all $y_1 \in Y_1, y_2 z_2 \in E_2$.

6. $f^p_{D_1 \Box D_2}((y_1, y_2)(z_1, z_2)) = f^p_{D_1}(y_1 z_1) \lor f^p_{C_2}(y_2)$, \hspace{1em} $f^n_{D_1 \Box D_2}((y_1, y_2)(z_1, z_2)) = f^n_{D_1}(y_1 z_1) \lor f^n_{C_2}(y_2)$, for all $y_1 z_1 \in E_1, y_2 \in Y_2$.

**Definition 3.7.** The direct product of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is denoted by the pair $G_1 \times G_2 = (C_1 \times C_2, D_1 \times D_2)$ such that,

$t^p_{C_1 \times C_2}(y) = t^p_{C_1}(y) \land t^p_{C_2}(y)$, \hspace{1em} $f^p_{C_1 \times C_2}(y) = f^p_{C_1}(y) \lor f^p_{C_2}(y)$,

$t^n_{C_1 \times C_2}(y) = t^n_{C_1}(y) \lor t^n_{C_2}(y)$, \hspace{1em} $f^n_{C_1 \times C_2}(y) = f^n_{C_1}(y) \land f^n_{C_2}(y)$,

for all $y \in Y_1 \times Y_2$.

1. $t^p_{D_1 \times D_2}((y_1, y_2)(z_1, z_2)) = t^p_{D_1}(y_1 z_1) \land t^p_{D_2}(y_2 z_2)$, \hspace{1em} $t^n_{D_1 \times D_2}((y_1, y_2)(z_1, z_2)) = t^n_{D_1}(y_1 z_1) \lor t^n_{D_2}(y_2 z_2)$, for all $y_1 z_1 \in E_1, y_2 z_2 \in E_2$.

2. $I^n_{D_1 \times D_2}((y_1, y_2)(z_1, z_2)) = I^n_{D_1}(y_1 z_1) \lor I^n_{D_2}(y_2 z_2)$, \hspace{1em} $f^n_{D_1 \times D_2}((y_1, y_2)(z_1, z_2)) = f^n_{D_1}(y_1 z_1) \land f^n_{D_2}(y_2 z_2)$, for all $y_1 z_1 \in E_1, y_2 z_2 \in E_2$.

3. $f^p_{D_1 \times D_2}((y_1, y_2)(z_1, z_2)) = f^p_{D_1}(y_1 z_1) \lor f^p_{D_2}(y_2 z_2)$, \hspace{1em} $f^n_{D_1 \times D_2}((y_1, y_2)(z_1, z_2)) = f^n_{D_1}(y_1 z_1) \lor f^n_{D_2}(y_2 z_2)$, for all $y_1 z_1 \in E_1, y_2 z_2 \in E_2$.

**Proposition 3.1.** Let $G_1$ and $G_2$ be any two bipolar neutrosophic graphs then $G_1 \cup G_2, G_1 \cap G_2, G_1 + G_2, G_1 \square G_2$ and $G_1 \times G_2$ are bipolar neutrosophic graphs.

**Definition 3.8.** A bipolar neutrosophic graph $G = (C, D)$ is called strong bipolar neutrosophic graph if

$t^p_{C}(y) = t^p_{C}(y) \land t^p_{C}(z)$, \hspace{1em} $I^n_{C}(y) = I^n_{C}(y) \lor I^n_{C}(z)$,

$t^n_{C}(y) = t^n_{C}(y) \lor t^n_{C}(z)$, \hspace{1em} $f^n_{C}(y) = f^n_{C}(y) \land f^n_{C}(z)$,

for all $y z \in supp(D)$.

**Definition 3.9.** A bipolar neutrosophic graph $G = (C, D)$ is called complete bipolar neutrosophic graph if

$t^p_{C}(y) = t^p_{C}(y) \land t^p_{C}(z)$, \hspace{1em} $I^n_{C}(y) = I^n_{C}(y) \lor I^n_{C}(z)$,

$t^n_{C}(y) = t^n_{C}(y) \lor t^n_{C}(z)$, \hspace{1em} $f^n_{C}(y) = f^n_{C}(y) \land f^n_{C}(z)$,

for all $y, z \in Y$.

**Definition 3.10.** The complement of a bipolar neutrosophic graph $G = (C, D)$ is defined as a pair $G^c = (C^c, D^c)$ such that, $C^c(y) = C(y)$, for all $y \in Y$, and the membership values of the edges in $G^c$ can be calculated as,
The term degree is also referred as \( \gamma \) by Definition 3.10, Similarly, it can be proved that

\[
\sum_{y \neq z} t^p_D(yz) = \frac{1}{2} \sum_{y \neq z} t^p_C(y) \land t^p_C(z), \quad \sum_{y \neq z} P^p_D(yz) = \frac{1}{2} \sum_{y \neq z} P^p_C(y) \land P^p_C(z),
\]

Remark 3.1. A bipolar neutrosophic graph \( G \) is called self complementary if \( G = G^c \).

Theorem 3.1. Let \( G \) be a self complementary bipolar neutrosophic graph then,

\[
\sum_{y \neq z} f^p_D(yz) = \frac{1}{2} \sum_{y \neq z} f^p_C(y) \lor f^p_C(z), \quad \sum_{y \neq z} I^p_D(yz) = \frac{1}{2} \sum_{y \neq z} I^p_C(y) \lor I^p_C(z),
\]

Theorem 3.2. Let \( G = (C, D) \) be a bipolar neutrosophic graph such that for all \( y, z \in Y \),

\[
t^p_D(yz) = \frac{1}{2} (t^p_C(y) \land t^p_C(z)), \quad P^p_D(yz) = \frac{1}{2} (P^p_C(y) \land P^p_C(z)), \quad f^p_D(yz) = \frac{1}{2} (f^p_C(y) \lor f^p_C(z)), \quad I^p_D(yz) = \frac{1}{2} (I^p_C(y) \lor I^p_C(z)), \quad f^p_D(yz) = \frac{1}{2} (I^p_C(y) \land I^p_C(z)).
\]

Then \( G \) is self complementary bipolar neutrosophic graph.

Proof. Let \( G^c = (C^c, D^c) \) be the complement of a bipolar neutrosophic graph \( G = (C, D) \) then, by Definition. 3.10,

\[
t'^p_D(yz) = t'^p_C(y) \land t'^p_C(z) - t^p_D(yz), \quad P'^p_D(yz) = P'^p_C(y) \land P'^p_C(z), \quad f'^p_D(yz) = f'^p_C(y) \lor f'^p_C(z), \quad I'^p_D(yz) = I'^p_C(y) \lor I'^p_C(z),
\]

Similarly, it can be proved that \( t'^n_D(yz) = t'^n_D(yz), P'^n_D(yz) = P'^n_D(yz), I'^n_D(yz) = I'^n_D(yz), f'^n_D(yz) = f'^n_D(yz) \) and \( f'^p_D(yz) = f'^p_D(yz) \). Hence, \( G \) is self complementary.

Definition 3.11. The degree of a vertex \( y \) in a bipolar neutrosophic graph \( G = (C, D) \) is denoted by deg(\( y \)) and defined by the 6–tuple as,

\[
\deg(y) = (\deg^p(y), \deg^n(y), \deg^p_I(y), \deg^n_I(y), \deg^p_E(y), \deg^n_E(y)),
\]

\[
= (\sum_{y \in E} t^p_D(yz), \sum_{y \in E} P^p_D(yz), \sum_{y \in E} f^p_D(yz), \sum_{y \in E} t^n_D(yz), \sum_{y \in E} I^n_D(yz), \sum_{y \in E} f^n_D(yz)).
\]

The term degree is also referred as neighborhood degree.
Definition 3.12. The closed neighborhood degree of a vertex $y$ in a bipolar neutrosophic graph is denoted by $\deg[y]$ and defined as, $\deg[y] = \deg(y) + C(y)$.

Definition 3.13. A bipolar neutrosophic graph $G$ is known as a regular bipolar neutrosophic graph if all vertices of $G$ have same degree. A bipolar neutrosophic graph $G$ is known as a totally regular bipolar neutrosophic graph if all vertices of $G$ have same closed neighborhood degree.

Theorem 3.3. A complete bipolar neutrosophic graph is regular.

Theorem 3.4. Let $G = (C, D)$ be a bipolar neutrosophic graph then $C = (t^D, P^D, f^D, t^n, I^n, f^n)$ is a constant function if and only if the following statements are equivalent:

1) $G$ is a regular bipolar neutrosophic graph,

2) $G$ is totally regular bipolar neutrosophic graph.

Proof. Assume that $C$ is a constant function and for all $y \in Y$,

$$t^D_C(y) = k_t, \; P^D_C(y) = k_P, \; f^D_C(y) = k_f, \; t^n_C(y) = k^n_t, \; I^n_C(y) = k^n_I, \; f^n_C(y) = k^n_f$$

where, $k_t, \; k_P, \; k_f, \; k^n_t, \; k^n_I, \; k^n_f$ are constants.

(1) $\Rightarrow$ (2) Suppose that $G$ is a regular bipolar neutrosophic graph and $\deg(y) = (p_t, p_P, p_f, n_t, n_P, n_f)$ for all $y \in Y$.

Now consider,

$$\deg[y] = \deg(y) + C(y) = (p_t + k_t, p_P + k_P, p_f + k_f, n_t + k^n_t, n_P + k^n_P, n_f + k^n_f),$$

for all $y \in Y$. Thus $G$ is a regular bipolar neutrosophic graph.

(2) $\Rightarrow$ (1) Suppose that $G$ is totally regular bipolar neutrosophic graph and for all $y \in Y$

$$\deg[y] = (p_t, p_P, p_f, n_t, n_P, n_f),$$

$$\Rightarrow \deg(y) + C(y) = (p_t + k_t, p_P + k_P, p_f + k_f, n_t + k^n_t, n_P + k^n_P, n_f + k^n_f),$$

for all $y \in Y$. Hence $C = (c_t - c_t, c_f - c_f, d_t - d_t, d_f - d_f), \; c_t - c_t, c_f - c_f, d_t - d_t, d_f - d_f), \; a constant function which completes the proof. \qed

Definition 3.14. A bipolar neutrosophic graph $G$ is said to be irregular if at least two vertices have distinct degrees. If all vertices do not have same closed neighborhood degrees then $G$ is known as totally irregular bipolar neutrosophic graph.

Theorem 3.5. Let $G = (C, D)$ be a bipolar neutrosophic graph and $C = (t^D, P^D, f^D, t^n, I^n, f^n)$ be a constant function then $G$ is an irregular bipolar neutrosophic graph if and only if $G$ is a totally irregular bipolar neutrosophic graph.

Proof. Assume that $G$ is an irregular bipolar neutrosophic graph then at least two vertices of $G$ have distinct degrees. Let $y$ and $z$ be two vertices such that $\deg(y) = (r_1, r_2, r_3, s_1, s_2, s_3)$ and
\[ \text{deg}(z) = (r'_1, r'_2, r'_3, s'_1, s'_2, s'_3) \text{ where, } r_i \neq r'_i, \text{ for some } i = 1, 2, 3. \]

Since, \( C \) is a constant function let \( C = (k_1, k_2, k_3, l_1, l_2, l_3) \). Therefore,

\[
\begin{align*}
\text{deg}[y] &= \text{deg}(y) + (k_1, k_2, k_3, l_1, l_2, l_3) \\
\text{deg}[y] &= (r_1 + k_1, r_2 + k_2, r_3 + k_3, s_1 + l_1, s_2 + l_2, s_3 + l_3) \\
\text{and } \text{deg}[z] &= (r'_1 + k_1, r'_2 + k_2, r'_3 + k_3, s'_1 + l_1, s'_2 + l_2, s'_3 + l_3).
\end{align*}
\]

Clearly \( r_i + k_i \neq r'_i + k_i \), for some \( i = 1, 2, 3 \) therefore \( y \) and \( z \) have distinct closed neighborhood degrees. Hence \( G \) is a totally irregular bipolar neutrosophic graph.

The converse part is similar.

**Definition 3.15.** If \( G = (C, D) \) is a bipolar neutrosophic graph and \( y, z \) are two vertices in \( G \) then we say that \( y \) dominates \( z \) if

\[
\begin{align*}
t^p_D(yz) &= t^p_C(y) \land t^p_C(z), \\
t^n_D(yz) &= t^n_C(y) \lor t^n_C(z), \\
t^l_D(yz) &= t^l_C(y) \lor t^l_C(z), \\
I^p_D(yz) &= I^p_C(y) \land I^p_C(z), \\
I^n_D(yz) &= I^n_C(y) \lor I^n_C(z), \\
I^l_D(yz) &= I^l_C(y) \lor I^l_C(z).
\end{align*}
\]

A subset \( D' \subseteq Y \) is a dominating set if for each \( z \in Y \setminus D' \) there exists \( y \in D' \) such that \( y \) dominates \( z \). A dominating set \( D' \) is minimal if for every \( y \in D', D' \setminus \{y\} \) is not a dominating set. The domination number of \( G \) is the minimum cardinality among all minimal dominating sets of \( G \), denoted by \( \lambda(G) \).

**Example 3.2.** Consider a bipolar neutrosophic graph as shown in Fig. 3.2. The set \{\( x, w \)\} is a minimal dominating set and \( \lambda(G) = 2 \).

![Figure 3.2: Bipolar neutrosophic graph G.](image)

**Theorem 3.6.** Let \( G_1 \) and \( G_2 \) be two bipolar neutrosophic graphs with \( D'_1 \) and \( D'_2 \) as dominating sets then following conditions hold,

1. If \( Y_1 \cap Y_2 = \emptyset \) then, \( \lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2) \).

2. If for every \( y \in Y_1 \cap Y_2 \neq \emptyset \), \( C_1(y) = C_2(y) \) and for \( yz \in E_1 \cap E_2 \), \( D_1(yz) = D_2(yz) \) then, \( \lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2) - |D'_1 \cap D'_2| \).

**Proof.** (1) The proof is obvious.

(2) Since \( D'_1 \) and \( D'_2 \) are dominating sets of \( G_1 \) and \( G_2 \), \( D'_1 \cup D'_2 \) is a dominating set of \( G_1 \cup G_2 \). Therefore, \( \lambda(G_1 \cup G_2) \leq |D'_1 \cup D'_2| \). It only remains to show that \( D'_1 \cup D'_2 \) is a minimal dominating
set. On contrary, assume that $D' = D'_1 \cup D'_2 \setminus \{y\}$ is a minimal dominating set of $G_1 \cup G_2$.

There are two cases:

**Case 1.** If $y \in D'_1$ and $y \notin D'_2$, then $D'_1 \setminus \{y\}$ is not a dominating set of $G_1$ which implies that $D'_1 \cup D'_2 \setminus \{y\} = D'$ is not a dominating set of $G_1 \cup G_2$. A contradiction, hence $D'_1 \cup D'_2$ is a minimal dominating set and

$$\lambda(G_1 \cup G_2) = |D'_1 \cup D'_2|,$$

$$\Rightarrow \lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2) - |D'_1 \cap D'_2|.$$

**Case 2.** If $y \in D'_2$ and $y \notin D'_1$, same contradiction can be obtained.

**Theorem 3.7.** If $G_1$ and $G_2$ are two bipolar neutrosophic graphs then the following conditions are satisfied,

1. If $Y_1 \cap Y_2 = \emptyset$ then, $\lambda(G_1 + G_2) = 2$.

2. If for every $y \in Y_1 \cap Y_2 \neq \emptyset$, $C_1(y) = C_2(y)$ and for $yz \in E_1 \cap E_2$, $D_1(yz) = D_2(yz)$ then, $\lambda(G_1 + G_2) = \min \{\lambda(G_1), \lambda(G_2), 2\}$.

**Proof.** (1). Let $y_1 \in Y_1$ and $y_2 \in Y_2$, since $G_1 + G_2$ is a bipolar neutrosophic graph, we have

$$t_{D_1 + D_2}^p(y_1 y_2) = t_{C_1 + C_2}^p(y_1) \land t_{C_1 + C_2}^p(y_2), \quad t_{D_1 + D_2}^n(y_1 y_2) = t_{C_1 + C_2}^n(y_1) \lor t_{C_1 + C_2}^n(y_2)$$

$$P_{D_1 + D_2}^p(y_1 y_2) = P_{C_1 + C_2}^p(y_1) \land P_{C_1 + C_2}^p(y_2), \quad P_{D_1 + D_2}^n(y_1 y_2) = P_{C_1 + C_2}^n(y_1) \lor P_{C_1 + C_2}^n(y_2)$$

$$F_{D_1 + D_2}^p(y_1 y_2) = F_{C_1 + C_2}^p(y_1) \lor F_{C_1 + C_2}^p(y_2), \quad F_{D_1 + D_2}^n(y_1 y_2) = F_{C_1 + C_2}^n(y_1) \land F_{C_1 + C_2}^n(y_2).$$

Hence any vertex of $G_1$ dominates all vertices of $G_2$ and similarly any vertex of $G_2$ dominates all vertices of $G_1$. So, $\{y_1, y_2\}$ is a dominating set of $G_1 + G_2$.

(2). If $D$ is a minimal dominating set of $G_1 + G_2$ then, $D$ is one of the following forms,

1. $D = D_1$ where, $\lambda(G_1) = |D_1|$,

2. $D = D_2$ where, $\lambda(G_2) = |D_2|$,

3. $D = \{y_1, y_2\}$ where, $y_1 \in Y_1$ and $y_2 \in Y_2$. $\{y_1\}$ and $\{y_2\}$ are not dominating sets of $G_1$ or $G_2$, respectively.

Hence, $\lambda(G_1 + G_2) = \min \{\lambda(G_1), \lambda(G_2), 2\}$.

**Theorem 3.8.** Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two bipolar neutrosophic graphs. If for $y_1 \in Y_1$, $C_1(y_1) > \emptyset$ where, $\emptyset = (0, 0, 0, 0, 0, 0)$, and $y_2$ dominates $z_2$ in $G_2$ then, $(y_1, y_2)$ dominates $(y_1, z_2)$ in $G_1 \Box G_2$.

**Proof.** Since $y_2$ dominates $z_2$ therefore,

$$t_{D_2}^p(y_2 z_2) = t_{C_2}^p(y_2) \land t_{C_2}^p(z_2), \quad t_{D_2}^n(y_2 z_2) = t_{C_2}^n(y_2) \land t_{C_2}^n(z_2), \quad f_{D_2}^p(y_2 z_2) = f_{C_2}^p(y_2) \lor f_{C_2}^p(z_2), \quad f_{D_2}^n(y_2 z_2) = f_{C_2}^n(y_2) \lor f_{C_2}^n(z_2),$$

$$t_{D_2}^p(y_2 z_2) = t_{C_2}^p(y_2) \lor t_{C_2}^p(z_2), \quad t_{D_2}^n(y_2 z_2) = t_{C_2}^n(y_2) \lor t_{C_2}^n(z_2), \quad f_{D_2}^p(y_2 z_2) = f_{C_2}^p(y_2) \land f_{C_2}^p(z_2), \quad f_{D_2}^n(y_2 z_2) = f_{C_2}^n(y_2) \land f_{C_2}^n(z_2).$$
For $y_1 \in Y_1$, take $(y_1, z_2) \in Y_1 \times Y_2$. By Definition 3.6,

$$t_{D_1 \square D_2}^p((y_1, y_2)(y_1, z_2)) = t_{C_1}^p(y_1) \land t_{D_2}^p(y_2, z_2),$$

$$= t_{C_1}^p(y_1) \land \{t_{C_2}^p(y_2) \land t_{C_2}^p(z_2)\},$$

$$= \{t_{C_1}^p(y_1) \land t_{C_2}^p(y_2)\} \land \{t_{C_1}^p(y_1) \land t_{C_2}^p(z_2)\},$$

$$= t_{C_1 \square C_2}^p(y_1, y_2) \land t_{C_1 \square C_2}^p(y_1, z_2).$$

Similarly, it can be proved that

$$t_{D_1 \square D_2}^p((y_1, y_2)(y_1, z_2)) = t_{C_1}^p(y_1) \lor t_{D_2}^p(y_2, z_2),$$

$$= t_{C_1}^p(y_1) \lor \{t_{C_2}^p(y_2) \lor t_{C_2}^p(z_2)\},$$

$$= \{t_{C_1}^p(y_1) \lor t_{C_2}^p(y_2)\} \lor \{t_{C_1}^p(y_1) \lor t_{C_2}^p(z_2)\},$$

$$= t_{C_1 \square C_2}^p(y_1, y_2) \lor t_{C_1 \square C_2}^p(y_1, z_2).$$

Hence $(y_1, z_2)$ dominates $(y_1, z_2)$ and the proof is complete. 

**Proposition 3.2.** If $G_1$ and $G_2$ are bipolar neutrosophic graphs and for $z_2 \in Y_2, C_2(z_2) > 0$ where, $\theta = (0, 0, 0, 0, 0, 0)$, $y_1$ dominates $z_1$ in $G_1$ then $(y_1, z_2)$ dominates $(z_1, z_2)$ in $G_1 \square G_2$.

**Theorem 3.9.** If $D_1'$ and $D_2'$ are minimal dominating sets of $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, respectively. Then $D_1' \times Y_2$ and $Y_1 \times D_2'$ are dominating sets of $G_1 \square G_2$ and

$$\lambda(G_1 \square G_2) \leq |D_1' \times Y_2| \lor |Y_1 \times D_2'|.$$  \hspace{1cm} (1)

**Proof.** To prove Inequality (1), we need to show that $D_1' \times Y_2$ and $Y_1 \times D_2'$ are dominating sets of $G_1 \square G_2$. Let $(z_1, z_2) \notin D_1' \times Y_2$ then, $z_1 \notin D_1'$. Since $D_1'$ is a dominating set of $G_1$, there exists $y_1 \in D_1'$ that dominates $z_1$. By Theorem 3.2, $(y_1, z_2)$ dominates $(z_1, z_2)$ in $G_1 \square G_2$. Since $(z_1, z_2)$ was taken to be arbitrary therefore, $D_1' \times Y_2$ is a dominating set of $G_1 \square G_2$. Similarly, $Y_1 \times D_2'$ is a dominating set if $G_1 \square G_2$. Hence the proof.

**Theorem 3.10.** Let $D_1'$ and $D_2'$ be the dominating sets of $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, respectively then, $D_1' \times D_2'$ is a dominating set of the direct product $G_1 \times G_2$ and

$$\lambda(G_1 \times G_2) = |D_1'| = |D_1' \times D_2' \cup \{(t_1, t_2) : t_1y_1 \in E_1, y_1 \in D_1', t_2 \in D_2'\}|.$$ \hspace{1cm} (2)

**Proof.** Let $(z_1, z_2) \in Y_1 \times Y_2 \setminus D_1' \times D_2'$ then there are two cases.

**Case 1:** If $z_1 \in Y_1 \setminus D_1'$ and $z_2 \in Y_2 \setminus D_2'$. Since, $D_1'$ and $D_2'$ are dominating sets there exist $y_1 \in D_1'$ and $y_2 \in D_2'$ such that $y_1$ dominates $z_1$ and $y_2$ dominates $z_2$. Consider,

$$t_{D_1 \times D_2}^p((y_1, y_2)(z_1, z_2)) = t_{D_1}^p(y_1z_1) \land t_{D_2}^p(y_2z_2),$$

$$= \{t_{C_1}^p(y_1) \land t_{C_1}^p(z_1)\} \land \{t_{C_2}^p(y_2) \land t_{C_2}^p(z_2)\},$$

$$= \{t_{C_1}^p(y_1) \land t_{C_2}^p(y_2)\} \land \{t_{C_1}^p(z_1) \land t_{C_2}^p(z_2)\},$$

$$= t_{C_1 \times C_2}^p(y_1, y_2) \land t_{C_1 \times C_2}^p(z_1, z_2).$$
Similarly, it can be proved for other truth, indeterminacy and falsity membership degrees. Hence $(y_1, y_2)$ dominates $(z_1, z_2)$.

**Case 2:** If $z_1 \in D_1'$ and $z_2 \in Y_2 \setminus D_2'$ then, there exists $(t_1, t_2) \in \{(t_1, t_2) : t_1z_1 \in E_1, z_1 \in D_1', t_2 \in D_2'\}$ such that $z_1$ dominates $t_1$ and $t_2$ dominates $z_2$. Consider,

$$
t^p_{D_1 \times D_2}((t_1, t_2)(z_1, z_2)) = t^p_{D_1}(t_1z_1) \land t^p_{D_2}(t_2z_2),
$$

$$
= \{t^p_{C_1}(t_1) \land t^p_{C_1}(z_1)\} \land \{t^p_{C_2}(t_2) \land t^p_{C_2}(z_2)\},
$$

$$
= \{t^p_{C_1}(t_1) \land t^p_{C_2}(t_2)\} \land \{t^p_{C_1}(z_1) \land t^p_{C_2}(z_2)\},
$$

$$
= t^p_{C_1 \times C_2}(t_1, t_2) \land t^p_{C_1 \times C_2}(z_1, z_2).
$$

On the same lines, the result can be obtained for other truth, indeterminacy and falsity membership degrees. Hence $(z_1, z_2)$ is dominated by $(t_1, t_2)$.

Since $(y_1, y_2)$ was taken to be arbitrary therefore, every element of $Y_1 \times Y_2 \setminus D_x'$ is dominated by some element of $D_x'$.

Clearly, no vertex in $\{(t_1, t_2) : t_1y_1 \in E_1, y_1 \in D_1', t_2 \in D_2'\}$ is dominated by any other vertex in $D_x'$. Therefore, it only remains to show that $D'_1 \times D'_2$ is minimal. On contrary, assume that $D'$ is a minimal such that $|D'| < |D'_1 \times D'_2|$. Let $(t_1, t_2) \in D'_1 \times D'_2$ such that $(t_1, t_2) \notin D'$ i.e., $t_1 \in D'_1$ and $t_2 \in D'_2$ then there exist $t'_1 \in Y_1 \setminus D'_1$ and $t'_2 \in Y_2 \setminus D'_2$ which are only dominated by $t_1$ and $t_2$, respectively. Hence no element other than $(t_1, t_2)$ dominates $(t'_1, t'_2)$ so $(t_1, t_2) \in D'$. A contradiction, thus $D'_1 \times D'_2$ is minimal. 

**Corollary 3.1.** If $G_1$ and $G_2$ are two bipolar neutrosophic graphs, $y_1$ dominates $z_1$ in $G_1$ and $y_2$ dominates $z_2$ in $G_2$ then $(y_1, z_1)$ dominates $(y_2, z_2)$ in $G_1 \times G_2$.

**Definition 3.16.** In a bipolar neutrosophic graph two vertices $y$ and $z$ are independent if

$$
t^p_D(yz) < t^p_C(y) \land t^p_C(z), \quad t^p_D(yz) < t^p_C(y) \land t^p_C(z), \quad f^p_D(yz) < f^p_C(y) \lor f^p_C(z), \quad f^p_D(yz) > f^p_C(y) \lor f^p_C(z),
$$

$$
t^p_D(yz) > t^p_C(y) \lor t^p_C(z), \quad f^p_D(yz) > f^p_C(y) \lor f^p_C(z), \quad f^p_D(yz) < f^p_C(y) \land f^p_C(z). \quad (3)
$$

An independent set $N$ of a bipolar neutrosophic graph is a subset $N$ of $Y$ such that for all $y, z \in N$, Equations (3) are satisfied. An independent set is maximal if for every $t \in Y \setminus N$, $N \cup \{t\}$ is not an independent set. An independent number is the maximal cardinality among all maximal independent sets of a bipolar neutrosophic graph. It is denoted by $\alpha(G)$.

**Theorem 3.11.** If $G_1$ and $G_2$ are bipolar neutrosophic graphs on $Y_1$ and $Y_2$, respectively such that $Y_1 \cap Y_2 = \emptyset$ then $\alpha(G_1 \cup G_2) = \alpha(G_1) + \alpha(G_2)$.

**Proof.** Let $N_1$ and $N_2$ be maximal independent sets of $G_1$ and $G_2$. Since $N_1 \cap N_2 = \emptyset$ therefore, $N_1 \cup N_2$ is a maximal independent set of $G_1 \cup G_2$. Hence $\alpha(G_1 \cup G_2) = \alpha(G_1) + \alpha(G_2)$. 

**Theorem 3.12.** Let $G_1$ and $G_2$ be two bipolar neutrosophic graphs then $\alpha(G_1 + G_2) = \alpha(G_1) \lor \alpha(G_2)$.

**Proof.** Let $N_1$ and $N_2$ be maximal independent sets. Since every vertex of $G_1$ dominates every vertex of $G_2$ in $G_1 + G_1$. Hence, maximal independent set of $G_1 + G_2$ is either $N_1$ or $N_2$. Thus, $\alpha(G_1 + G_2) = \alpha(G_1) \lor \alpha(G_2)$. 

12
Theorem 3.13. If $N_1$ and $N_2$ are maximal independent sets of $G_1$ and $G_2$, respectively and $Y_1 \cap Y_2 = \emptyset$. Then $\alpha(G_1 \square G_2) = |N_1 \times N_2| + |N|$ where, $N = \{(y_i, z_i) : y_i \in Y_1 \setminus N_1, z_i \in Y_2 \setminus N_2, y_iz_{i+1} \in E_1, z_iz_{i+1} \in E_2, \ i = 1, 2, 3, \ldots \}$.

Proof. $N_1$ and $N_2$ are maximal independent sets of $G_1$ and $G_2$, respectively. Clearly, $N_1 \times N_2$ is an independent set of $G_1 \square G_2$ as no vertex of $N_1 \times N_2$ dominates any other vertex of $N_1 \times N_2$. Consider the set of vertices $N = \{(y_i, z_i) : y_i \in Y_1 \setminus N_1, z_i \in Y_2 \setminus N_2, y_iz_{i+1} \in E_1, z_iz_{i+1} \in E_2 \}$. It can be seen that no vertex $(y_i, z_i) \in N$ for each $i = 1, 2, 3, \ldots$ dominates $(y_{i+1}, z_{i+1}) \in N$ for each $i = 1, 2, 3, \ldots$. Hence $N' = (N_1 \times N_2) \cup N$ is an independent set of $G_1 \square G_2$.

Assume that $S = N' \cup \{(y_i, z_j)\}$, for some $i \neq j$, $y_i \in Y_1 \setminus N_1$ and $z_j \in Y_2 \setminus N_2$, is a maximal independent set. Without loss of generality, assume that $j = i + 1$ then, $(y_i, z_j)$ is dominated by $(y_i, z_i)$. A contradiction, hence $N'$ is a maximal independent set and $\alpha(G_1 \square G_2) = |N'| = |N_1 \times N_2| + |N|$. \hfill $\blacksquare$

Theorem 3.14. If $D'_\times$ is a minimal dominating sets of $G_1 \times G_2$ then, $Y_1 \times Y_2 \setminus D'_\times$ is a maximal independent set of $G_1 \times G_2$ and $\alpha(G_1 \times G_2) = n_1n_2 - \lambda(G_1 \times G_2)$ where, $n_1$ and $n_2$ are the number of vertices in $G_1$ and $G_2$.

The proof is obvious.

Theorem 3.15. An independent set of a bipolar neutrosophic graph $G = (C, D)$ is maximal if and only if it is independent and dominating.

Proof. If $N$ is a maximal independent set of $G$, then for every $y \in Y \setminus N$, $N \cup \{y\}$ is not an independent set. For every vertex $y \in Y \setminus N$, there exists some $z \in N$ such that

$$
t_D^p(yz) = t_C^p(y) \land t_C^p(z), \quad t_D^n(yz) = t_C^n(y) \lor t_C^n(z), \quad f_D^p(yz) = f_C^p(y) \lor f_C^p(z), \quad f_D^n(yz) = f_C^n(y) \land f_C^n(z).
$$

Thus $y$ dominates $z$ and hence $N$ is both independent and dominating set.

Conversely, assume that $D$ is a maximal independent and dominating set but not maximal independent set. So there exists a vertex $y \in Y \setminus N$ such that $N \cup \{y\}$ is an independent set i.e., no vertex in $N$ dominates $y$, a contradiction to the fact that $N$ is a dominating set. Hence $N$ is maximal. \hfill $\blacksquare$

Theorem 3.16. Any maximal independent set of a bipolar neutrosophic graph is a minimal dominating set.

Proof. If $N$ is a maximal independent set of a bipolar neutrosophic graph then by Theorem 3.15, $N$ is a dominating set. Assume that $N$ is not a minimal dominating set then, there always exist at least one $z \in N$ for which $N \setminus \{z\}$ is a dominating set. On the other hand if $N \setminus \{z\}$ dominates $Y \setminus \{N \setminus \{z\}\}$, at least one vertex in $N \setminus \{z\}$ dominates $z$. A contradiction to the fact that $N$ is an independent set of bipolar neutrosophic graph $G$. Hence $N$ is a minimal dominating set. \hfill $\blacksquare$

4 Multi-criteria decision making methods

Multiple criteria decision making refers to making decisions in the presence of multiple, usually conflicting criteria. Multi-criteria decision making problems are common in everyday life. In this section, we present multi-criteria decision making methods for the identification of risk in
decision support systems. The method is explained by an example for prevention of accidental hazards in chemical industry. The application of domination in bipolar neutrosophic graphs is given for the construction of transmission stations.

(1) An outranking approach for safety analysis using bipolar neutrosophic information

The proposed methodology can be implemented in various fields in different ways e.g., multi-criteria decision making problems with bipolar neutrosophic information. However, our main focus is the identification of risk assessments in industry which is described in the following steps. The bipolar neutrosophic information consists of a group of risks\alternatives $R = \{r_1, r_2, \cdots, r_n\}$ evaluated on the basis of criteria $C = \{c_1, c_2, \cdots, c_m\}$. Here $r_i$, $i = 1, 2, \cdots, n$ is the possibility for the criteria $c_k$, $k = 1, 2, \cdots, m$ and $r_{ik}$ are in the form of bipolar neutrosophic values. This method is suitable if we have a small set of data and experts are able to evaluate the data in the form of bipolar neutrosophic information. Take the values of $r_{ik}$ as $r_{ik} = (t_{ip}, I_{ip}, f_{ip}, t_{in}, I_{in}, f_{in})$.

**Step 1.** Construct the table of the given data.

**Step 2.** Determine the average values using the following bipolar neutrosophic average operator,

$$A_i = \frac{1}{n} \left( \sum_{j=1}^{m} t_{ij}^p - m \prod_{j=1}^{m} t_{ij}^p, \prod_{j=1}^{m} f_{ij}^p, \prod_{j=1}^{m} f_{ij}^n, \sum_{j=1}^{m} t_{ij}^n - m \prod_{j=1}^{m} t_{ij}^n, \sum_{j=1}^{m} f_{ij}^n - m \prod_{j=1}^{m} f_{ij}^n \right), \quad (4)$$

for each $i = 1, 2, \cdots, n$.  

**Step 3.** Construct the weighted average matrix.

Choose the weight vector $w = (w_1, w_2, \cdots, w_n)$ . According to the weights for each alternative, the weighted average table can be calculated by multiplying each average value with the corresponding weight as:

$$\beta_i = A_i w_i, \quad i = 1, 2, \cdots, n.$$  

**Step 4.** Calculate the normalized value for each alternative\risk $\beta_i$ using the formula,

$$\alpha_i = \sqrt{(t_{ip})^2 + (I_{ip})^2 + (f_{ip})^2 + (-1 + t_{in})^2 + (-1 + I_{in})^2 + (-1 + f_{in})^2}, \quad (5)$$

for each $i = 1, 2, \cdots, n$. The resulting table indicate the preference ordering of the alternatives\risks. The alternative\risk with maximum $\alpha_i$ value is most dangerous or more preferable.

**Example 4.1.** Chemical industry is a very important part of human society. These industries contain large amount of organic and inorganic chemicals and materials. Many chemical products have a high risk of fire due to flammable materials, large explosions and oxygen deficiency etc. These accidents can cause the death of employs, damages to building, destruction of machines and transports, economical losses etc. Therefore, it is very important to prevent these accidental losses by identifying the major risks of fire, explosions and oxygen deficiency. A manager of a chemical industry Y wants to prevent such types of accidents that caused the major loss to company in the past. He collected data from witness reports, investigation teams and near by chemical industries and found that the major causes could be the chemical reactions, oxidizing materials, formation of toxic substances, electric hazards, oil spill, hydrocarbon gas leakage and energy systems. The witness reports, investigation teams and industries have
different opinions. There is a bipolarity in people’s thinking and judgement. The data can be considered as bipolar neutrosophic information. The bipolar neutrosophic information about company Y old accidents are given in Table. 3 and Table. 4.

Table 3: Bipolar neutrosophic Data

<table>
<thead>
<tr>
<th></th>
<th>Fire</th>
<th>Oxygen Deficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemical Exposures</td>
<td>(0.5,0.7,0.2,-0.6,-0.3,-0.7)</td>
<td>(0.1,0.5,0.7,-0.5,-0.2,-0.8)</td>
</tr>
<tr>
<td>Oxidizing materials</td>
<td>(0.9,0.7,0.2,-0.8,-0.6,-0.1)</td>
<td>(0.3,0.5,0.2,-0.5,-0.5,-0.2)</td>
</tr>
<tr>
<td>Toxic vapour cloud</td>
<td>(0.7,0.3,0.1,-0.4,-0.1,-0.3)</td>
<td>(0.6,0.3,0.2,-0.5,-0.3,-0.3)</td>
</tr>
<tr>
<td>Electric Hazard</td>
<td>(0.3,0.4,0.2,-0.6,-0.3,-0.7)</td>
<td>(0.9,0.4,0.6,-0.1,-0.7,-0.5)</td>
</tr>
<tr>
<td>Oil Spill</td>
<td>(0.7,0.5,0.3,-0.4,-0.2,-0.2)</td>
<td>(0.2,0.2,0.2,-0.7,-0.4,-0.4)</td>
</tr>
<tr>
<td>Hydrocarbon gas leakage</td>
<td>(0.5,0.3,0.2,-0.5,-0.2,-0.2)</td>
<td>(0.3,0.2,0.3,-0.7,-0.4,-0.3)</td>
</tr>
<tr>
<td>Ammonium Nitrate</td>
<td>(0.3,0.2,0.3,-0.5,-0.6,-0.5)</td>
<td>(0.9,0.2,0.1,0.0,-0.6,-0.5)</td>
</tr>
</tbody>
</table>

Table 4: Bipolar neutrosophic Data

<table>
<thead>
<tr>
<th></th>
<th>Large Explosion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemical Exposures</td>
<td>(0.6,0.2,0.3,-0.4,0.0,-0.1)</td>
</tr>
<tr>
<td>Oxidizing materials</td>
<td>(0.9,0.5,0.5,-0.6,-0.5,-0.2)</td>
</tr>
<tr>
<td>Toxic vapour cloud</td>
<td>(0.5,0.1,0.2,-0.6,-0.2,-0.2)</td>
</tr>
<tr>
<td>Electric Hazard</td>
<td>(0.7,0.6,0.8,-0.7,-0.5,-0.1)</td>
</tr>
<tr>
<td>Oil Spill</td>
<td>(0.9,0.2,0.7,-0.1,-0.6,-0.8)</td>
</tr>
<tr>
<td>Hydrocarbon gas leakage</td>
<td>(0.8,0.2,0.1,-0.1,-0.9,-0.2)</td>
</tr>
<tr>
<td>Ammonium Nitrate</td>
<td>(0.6,0.2,0.1,-0.2,-0.3,-0.5)</td>
</tr>
</tbody>
</table>

By applying the bipolar neutrosophic average operator on Table. 3 and Table. 4, the average values are given in Table. 5.

Table 5: Bipolar neutrosophic average values

<table>
<thead>
<tr>
<th></th>
<th>Average Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemical Exposures</td>
<td>(0.39,0.023,0.014,-0.04,-0.167,-0.515)</td>
</tr>
<tr>
<td>Oxidizing materials</td>
<td>(0.619,0.032,0.001,-0.08,-0.483,-0.165)</td>
</tr>
<tr>
<td>Toxic vapour cloud</td>
<td>(0.53,0.003,0.001,-0.04,-0.198,-0.261)</td>
</tr>
<tr>
<td>Electric Hazard</td>
<td>(0.57,0.032,0.032,-0.014,-0.465,-0.422)</td>
</tr>
<tr>
<td>Oil Spill</td>
<td>(0.558,0.007,0.014,-0.009,-0.384,-0.445)</td>
</tr>
<tr>
<td>Hydrocarbon gas leakage</td>
<td>(0.493,0.004,0.002,-0.011,-0.543,-0.229)</td>
</tr>
<tr>
<td>Ammonium Nitrate</td>
<td>(0.546,0.003,0.001,0.00,-0.464,-0.417)</td>
</tr>
</tbody>
</table>

With regard to the weight vector (0.35, 0.80, 0.30, 0.275, 0.65, 0.75, 0.50) associated to each cause of accident, the weighted average values are obtained by multiplying each average value with corresponding weight and are given in Table. 6.
### Table 6: Bipolar neutrosophic weighted average table

<table>
<thead>
<tr>
<th>Weighted Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemical Exposures</td>
</tr>
<tr>
<td>(0.1365,0.0081,0.0049,-0.0140,-0.0585,-0.1803)</td>
</tr>
<tr>
<td>Oxidizing materials</td>
</tr>
<tr>
<td>(0.4952,0.0256,0.0008,-0.0640,-0.3864,-0.1320)</td>
</tr>
<tr>
<td>Toxic vapour cloud</td>
</tr>
<tr>
<td>(0.1590,0.0009,0.0003,-0.012,-0.0594,-0.0783)</td>
</tr>
<tr>
<td>Electric Hazard</td>
</tr>
<tr>
<td>(0.2850,0.0160,0.0160,-0.0070,-0.2325,-0.2110)</td>
</tr>
<tr>
<td>Oil Spill</td>
</tr>
<tr>
<td>(0.1535,0.0019,0.0039,-0.0025,-0.1056,-0.1224)</td>
</tr>
<tr>
<td>Hydrocarbon gas leakage</td>
</tr>
<tr>
<td>(0.3205,0.0026,0.0013,-0.0072,-0.3530,-0.1489)</td>
</tr>
<tr>
<td>Ammonium Nitrate</td>
</tr>
<tr>
<td>(0.4095,0.0023,0.0008,0.0,-0.3480,-0.2110)</td>
</tr>
</tbody>
</table>

Using Formula 5, the resulting normalized values are shown in Table 8.

### Table 7: Normalized values

<table>
<thead>
<tr>
<th>Normalized value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemical Exposures</td>
</tr>
<tr>
<td>1.5966</td>
</tr>
<tr>
<td>Oxidizing materials</td>
</tr>
<tr>
<td>1.5006</td>
</tr>
<tr>
<td>Toxic vapour cloud</td>
</tr>
<tr>
<td>1.6540</td>
</tr>
<tr>
<td>Electric Hazard</td>
</tr>
<tr>
<td>1.6090</td>
</tr>
<tr>
<td>Oil Spill</td>
</tr>
<tr>
<td>1.4938</td>
</tr>
<tr>
<td>Hydrocarbon gas leakage</td>
</tr>
<tr>
<td>1.6036</td>
</tr>
<tr>
<td>Ammonium Nitrate</td>
</tr>
<tr>
<td>1.5089</td>
</tr>
</tbody>
</table>

The accident possibilities can be placed in the following order: toxic vapour cloud $\succ$ electric hazard $\succ$ hydrocarbon gas leakage $\succ$ chemical exposures $\succ$ ammonium nitrate $\succ$ oxidizing materials $\succ$ Oil spill where, the symbol $\succ$ represents partial ordering of objects. It can be easily seen that the formation of toxic vapour clouds, electrical and energy systems and hydrocarbon gas leakage are the major dangers to the chemical industry. There is a very little danger due to oil spill. Chemical exposures, oxidizing materials and ammonium nitrate has an average accidental danger. Therefore, industry needs special precautions to prevent the major hazards that could happen due the formation of toxic vapour clouds.

### (2) Domination in bipolar neutrosophic graphs

Domination has a wide variety of applications in communication networks, coding theory, fixing surveillance cameras, detecting biological proteins and social networks etc. Consider the example of a TV channel that wants to set up transmission stations in a number of cities such that every city in the country get access to the channel signals from at least one of the stations. To reduce the cost for building large stations it is required to set up minimum number of stations. This problem can be represented by a bipolar neutrosophic graph in which vertices represent the cities and there is an edge between two cities if they can communicate directly with each other. Consider a network of ten cities $\{C_1, C_2, \ldots, C_{10}\}$. In the bipolar neutrosophic graph, the positive degree of each vertex represents the level of truth, indeterminacy and falsity of strong signals it can transmit to other cities and the negative degree of each vertex represents the level of truth, indeterminacy and falsity of weaker signals it can transmit to other cities.
The bipolar neutrosophic value of each edge represents the degree of truth, indeterminacy and falsity of strong and weak communication between the cities. The graph is shown in Fig. 4.1. $D = \{C_8, C_{10}\}$ is the minimum dominating set. It is concluded that building only two large transmitting stations in $C_8$ and $C_{10}$, a high economical benefit can be achieved.
The method of calculating the minimum number of stations is described in the following Algorithm 1.

Table 8: Algorithm for the selection of minimum locations

<table>
<thead>
<tr>
<th>Algorithm 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Begin</td>
</tr>
<tr>
<td>2. Enter the membership values $B(x_i)$ of $n$ number of possible locations $A = {C_1, C_2, \ldots, C_n}$.</td>
</tr>
<tr>
<td>3. Input the adjacency matrix $[C_{ij}]_{n \times n}$ of transmission stations.</td>
</tr>
<tr>
<td>4. $k = 0$</td>
</tr>
<tr>
<td>5. $D = \emptyset$</td>
</tr>
<tr>
<td>6. do $i$ from 1 $\rightarrow$ n</td>
</tr>
<tr>
<td>7. do $j$ from $i + 1$ $\rightarrow$ n</td>
</tr>
<tr>
<td>8. if $C_{ij} = A(C_i) \cap A(C_j)$ then</td>
</tr>
<tr>
<td>9. $C_i \in D$, $k = k + 1$, $x_k = C_i$</td>
</tr>
<tr>
<td>10. end if</td>
</tr>
<tr>
<td>11. end do</td>
</tr>
<tr>
<td>12. end do</td>
</tr>
<tr>
<td>13. Arrange $X \setminus D = {x_{k+1}, x_{k+2}, \ldots, x_n} = J$</td>
</tr>
<tr>
<td>14. do $i$ from 1 $\rightarrow$ k</td>
</tr>
<tr>
<td>15. $D' = D \setminus {x_i}$</td>
</tr>
<tr>
<td>16. if $D'$ is a dominating set then</td>
</tr>
<tr>
<td>17. $D = D'$</td>
</tr>
<tr>
<td>18. $J = J \cup {x_i}$</td>
</tr>
<tr>
<td>19. end if</td>
</tr>
<tr>
<td>20. end do</td>
</tr>
<tr>
<td>21. if $D \cup J = Y$ then</td>
</tr>
<tr>
<td>22. print: $D$ is a minimal dominating set.</td>
</tr>
<tr>
<td>23. else</td>
</tr>
<tr>
<td>24. print: There is no dominating set.</td>
</tr>
<tr>
<td>25. end if</td>
</tr>
</tbody>
</table>

5 Comments on Broumi et al.’s Bipolar Neutrosophic Graphs

Broumi et al. [11] defined bipolar single-valued neutrosophic graphs in the following way:

**Definition 5.1.** [11] Let $A = (T_A, I_A, F_A, T^N_A, I^N_A, F^N_A)$ and $B = (T_B, I_B, F_B, T^N_B, I^N_B, F^N_B)$ be bipolar single valued neutrosophic graph on set $Y$. If $B = (T^P_B, I^P_B, F^P_B, T^N_B, I^N_B, F^N_B)$ is a bipolar single valued neutrosophic relation on $A = (T_A, I_A, F_A, T^N_A, I^N_A, F^N_A)$ then

$$T^P_B(xy) \leq T^P_A(x) \wedge T^P_A(y), \quad I^P_B(xy) \geq I^P_A(x) \vee I^P_A(y), \quad F^P_B(xy) \geq F^P_A(x) \vee F^P_A(y),$$

$$T^N_B(xy) \geq T^N_A(x) \vee T^N_A(y), \quad I^N_B(xy) \leq I^N_A(x) \wedge I^N_A(y), \quad F^N_B(xy) \leq F^N_A(x) \wedge F^N_A(y),$$

for all $x, y \in Y$.

Broumi et al. [11] defined complement of a bipolar neutrosophic graph as follows:
Figure 5.1: Bipolar single valued neutrosophic graph $G$ according to [11]

Definition 5.2. [11] The complement of a bipolar neutrosophic graph $G = (A, B)$ is a bipolar single valued neutrosophic graph $\overline{G} = (\overline{A}, \overline{B})$ where, $\overline{A} = A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N)$, $\overline{B} = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N)$ and is defined by

$$\overline{T}_B^P(xy) = T_A^P(x) \land T_A^P(y) - T_B^P(xy), \quad \overline{I}_B^P(xy) = I_A^P(x) \lor I_A^P(y) - I_B^P(xy),$$

$$\overline{F}_B^P(xy) = F_A^P(x) \lor F_A^P(y) - F_B^P(xy), \quad \overline{T}_B^N(xy) = T_A^N(x) \lor T_A^N(y) - T_B^N(xy),$$

$$\overline{I}_B^N(xy) = I_A^N(x) \land I_A^N(y) - I_B^N(xy), \quad \overline{F}_B^N(xy) = F_A^N(x) \land F_A^N(y) - F_B^N(xy),$$

for all $xy \in Y \times Y$.

We illustrate Definitions 5.1-5.2 [11] by the following example.

Example 5.1. The complement $\overline{G}$ of $G$ is obtained by using Definitions 5.1-5.2 [11] as shown in Fig. 5.2. It can be seen from Fig. 5.2 that $P_B^p(yz)$, $P_B^p(yz)$, $I_B^p(yt)$, $I_B^p(yt) \not\in [0, 1]$ and $I_B^n(yz)$, $I_B^n(yz)$, $I_B^n(yt)$, $I_B^n(yt) \not\in [-1, 0]$. Hence $\overline{G}$ is not a bipolar single valued neutrosophic graph.

6 Conclusions

Bipolar fuzzy graph theory has many applications in science and technology. A bipolar neutrosophic graph is a generalization of the notion bipolar fuzzy graph. We have introduced the idea of bipolar neutrosophic graph and operations on bipolar neutrosophic graphs. We have investigated the dominating and independent sets of certain graph products. Two applications of bipolar neutrosophic sets and bipolar neutrosophic graphs are studied in chemical industry and construction of radio channels. We are planning to extend our research of fuzzification to (1) Bipolar fuzzy rough graphs; (2) Bipolar fuzzy rough hypergraphs, (3) Bipolar fuzzy rough neutrosophic graphs, and (4) Decision support systems based on bipolar neutrosophic graphs.
References


