On a Q-Smarandache Implicative Ideal with Respect to an Element of a Q-Smarandache BH-algebra

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Abstract
In this paper, we define the concept of a Q-Smarandache implicative ideal with respect to an element of a Q-Smarandache BH-algebra. We state and prove some theorems which determine the relationships among this notion and other types of ideals of a Q-Smarandache BH-algebra.

1. INTRODUCTION
The notion of BCK-algebra and BCI-algebra was formulated first in 1966 by Y. Imai and K. Iseki as a generalization of the concept of set-theoretic difference and propositional calculus [4]. In 1983, Q.P. Hu and X. Li introduced the notion of BCH-algebra which are generalization of BCKBCI-algebra [5]. In 1998, Y. B. Jun, E. H. Roh and H. S. Kim introduced the notion of BH-algebra, which is a generalization of BCH-algebra [7]. In 2009, A. B. Saeid and A. Namdar introduced the notion of a Q-Smarandache BCH-algebra and Q-Smarandache ideal of a Q-Smarandache BCH-algebra, these notion were generalized to BH-algebra in 2012 by H. H. Abbass and S. A. Neamah introduced the notion of an implicative ideal with respect to an element of a BH-algebra[1]. In this paper, a new type of a Q-Smarandache ideal of Q-Smarandache BH-algebra, namely a Q-Smarandache implicative ideal with respect to an element is introduced some related properties investigated.

2. PRELIMINARIES
In this section, we review some basic concepts about a BCK-algebra, BCI-algebra, BCH-algebra, BH-algebra, Smarandache BH-algebra, (ideal, positive implicative and implicative ideal with respect to an element) of a BH-algebra and Q-Smarandache ideal of a Q-Smaradache BH-algebra, with some theorems and propositions.

Definition (2.1) :[8]
A BCI-algebra is an algebra \((X, *, 0)\), where \(X\) is a nonempty set, "*" is a binary operation and 0 is a constant, satisfying the following axioms:
i. \((x*y)*(x*z))*(z*y) = 0\), for all \(x, y, z \in X\).
ii. \((x*(x*y))*y = 0\), for all \(x, y \in X\).
iii. \(x * x = 0\), for all \(x \in X\).
iv. \(x * y = 0 \) and \(y * x = 0\) imply \(x = y\), for all \(x, y \in X\).

**Definition (2.2) :**[4]

A **BCK-algebra** is a BCI-algebra satisfying the axiom: \(0 * x = 0\), for all \(x \in X\).

**Definition (2.3):**[5]

A **BCH-algebra** is an algebra \((X,*,0)\), where \(X\) is a nonempty set, "*" is a binary operation and 0 is a constant, satisfying the following axioms:

i. \(x * x = 0\), \(\forall x \in X\).
ii. \(x * y = 0 \) and \(y * x = 0\) imply \(x = y\), \(\forall x, y \in X\).
iii. \((x * y) * z = (x * z) * y\), \(\forall x, y, z \in X\).

**Definition (2.4) :**[7]

A **BH-algebra** is a nonempty set \(X\) with a constant 0 and a binary operation "*" satisfying the following conditions:

i. \(x*x=0\), \(\forall x \in X\).
ii. \(x*y=0 \) and \(y*x =0\) imply \(x = y\), \(\forall x, y \in X\).
iii. \(x*0 =x\), \(\forall x \in X\).

**Definition (2.5) :**[3]

A bounded **BCK-algebra** satisfying the identity \(x * (y * x) = x, \forall x, y \in X\).

**Definition (2.6) :**[7]

Let \(I\) be a nonempty subset of a BH-algebra \(X\). Then \(I\) is called an **ideal** of \(X\) if it satisfies:

i. \(0 \in I\).
ii. \(x*y \in I\) and \(y \in I\) imply \(x \in I\).

**Definition (2.7):**[1]

A nonempty subset \(I\) of a BH-algebra \(X\) is called an **implicative ideal with respect to an element \(b\) of a BH- Algebra** (or briefly **b-implicative ideal**), \(b \in X\), if

i. \(0 \in I\).
ii. \(((x*(y*x))*z)*b \in I\) and \(z \in I\) imply \(x \in I, \forall x, y, z \in X\).

**Definition (2.8):**[6]

A BH-algebra \((X,*,0)\) is said to be a **positive implicative** if it satisfies for all \(x, y \) and \(z \in X\), 
\((x*z)*(y*z) = (x*y)*z\).

**Definition (2.9):**[2]

A **Smarandache BH-algebra** is defined to be a BH-algebra \(X\) in which there exists a proper subset \(Q\) of \(X\) such that:

i. \(0 \in Q\) and \(|Q| \geq 2\).
ii. \(Q\) is a BCK-algebra under the operation of \(X\).

**Definition (2.10):**[2]

Let \(X\) be a Smarandache BH-algebra. A nonempty subset \(I\) of \(X\) is called a **Smarandache ideal of \(X\) related to \(Q\)** (or briefly, **Q-Smarandache ideal** of \(X\)) if it satisfies:

i. \(0 \in I\).
ii. \(\forall y \in I\) and \(x*y \in I \Rightarrow x \in I, \forall x \in Q\).
Proposition (2.11) : [2]
Let \( \{ I_i, i \in \lambda \} \) be a family of Q-Smarandache ideals of a Smarandache BH-algebra X. Then \( \bigcap_{i \in \lambda} I_i \) is a Q-Smarandache ideal of X.

Proposition (2.12) : [2]
Let \( \{ I_i, i \in \lambda \} \) be a chain of a Q-Smarandache ideals of a Smarandache BH-algebra X. Then \( \bigcup_{i \in \lambda} I_i \) is a Q-Smarandache ideal of X.

Proposition (2.13) : [2]
Let X be a Smarandache BH-algebra. Then every ideal of X is a Q-Smarandache ideal of X.

Theorem (2.14) : [2]
Let \( Q_1 \) and \( Q_2 \) be BCK-algebras contained in a Smarandache BH-algebra X and \( Q_1 \subseteq Q_2 \). Then every Smarandache ideal of X related to \( Q_2 \) is a Smarandache ideal of X related to \( Q_1 \).

3. THE MAIN RESULTS
In this section, we introduce the concept of a Q-Smarandache implicative ideal of a Q-Smarandache BH-algebra. Also, we state and prove some theorems and examples about these concepts.

Definition (3.1):
Let I be a Q-Smarandach ideal of a Q-Smarandache BH-algebra X and \( b \in X \). Then I is called a Q-Smarandache implicative ideal with respect to b (denoted by a Q-Smarandache b-implicative ideal) if:
\[
((x*(y*x))*z)*b \in I \text{ and } z \in I \text{ imply } x \in I, \forall x, y \in Q.
\]

Example (3.2):
Consider the Q-Saramdache BH-algebra X = \{0, 1, 2, 3\} with the binary operation "*" defined by the following table:

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where \( Q = \{0, 2\} \) is a BCK-algebra. The Q-Smarandache ideal I = \{0, 1\} is a Q-Smarandache 0-implicative ideal of X, so I be a Q-Smarandache 1,3-implicative ideal of X, but it is not a Q-Smarandache 2-implicative ideal of X. Since, \( x=2, y=2, z=0 \), \( ((2*(2*2))*0)*2=((2*0)*2=2*2=0 \in 1 \), but \( x=2 \notin 1 \).

Proposition (3.3):
Let X be a Q-Smarandache BH-algebra. Then every b-implicative ideal of X is a Q-Smarandache b-implicative ideal of X, \( \forall b \in X \).
Proof:
Let $I$ be a $b$-implicative ideal of $X$, $\forall b \in X$.
Now, let $x,y \in Q$ and $z \in I$ such that $(x^*(y^*x))^* z \in I$ and $z \in I$.
Since $x,y \in Q \Rightarrow x,y \in X$. [Since $Q \subseteq X$]
Now, we have
$(x^*(y^*x))^* b \in I$ and $z \in I$.
$\Rightarrow x \in I$. [Since $I$ is a $b$-implicative ideal of $X$, by Definition (2.7) (ii)]
Therefore, $I$ is a $Q$-Smarandache $b$-implicative ideal of $X$. $\blacksquare$

Remark (3.4):
The following example shows that converse of Proposition (3.3) is not correct in general.

Example (3.5):
Consider the $Q$-Smarandache BH-algebra $X=\{0,1,2,3\}$ with binary operation "$*$" defined by the following table:

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where $Q=\{0,1\}$ is a BCK-algebra.
The $Q$-Smarandache ideal $I=\{0,2\}$ is a $Q$-Smarandache 2-implicative ideal of $X$, but it is not an 2-implicative ideal of BH-algebra. Since, $x=3$, $y=0$, $z=2$, 
$((3^*(0^*3))^*2=2^*2=3 \in I,$ but $3 \notin I$.

Theorem (3.6):
Let $(N,^*)$ be a $Q$-Smarandache BH-algebra, where $N=\{0,1,2,\ldots\}$, "$^*$" be a binary operation defined on $N$ by:

$x^*y=\begin{cases} x & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$, $\forall x,y \in N$

and $Q=\{4k \mid k \in N\}$ is a BCK-algebra. Then $I=\{2k \mid k \in N\}$ is a $Q$-Smarandache $b$-implicative ideal of $N$, $\forall b \in I$.

Proof:
It is clear $I$ is a $Q$-Smarandache ideal of $N$.
Now, let $x,y \in Q$ and $z,b \in I$ such that $(x^*(y^*x))^* z \in I$ and $z \in I$.
$\Rightarrow (x^*(y^*x))^* z \in I$. [Since $I$ is a $Q$-Smarandache ideal of $X$]
$\Rightarrow (x^*(y^*x)) \in I$. [Since $I$ is a $Q$-Smarandache ideal of $X$]

Case 1: if $x = y$, then $x^*(y^*x)=x^*(x^*x)=x^*0=x$
[Since $Q$ is a BCK-algebra; $x^*x=0$ and $x^*0=x$, $\forall x \in Q$]
$\Rightarrow x \in I$. [Since $x^*(y^*x) \in I$]

Case 2: if $x \neq y$, then $x^*(y^*x) = x^*y = x$. [Since $x^*y = x$]
Therefore, I is a Q-Smarandach b-implicative ideal of X, ∀ b ∈ I. ■

**Theorem (3.7):**
Let Q₁ and Q₂ be a two BCK-algebras contained in Q₂-Smarandache BH-algebra X such that Q₁ ⊆ Q₂ and b ∈ X. Then every a Q₂-Smarandache b-implicative ideal of X is a Q₁-Smarandache b-implicative ideal of X.

**Proof:**
Let ı be a Q₂-Smarandache b-implicative ideal of X.
⇒ I is a Q₂-Smarandache ideal of X. [By Definition (3.1)]
⇒ I is a Q₁-Smarandache ideal of X. [By Theorem (2.14)]
Now, let x, y ∈ Q₁ and z ∈ I such that ((x*(y*x)) * z)*b ∈ I.
Since x, y ∈ Q₁ ⇒ x, y ∈ Q₂. [Since Q₁ ⊆ Q₂]
Now, we have
((x*(y*x)) * z)*b ∈ I and x, y ∈ Q₂, z ∈ I.
⇒ x ∈ I. [Since I is a Q₂-Smarandache b-implicative ideal of X]
Therefore, I is a Q₁-Smarandache b-implicative ideal of X. ■

**Remark (3.8):**
The converse of Theorem (3.7) is not correct in general as in the following example.

**Example (3.9):**
Consider the Q-Smarandache BH-algebra X={0,1,2,3,4} with binary operation "*" defined by the following table:

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</table>

where Q₁={0,1}, Q₂={0,1,3} are two BCK-algebras such that Q₁ ⊆ Q₂. The Q-Smarandache ideal I={0,1,4} is a Q₁-Smarandache 4-implicative ideal of X, but it is not Q₂-Smarandache 4-implicative ideal of X. Since, x=3, y=0, z =1, ((3*(0*3))*1)*4=((3*0)*1)*4=(3*1)*4=1*4=1 ∈ I, but x=3 ∉ I.

**Theorem (3.10):**
Let I be a Q-Smarandache ideal of a Q-Smarandache BH-algebra X. Then I is a Q-Smarandache b-implicative ideal of X if and only if for all x, y ∈ X and b ∈ I, x*(y*x) ∈ I imply x ∈ I.
Proof:
Let I be a Q-Smarandache b-implicative ideal of X, ∀ b ∈ I.
Now, let x*(y*x) ∈ I.
Then x*(y*x) = (x*(y*x)) *0 =((x*(y*x)) *0 )*0.
[Since Q is a BCK-algebra; x*0 = x, ∀x ∈ Q]
Then, we have
((x*(y*x)) *0)) * 0 ∈ I and 0 ∈ I implies that x ∈ I. [Since I is a Q-Smarandache 0-implicative ideal of X ]

Conversely, suppose that I is a Q-Smarandache ideal of X and the condition is satisfied.
Let x, y ∈ Q and z, b ∈ I such that ((x*(y*x)) * z)*b ∈ I.
⇒ (x*(y*x))*z ∈ I. [Since I is a Q-Smarandache ideal of X, by Definition (2.10)(ii)]
⇒ x*(y*x) ∈ I. [Since I is a Q-Smarandache ideal of X, by Definition (2.10)(ii)]
⇒ x ∈ I. [By hypothesis ]
Therefore, I is a Q-Smarandache b-implicative ideal of X.

Theorem (3.11):
Let X be a positive implicative Q-Smarandache BH-algebra and I be a Q-Smarandache ideal of X such that Q * I ⊆ I. Then I is a Q-Smarandache b-implicative ideal of X, ∀ b ∈ I.

Proof:
Let I be a Q-Smarandache ideal of X such that Q * I ⊆ I.
Now, let x, y ∈ Q and z, b ∈ I such that ((x*(y*x)) * z)*b ∈ I.
⇒ (x*(y*x))*z ∈ I. [Since I is a Q-Smarandache ideal of X, by Definition (2.10)(ii)]
But (x*(y*x))*z = (x*z) *((y*x)*z). [Since X is a positive implicative BH-algebra ]
Now x, y ∈ Q ⇒ y*x ∈ Q, so (y*x)*z ∈ I. [Since Q * I ⊆ I ]
So, we have
(x*z) *((y*x)*z) ∈ I and ((y*x)*z) ∈ I.
⇒ x*z ∈ I. [Since I is Q-Smarandache ideal of X ]
⇒ x ∈ I. [Since I is Q-Smarandache ideal of X ]
Therefore, I is a Q-Smarandache b-implicative ideal of X.

Theorem (3.12):
Let X be a bounded Q-Smarandache BH-algebra such that Q is a bounded BCK-algebra and I be a Q-Smarandache ideal of X. Then I is a Q-Smarandache b-implicative ideal of X, ∀ b ∈ I.

Proof:
Let I be a Q-Smarandache ideal of X.
Now, let x, y ∈ Q and z, b ∈ I such that ((x*(y*x)) *z)*b ∈ I.
⇒ (x*(y*x))*z ∈ I. [Since I is a Q-Smarandache ideal of X ]
⇒ (x*(y*x)) ∈ I. [Since I is a Q-Smarandache ideal of X ]
⇒ x ∈ I. [Since Q is a bounded BCK algebra, by Definition (2.5)]
Therefore, I is a Q-Smarandache b-implicative ideal of X.

Theorem (3.13):
Let X be a Q-Smarandache BH-algebra and satisfies the following condition:
∀ x, y ∈ Q, x*y = x with x ≠ y
and I be a Q-Smarandache ideal of X. Then I is a Q-Smarandache b-implicative ideal of X, ∀ b ∈ I.

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Proof:
Let I be a Q-Smarandache ideal of X.
Now, let \( x, y \in Q \) and \( z, b \in I \) such that \((x*(y*x))*z)*b \in I\).
\( \Rightarrow (x*(y*x))*z \in I \). [Since I is a Q-Smarandache ideal of X ]
\( \Rightarrow x*(y*x) \in I \). [Since I is a Q-Smarandache ideal of X ]

Now, we have two cases:

Case 1: if \( x=y \), then \( x*(0*x) = x*0 = x \).
[ Since Q is a BCK-algebra; \( x*0 = x \) and \( 0*x = 0 \) ]
\( \Rightarrow x \in I \). [Since \( x*(y*x) \in I \)]
\( \Rightarrow \) I is a Q-Smarandache b-implicative ideal of X.

Case 2: if \( x \neq y \), then \( x*(y*x) = x*y = x \). [Since \( x*y = x \)]
\( \Rightarrow x \in I \). [Since \( x*(y*x) \in I \)]
Therefore, I is a Q-Smarandache b-implicative ideal of X. ■

Theorem (3.14):
Let \( X \) be a Q-Smarandache BH-algebra and satisfies the condition:
\( \forall x, y \in Q ; \ x = x *(y *x) \), and I be a Q-Smarandache ideal of X. Then I is a Q-Smarandache b-implicative ideal of X, \( \forall b \in I \).

Proof:
Let I be a Q-Smarandache ideal of X.
Now, let \( x, y \in Q \) and \( z, b \in I \) such that \((x*(y*x))*z)*b \in I\).
\( \Rightarrow (x*(y*x))*z \in I \). [Since I is a Q-Smarandache ideal of X ]
\( \Rightarrow x*(y*x) \in I \). [Since I is a Q-Smarandache ideal of X ]

Case 1: if \( y = 0 \), then \( x*(0*x) = x*0 = x \).
[ Since Q is a BCK-algebra; \( x*0 = x \), \( 0*x = 0 \), \( \forall x \in Q \) ]
\( \Rightarrow x \in I \).
Hence I is a Q-Smarandache implicative ideal of X. ■

Case 2: if \( y \neq 0 \), then \( x*(y*x) = x \). [By condition; \( x = x *(y *x) \)]
\( \Rightarrow x \in I \). [Since \( x*(y*x) \in I \)]
Therefore, I is a Q-Smarandache b-implicative ideal of X. ■

Proposition (3.15):
Let \( \{ I_i : i \in \lambda \} \) be family of a Q-Smarandache b-implicative ideals of a Q-Smarandache BH-algebra. Then \( \bigcap I_i \) is a Q-Smarandache b-implicative ideal of X.

Proof:
Let \( x, y \in Q \) and \( z \in \bigcap I_i \) such that \((x*(y*x))*z)*b \in \bigcap I_i \).
\( \Rightarrow (x*(y*x))*z \in I_i \) and \( z \in I_i , \forall i \in \lambda \).
\( \Rightarrow x \in I_i , \forall i \in \lambda . \)
[Since I_i is a Q-Smarandache b-implicative ideal of X, \( \forall i \in \lambda \) ]
\( \Rightarrow x \in \bigcap I_i \).
Therefore, \( \bigcap I_i \) is a Q-Smarandache b-implicative ideal of X. ■
Remark (3.16):
The union of a Q-Smarandache implicatives ideals with respect to an element b of a Q-Smarandache BH-algebra may not be a Q-Smarandache implicative ideal of X as in the following example.

Example (3.17):
Consider the Q-Smarandache BH-algebra X={0,1,2,3,4,5} with binary operation "∗" defined by the following table:

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where Q={0,2} is a BCK-algebra. I={0,1} and J={0,5} are two a Q-Smarandache 0-implicative ideals of X, but I ∪ J= {0,1,5} is not a Q-Smarandache 0-implicative ideal of X, since x=2 ,y=0, z=5, ((2*(0*2))*5)*0=((2*0)*5)*0 = (2*5)*0 = 1*0=1 ∈ I, but 2 ∉ I ∪ J.

Proposition (3.18):
Let {I_i, i ∈ λ} be chain of a Q-Smarandache b-implicative ideal of a Q-Smarandache BH-algebra X. Then ∪ I_i is a Q-Smarandache b-implicative ideal of X.

Proof:
Since {I_i, i ∈ λ} is a chain of a Q-Smarandache ideal of X. Then ∪ I_i is a Q-Smarandache ideal of X.

Let x,y ∈ Q and z ∈ ∪ I_i such that ( (x*(y*x))*z )*b ∈ ∪ I_i and z ∈ ∪ I_i. 

There exist I_j, I_k ∈ {I_i, i ∈ λ} such that ( (x*(y*x))*z )*b ∈ I_j and z ∈ I_k. 
⇒ either I_j ⊆ I_k or I_k ⊆ I_j. [Since {I_i} are chain]
⇒ either ( (x*(y*x))*z )*b ∈ I_j and z ∈ I_k or ( (x*(y*x))*z )*b ∈ I_k and z ∈ I_j.
⇒ either x ∈ I_j or x ∈ I_k. [Since I_j and I_k are Q-Smarandache b-implicative ideal of X]
⇒ x ∈ ∪ I_i .

Therefore, ∪ I_i is a Q-Smarandache b-implicative ideal of X.
References:


