On a Q-Smarandache Fuzzy Commutative Ideal of a Q-Smarandache BH-algebra

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Abstract

In this paper, the notions of Q-Smarandache fuzzy commutative ideal and Q-Smarandache fuzzy sub-commutative ideal of a Q-Smarandache BH-Algebra are introduced, examples and related properties are investigated. Also, the relationships among these notions and other types of Q-Smarandache fuzzy ideal of a Q-Smarandache BH-Algebra are studied.

Mathematics Subject Classification: 06F35, 03G25, 08A72

Keywords: BCK-algebra, BCH-algebra, BH-algebra, Q-Smarandache BH-algebra, Q-Smarandache fuzzy ideal of Q-Smarandache BH-algebra

1 Introduction

The concept of BCK-algebra was introduced by Y. Imai and K. Iseki [18]. In 1995 the concept of n-fold commutative BCK-algebras has been introduced [7]. In 1998, Y.B. Jun, E.H. Roh and H.S. Kim introduced the
notion of BH-algebra, which is a generalization of BCH/BCI/BCK-algebra [15]. In 2005, Y.B. Jun introduced the notion of a Smarandache BCI-algebra, Smarandache ideal of a Smarandache BCI-algebra [13]. In 2009, A.B. Saeid and A. Namdar, introduced the notion of a Q-Smarandache BCH-algebra and Q-Smarandache ideal of Q-Smarandache BCH-algebra [1]. In 2015, H.H. Abbass and H.K. Gatea introduced the notion Q-Smarandache Sub-Commutative ideal of a Q-Smarandache BH-Algebra [4]. In this paper we introduce the notion of Q-Smarandache fuzzy Commutative ideal and Q-Smarandache fuzzy Sub-Commutative ideal of a Q-Smarandache BH-Algebra. In this paper X denotes Q-Smarandache BH-Algebra.

2 Preliminary Notes

In this section, some basic concepts about a BH-algebra, a Q-Smarandache BH-algebra, a Q-Smarandach ideal in ordinary and fuzzy sences, Q-Smarandache sub-commutative ideal and Q-Smarandache commutative ideal of a Q-Smarandache BH-algebra are given.

**Definition 2.1.** [14]. A BCI-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$, where $X$ is a nonempty set, $*$ is a binary operation and $0$ is a constant, satisfying the following axioms: for all $x, y, z \in X$:

i. $((x * y) * (x * z)) * (z * y) = 0$,

ii. $(x * (x * y)) * y = 0$,

iii. $x * x = 0$,

iv. $x * y = 0$ and $y * x = 0$ imply $x = y$.

**Definition 2.2.** [11]. BCK-algebra is a BCI-algebra satisfying the axiom: $0 * x = 0$ for all $x \in X$.

**Definition 2.3.** [15]. A BH-algebra is a nonempty set $X$ with a constant 0 and a binary operation $*$ satisfying the following conditions:

i. $x * x = 0$, $\forall x \in X$.

ii. $x * y = 0$ and $y * x = 0$ imply $x = y$, $\forall x, y \in X$.

iii. $x * 0 = x$, $\forall x \in X$.

**Definition 2.4.** [2].
A BCK-algebra $X$ is called commutative if $x * (x * y) = y * (y * x), \forall x, y \in X$. 


Lemma 2.5. [2]
In a BCI-algebra $X$ the following conditions are equivalent:

i. $x * y = x * (y * (y * x))$, $\forall x, y \in X$.

ii. $X$ is a commutative BCK-algebra

Definition 2.6. [6]. A Q-Smarandache BH-algebra is defined to be a BH-algebra $X$ in which there exists a proper subset $Q$ of $X$ such that

i. $0 \in Q$ and $|Q| \geq 2$.

ii. $Q$ is a BCK-algebra under the operation of $X$.

Definition 2.7. [4]. A Q-Smaradache BH-algebra is said to be a Q-Smaradache implicative BH-algebra if it satisfies the condition, $(x*(x*y))*(y*x)=y*(y*x)$ $\forall x, y \in Q$.

Definition 2.8. [4]. A Q-Smaradache BH-algebra $X$ is called a Q-Smaradache medial BH-algebra if $x*(x*y)=y, \forall x, y \in Q$.

Definition 2.9. [6]. A nonempty subset $I$ of $X$ is called a Q-Smaradache ideal of $X$, denoted by a Q-S.I of $X$ if it satisfies:

$(J_1)$ $0 \in I$.

$(J_2)$ $\forall y \in I$ and $x * y \in I \implies x \in I, \forall x \in Q$.

Definition 2.10. [4]. A subset $I$ of a BH-algebra $X$ is called commutative ideal of $X$ if it satisfies $(J_1)$ and:

$(J_3)$ $(x * y) * z \in I$ and $z \in I \Rightarrow x * (y * (y * x)) \in I, \forall x, y, z \in X$.

Definition 2.11. [4]. A subset $I$ of a Q-Smaradache BH-algebra $X$ is called a Q-Smaradache commutative ideal of $X$ if it satisfies $(J_1)$ and:

$(J_4)$ $(x * y) * z \in I$ and $z \in I \Rightarrow x * (y * (y * x)) \in I, \forall x, y \in Q$ and $z \in X$.

Definition 2.12. [4]. A nonempty subset $I$ of a BH-algebra $X$ is called sub-commutative ideal of $X$ if it satisfies $(J_1)$ and:

$(J_5)$ $(y * (x * (x * y))) * z \in I$ and $z \in I \implies x * (x * y) \in I, \forall x, y \in Q, z \in X$.

Definition 2.13. [12] A fuzzy subset $A$ of a BH-algebra $X$ is said to be a fuzzy ideal if and only if:
(I₁) \( A(0) \geq A(x), \forall x \in X \).

(II) \( A(x) \geq \min\{A(x \ast y), A(y)\}, \forall x, y \in X \).

**Definition 2.14.** [16] Let \( X \) be a BCK-algebra. A fuzzy set \( A \) in \( X \) is called a fuzzy commutative ideal of \( X \) if it satisfies

(III) \( A((x \ast (y \ast (y \ast x)))) \geq \min\{((x \ast y) \ast z), (z)\} \quad \forall x, y, z \in X \).

We generalize the concept of a Q-Smarandache fuzzy commutative ideal to the Q-Smarandache BH-algebra.

**Definition 2.15.** A fuzzy subset \( A \) of a BH-algebra \( X \) is called a fuzzy commutative ideal of \( X \), denoted by a F.C.I if it satisfies

(IV) \( A((x \ast (y \ast (y \ast x)))) \geq \min\{((x \ast y) \ast z), (z)\} \quad \forall x, y, z \in X \).

**Definition 2.16.** [10]. Let \( A \) be a fuzzy set in \( X \), \( \forall \alpha \in [0, 1] \), the set.

\( A_\alpha = \{x \in X, A(x) \geq \alpha\} \) is called a level subset of \( A \).

Note that, \( A_\alpha \) is a subset of \( X \) in the ordinary sense.

**Definition 2.17.** [6]. A fuzzy subset \( A \) of \( X \) is said to be a Q-Smarandache fuzzy ideal of \( X \), denoted by a Q-S.F.I of \( X \):

(F₁) \( A(0) \geq A(x), \forall x \in X \).

(F₂) \( A(x) \geq \min\{A(x \ast y), A(y)\}, \forall x, y \in X \).

3 Main results

In this section, we introduce the concepts of a Q-Smarandache fuzzy commutative ideal and Q-Smarandache fuzzy sub-commutative ideal of a Q-Smarandache BH-algebra, and also we study some properties of them.

**Definition 3.1.** A fuzzy subset \( A \) of a \( X \) is called a Q-Smarandache fuzzy commutative ideal of \( X \), denoted by a Q-S.F.C.I if it satisfies (F₁) and,

(F₃) \( A(x \ast (y \ast (y \ast x))) \geq \min\{A((x \ast y) \ast z), A(z)\}, \text{ for all } x, y \in Q, z \in X \).

**Example 3.2.** Consider \( X = \{0, 1, 2, 3\} \) with binary operation "*" defined by the following table:

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where \( Q = \{0, 1\} \) is a BCK-algebra. The fuzzy subset \( A \) defined by \( A(0) = A(1) = A(2) = 0.6 \) and \( A(3) = 0.3 \).
**Proposition 3.3.** Every Q-S.F.C.I of $X$ is Q-S.F.I of $X$

*Proof.* Let $A$ be a Q-S.F.C.I of $X$ to prove that $A$ is a Q-S.F.I. by Definition (3.1) the condition $(F_1)$ is satisfied. Now, let $x \in Q$ and $y \in X$. we have $x = x \ast (0 \ast (0 \ast x))$. It follows that $A(x) = A(x \ast (0 \ast (0 \ast x))) \geq \min \{A(x \ast 0) \ast y), A(y)\}[by \ 0 \ast x = 0 and \ x \ast 0 = x]$ implies that $A(x) \geq \min \{A(x \ast y), A(y)\}$. Hence $A$ is Q-S.F.I of $X$.

**Remark 3.4.** In the following example, we see that the converse of theorem 3.3 may not be true in general.

**Example 3.5.** Consider $X = \{0, 1, 2, 3, 4\}$ with binary operation ",*" defined by the following table:

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Where $Q = \{0, 2, 3\}$ is a BCK-algebra. The fuzzy subset $A$ defined by $A(0) = 0.7, A(1) = 0.5$ and $A(2) = A(3) = A(4) = 0.3$ is a Q-S.F.I of $X$, but $A$ is not a Q-S.F.C.I since if $x = 2, y = 3, z = 0$, then

$$A(2 \ast (3 \ast (3 \ast 2))) = 0.3 \not\in \min \{A((2 \ast 3) \ast 0), A(0)\} = 0.7$$

**Theorem 3.6.** Let $A$ be a Q-S.F.I of $X$. Then $A$ is a Q-S.F.C.I of $X$ if and only if the level subset $A_\alpha$ is a Q-S.C.I of $X$, $\forall \alpha \in [0, A(0)]$, such that $A(0) = \sup_{x \in X} A(x)$.

*Proof.* Let $A$ be a Q-S.F.C.I of $X$ and $\alpha \in [0, A(0)]$. To prove $A_\alpha$ is a Q-S.C.I of $X$. It is clear that $A(0) \geq \alpha$. So $0 \in A_\alpha$. Hence $A_\alpha$ satisfies $I_1$. Now, let $x, y \in Q, z \in X$ such that $(x \ast y) \ast z \in A_\alpha$ and $z \in A_\alpha$, it follows that $A((x \ast y) \ast z) \geq \alpha$ and $A(z) \geq \alpha$ thus $\min \{A((x \ast y) \ast z), A(z)\} \geq \alpha$. But $A(x \ast (y \ast (x \ast z))) \geq \min \{A((x \ast y) \ast z), A(z)\}[Since A is a Q-S.F.C.I of X]. By definition 3.1(F_3) so $A(x \ast (y \ast (y \ast x))) \geq \alpha \Rightarrow (x \ast (y \ast (y \ast x))) \in A_\alpha$. Therefore, $A_\alpha$ is a Q-S.C.I of $X$.

Conversely, let $A_\alpha$ be a Q-S.C.I. of $X, and \forall \alpha \in [0, A(0)]$. It is clear that $A(0) \geq A(x) \forall x \in X$. Now, let $x, y \in Q, z \in X \alpha = \min \{A((x \ast y) \ast z), A(z)\}$. Then $A((x \ast y) \ast z) \geq \alpha$ and $A(z) \geq \alpha$, it follows that $((x \ast y) \ast z) \in A_\alpha$ and $z \in A_\alpha$, thus $(x \ast (y \ast (y \ast x))) \in A_\alpha[Since A_\alpha is a Q-S.C.I of X] \Rightarrow A(x \ast (y \ast (y \ast x))) \geq \alpha$, we get $A(x \ast (y \ast (y \ast x))) \geq \min \{A((x \ast y) \ast z), A(z)\}$. Therefore, $A$ is a Q-S.F.C.I of $X$.■
**Proposition 3.7.** Let $A$ be a Q-S.F.I of $X$. Then $A$ is a Q-S.F.C.I if and only if $\forall \ x, \ y \in Q; \ A(x*y*(y*x)) \geq A(x*y)$ \hspace{1cm} (b_1)

**Proof.** Let $A$ be a Q-S.F.C.I. Then $A(x*y*(y*x)) \geq A(x*y)$ \hspace{1cm} (F_3). We obtain $A(x*y*(y*x)) \geq min\{A(x*y)*z), A(z)\}$ [By condition (b_1)]. Therefore, $A$ is a Q-S.F.C.I of $X$.

**Theorem 3.8.** Let $A$ be a Q-S.F.I of a commutative Q- Smarandache BH-algebra $X$ such that $Q$ is a commutative BCK-algebra. Then $A$ is a Q-S.F.C.I of $X$.

**Proof.** Let $A$ be a Q-S.F.I of $X$. To prove that $A$ is Q-S.F.C.I. By Definition (2.17) the condition $(F_1)$ is satisfied. Now, let $x, y \in Q$ and $z \in X$. Then $A(x*y) \geq min\{A(x*y)*z), A(z)\}$ [by condition $\forall x \in X$. Hence the condition (b_1) is satisfied Conversely,

Let $A$ be a Q-S.F.I and $x, y \in Q, z \in X$. Then $A(x*y) \geq min\{A(x*y)*z), A(z)\}$ \hspace{1cm} (b_1).

**Definition 3.9.** Let $n$ be a positive integer. A nonempty subset $I$ of $X$ is called a Q-Smarandache $n$-fold commutative ideal of $X$, denoted by a Q-S.$n$-fold C.I of $X$ if it satisfies $(J_1)$ and :

$(J_3) \ (x*y^n)*z \in I \text{ and } z \in I \Rightarrow x*(y^n*(y^n*x)) \in I, \forall x, y \in Q$ and $z \in X.$

**Example 3.10.** Consider $X = \{0,1,2,3,4\}$ with binary operation "*" defined by the following table:

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where $Q=\{0,1\}$ is a BCK-algebra. Then $I = \{0,1,2\}$ is $A$ is a Q-S.2-fold C.I.

**Definition 3.11.** Let $n$ be a positive integer. A fuzzy subset $A$ of a $X$ is called a Q-Smarandache fuzzy $n$-fold commutative ideal of $X$, denoted by a Q-S.$n$-fold C.I of $X$ if it satisfies $(F_1)$ and,

$(F_4) \ A(x*(y^n*(y^n*x)) \geq min\{A(x*y^n)*z), A(z)\}, \forall x, y \in Q, z \in X.$
Example 3.12. Consider \( X = \{0, 1, 2, 3, 4\} \) with binary operation "\(*" defined by the following table:

\[
\begin{array}{c|ccccc}
* & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 \\
2 & 2 & 0 & 0 & 2 & 2 \\
3 & 3 & 4 & 3 & 0 & 3 \\
4 & 4 & 4 & 2 & 0 & 0 \\
\end{array}
\]

where \( Q=\{0,1\} \) is a BCK-algebra. The fuzzy subset \( A \) defined by
\[
A(0) = A(1) = A(2) = 0.8 \quad \text{and} \quad A(3) = A(4) = 0.5
\]
A is a Q-S.F.2-fold.C.I.

Proposition 3.13. Every Q-S.n-fold.F.C.I of \( X \) is Q-S.F.I of \( X \)

Proof. let \( A \) Q-S.F.C.I of \( X \) To prove that \( A \) is Q-S.F.I. by Defintion (3.11) the condition \(( F_1)\) is satisfied .Now, let \( x \in Q \) and \( y \in X \). we have \( x = (x * (0^n *(0^n *x)) \) it follows that \( A(x) = A(x * (0^n *(0^n *x)) \geq \min\{A(x * 0^n )* y), A(y)\}\) by \( 0 * x = 0 \) and \( x * 0 = x \) implies that \( A(x) \geq \min\{A(x * y), A(y)\}\). Hence \( A \) is Q-S.F.I of \( X \).

Remark 3.14. In the following example, we see that the converse of Propo-sition 3.13 may not be true in general.

Example 3.15. Consider \( X = \{0, 1, 2, 3, 4\} \) with binary operation "\(*" defined by the following table:

\[
\begin{array}{c|ccccc}
* & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 4 \\
1 & 1 & 0 & 0 & 1 & 1 \\
2 & 2 & 0 & 0 & 3 & 3 \\
3 & 3 & 3 & 3 & 0 & 4 \\
4 & 4 & 4 & 4 & 4 & 0 \\
\end{array}
\]

where \( Q=\{0,1,2\} \) is a BCK- algebra. The fuzzy subset \( A \) defined by
\[
A(0) = 0.8 \quad \text{and} \quad A(1) = A(2) = A(3) = A(4) = 0.5
\]
is Q-S.F.I of \( X \), but it is not 1-fold Q-S.F.C.I of \( X \). Since \( x=1, y=2, z=0 \)

\[
A(1 * (2 * (2 * 1)) = 0.5 \n\not\geq \min\{A((1 * 2) * 0), A(0)\} = 0.8
\]

Theorem 3.16. Let \( A \) be a Q-S.F.I of \( X \).Then \( A \) is a Q-S.F.n-fold C.I if and only if

\[
\forall x, y \in Q, \quad A(x * (y^n *(y^n *x))) \geq A(x * y^n) \quad (b_2)
\]
Proof. Let $A$ be a Q-S.F.n-fold C.I of $X$ and $x, y \in Q$

\[ A(x \ast (y^n \ast (y^n \ast x)) \geq \min\{A((x \ast y^n) \ast 0), A(0)\} \text{[By definition 3.11 (F_4)]} \]

\[ \implies A(x \ast (y^n \ast (y^n \ast x)) \geq A(x \ast y^n)[\text{Since } x \ast 0 = x, A(0) \geq A(x) \forall x \in X] \]

\[ \implies \text{The condition } (b_2) \text{ is satisfied.} \]

Conversely, let $A$ be a Q-S.F.I of $X$, $x, y \in Q$ and $x \in X$. Then

\[ A(x \ast y^n) \geq \min\{A((x \ast y^n) \ast z), A(z)\}[\text{Since } A \text{ is a Q-S.F.I of } X] \]

\[ \implies A(x \ast (y^n \ast (y^n \ast x)) \geq \min\{A((x \ast y^n) \ast z), A(z)\} \text{ [By condition(b_2)]} \]

Therefore, $A$ is a Q-S.F.n-fold C.I of $X$

**Definition 3.17.** A fuzzy subset $A$ of $X$ is called a Q-Smarandache fuzzy sub-commutative ideal of $X$, denoted by a Q-S.F.S.C.I of $X$ if it satisfies

(F_1) \ $A(x \ast (x \ast y)) \geq \min\{A(y \ast (y \ast (x \ast y))) \ast z), A(z)\} \ \forall \ x, y \in Q, \ z \in X.$

**Example 3.18.** Consider $X = \{0, 1, 2, 3\}$ with binary operation "*" defined by the following table:

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where $Q = \{0,1\}$ is a BCK-algebra. The fuzzy subset $A$ defined by $A(0) = A(1) = A(2) = 0.6 \text{ and } A(3) = 0.3$ A is Q-S.F.C.I of $X$.

**Theorem 3.19.** Let $A$ be a Q-S.F.S.C.I of $X$. Then $A$ is a Q-S.F.I of $X$.

Proof. Let $A$ be a Q-S.F.S.C.I of $X$. It is clear that the condition $(F_1)$ is satisfied. Now, let $x \in Q$ and $y \in X$, we have $A(x \ast (x \ast x)) \geq \min\{A(x \ast (x \ast (x \ast x))) \ast y), A(y)\}$. [By Definition 3.17 $(F_5)$] it follows that $A(x \ast 0) \geq \min\{A(x \ast (x \ast 0) \ast y), A(y)\}[\text{Since } Q \text{ is a BCK-algebra } x \ast x = 0] \implies A(x) \geq \min\{A(x \ast y), A(y)\}[\text{Since } Q \text{ is a BCK-algebra } x \ast 0 = x]$. Hence $A$ is a Q-S.F.I of $X$.

**Remark 3.20.** In the following example shows that the converse of theorem 3.19 may not be true in general.
On a Q-Smarandache fuzzy commutative ideal

Example 3.21. Consider \( X = \{0, 1, 2, 3\} \) with binary operation "\(*" defined by the following table:

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\times & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 \\
2 & 2 & 2 & 0 & 1 \\
3 & 3 & 3 & 3 & 0 \\
\hline
\end{array}
\]

Where \( Q=\{0,1,2\} \) is a BCK-algebra. The fuzzy subset \( A \) defined by \( A(0) = A(3) = 0.9, \) and \( A(1) = A(2) = 0.5 \) is a Q-S.F.I of \( X, \) but it is not a Q-S.F.S.C.I. Since, \( x=1, \ y=2, \ z=0 \)

\[
A(1 \ast (1 \ast 2)) \not\geq \min\{A(2 \ast (2 \ast (1 \ast 2))) \ast 0), A(0)\}
\]

Theorem 3.22. Let \( A \) be a Q-S.F.I of \( X. \) Then \( A \) is a Q-S.F.S.C.I of \( X \) if and only if it is \( \forall \ x, y \in Q, \ A(x \ast (x \ast y)) \geq A(y \ast (y \ast (x \ast y))) \) \( (b_3) \)

Proof. Suppose \( A \) is a Q-S.F.S.C.I of \( X. \) Let \( x, y \in Q. \) Then \( A(x \ast (x \ast y)) \geq \min\{A(y \ast (y \ast (x \ast y))) \ast 0), A(0)\} \) \( \) [By definition 3.17(\( F_5 \))] it follows that \( A(x \ast (x \ast y)) = \min\{A(y \ast (y \ast (x \ast y))), A(0)\} \) \( \) [Since \( X; x \ast 0 = x\) implies that \( A(x \ast (x \ast y)) \geq A(y \ast (y \ast (x \ast y))) \) \( \) \( \) \( A(0) \geq A(x) \forall x \in X. \) By definition 3.17(\( F_1 \)). Hence The condition \( (b_3) \) is satisfied.

Conversely, Let \( A \) be a Q-S.F.I of \( X \) and the condition \( (b_2) \) satisfied. To prove that \( A \) is a Q-S.F.S.C.I. By Definition (2.17) the condition \( (F_2) \) is satisfied. Now, let \( x, y \in Q \) and \( z \in X \) we have \( A(y \ast (y \ast (x \ast (x \ast y)))) \geq \min\{A(y \ast (y \ast (x \ast (x \ast y)))) \ast z), A(z)\} \) \( \) [Since \( A \) is a Q-S.F.I of \( X, \) by Definition 2.17 \( (F_2) \)] implies that \( A(x \ast (x \ast y)) \geq \min\{A(y \ast (y \ast (x \ast (x \ast y)))) \ast z), A(z)\} \) \( \) [By \( b_3 \) \]. Hence \( A \) is a Q-S.F. S.C.I of \( X. \)

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