Non-power-function metric: a generalized fractal

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This short note proposes a general time-space metric by an extension of the power-function based fractal concept to the structural-function fabric. The structural function can be an arbitrary-function to describe complex metric underlying physical systems. We call such a metric Structal, and the fractal is its special case. This work is inspired by our recent work on the structural derivative, in which the structural function takes into account the significant influence of time-space fabric of a complex system on its physical behaviors, in particular, the ultra-slow diffusion. Based on the structal concept, this communication suggests the structural time-space transformation and introduces the general diffusion model. In addition, the statistics implication of the structal and the structural derivative model is also briefly discussed.

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1. Introduction

The Hausdorff dimension of metric spacetime is the most known definition of the fractal. It is a generalization of the classical integer-dimension concept. Namely, the metric of space is described by

$$M = r^d, \tag{1}$$

where r represents the Euclidean distance between two points, and the dimensionality d is not necessarily an integer number but can be an arbitrary real or even a complex number [1]. The power law function definition (1) of metric has since found applications in many diverse fields and has profound significance in sciences and engineering.

It, however, is observed that many real world problems cannot well be described by the classical fractal concept. And some variants, e.g., multifractal, have since been proposed, which allows the dimensionality varies with time and space. But nevertheless all such fractal variants keep the power function metric as shown in formula (1).

This note extends the power-function fractal metric (1) to the structural fabric as

$$M = R(r)\,, \tag{2}$$

where R denotes the structural function and is not necessary an power function and can be an arbitrary function. We call such the structural fabric "Structal". For instance, $R(r) = r^{b1} + r^{b2} + r^{b3}$ is a combination of fractals but in essence is not a fractal but can be found in many real-world systems, where exponent b can be an arbitrary number.

2. Derivative and differential operator on structal

This section is concerned with time and space calculus on structal. First, consider the spatial operator and take the Laplacian operator as an illustrative example.

By applying the implicit calculus equation modeling approach [2], we can define the Laplacian on fractal by using its fractal fundamental solution [3,4]

$$u_d^*(r) = \frac{1}{(d-2)Q_d(1)} r^{2-d}\,, \tag{3}$$

where $Q_n(1) = 2\pi^{n/2}/\Gamma(n/2)$, r represents the Euclidean distance, d is the fractal dimension. Concerning the Laplacian on structal, its fundamental solution can be stated as

$$u_S^*(r) = H(r)\,, \tag{4}$$

where H depends on the structal metric. By using this fundamental solution (4), we can define the Laplacian on structal.

The Hausdorff derivative on time fractal α is given by [5]

$$\frac{dg(t)}{dt^{\alpha}} = \lim_{t' \to t} \frac{g(t) - g(t')}{t^{\alpha} - t'^{\alpha}}. \tag{5}$$

The Hausdorff derivative can well model the so-called anomalous diffusion, which is characterized by the time evolution of the mean square displacement of diffusing particle movements

$$\langle \Delta x^2 \rangle \propto \Delta t^{\alpha}, \tag{6}$$

where Δx represents distance, Δt denotes time interval, and the brackets means the mean value of random variables. α is a positive real number and fractal, and $\alpha \neq 1$ indicates anomalous diffusion.

As a further development of the above fractal derivative (5), the structural derivative was recently introduced to model the ultra-slow diffusion behaviors [6]. On time structal, the structural derivative is defined as

$$\frac{df(t)}{d_S t} = \lim_{t' \to t} \frac{f(t) - f(t')}{S(t) - S(t')}, \tag{7}$$

where $S(t)$ is the structural function of time structal metric. When $S(t)=t$, the definition (7) is reduced to the classical first order derivative, and when $S(t)=t^{\alpha}$, it is the fractal derivative definition (5).

In the above ultra-slow diffusion case, the structural function in time structal is the inverse Mittag-Leffler (ML) function [4,6]. The corresponding displacement and time relationship is given by

$$\langle \Delta x^2 \rangle \propto E_{\eta}^{-1}(1 + \Delta t), \tag{8}$$

where E_{η}^{-1} is the inverse function of the single-parameter ML function with $0<\eta<1$. The structural function and derivative in space structal for modeling ultra-slow diffusion can also be developed in the same manner.

3. Speculative results

This section presents some speculative results on structal metric.

To solve the diverging square moment in anomalous diffusion, the present author [5] uses the Hausdorff time-space metric to introduce the following scaling transforms

$$\begin{cases} \Delta\hat{x} = \Delta x^{\beta} \\ \Delta\hat{t} = \Delta t^{\alpha} \end{cases}, \qquad 0 \prec \alpha, \beta \le 1. \tag{9}$$

By analogy with the above fractal metric transform, the structal transform is proposed as

$$\begin{cases} \Delta\hat{x} = \mathrm{G}(\Delta x) \\ \Delta\hat{t} = \mathrm{P}(\Delta t) \end{cases}, \tag{10}$$

where the structural functions G and P characterize the space and time metrics. Under the structal transform (10), the diffusion of the mean square displacement over time is recast as

$$\left\langle \Delta\hat{x}^2 \right\rangle \propto \Delta\hat{t}\,. \tag{11}$$

4. Statistics on structal

The fractal has important applications in statistics and signal processing such as 1/f noise analysis. It is expected that the structal has rich implications on statistics and probability as well. For example, the time structal has successfully used to describe ultra-slow diffusion with the inverse ML function as the time structural function. The inverse ML function characterizes the ultra-slow power law decay (memory) and can find applications in time series analysis.

The solution of diffusion equation on structal can also lead to new statistics.

5. Remarks

Time and space are fundamental mathematical and physical quantities in nature. This study generalizes the fractal to non-power function metric “Structal” and consequently proposes the structal definition of derivative and differential operators as well as the structal time-space transform and the general diffusion model.

It is worthy of noting that the structal is different from the so-called multifractal [7], which depicts the fractal metric varies with time and space variables but remains a power-function metric. Unlike the structural derivative, the varying-order fractional and fractal derivatives can be used to describe multifractal systems.

Nowadays many artificial materials are invented and manufactured such as metamaterials, which have the metric to process specific-purpose function. The structal concept and methodology could help to develop and analyze micro- and meso-structures of such materials. In addition, the complex network is another possible application field of the structal methodology, where the corresponding statistics methods may play an important role.

References

1. B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman, San Francisco, 1982.
2. W. Chen, Implicit calculus modeling for simulation of complex scientific and engineering problems, *Comput. Aided E* **23**, 1-6 2014. (in Chinese)
3. W. Chen, F. Wang, X. Yang, *Definition of calculus operator on fractal and its applications*, *Comput. Aided E.*, **26**, 1-5, 2016. (in Chinese)
4. W. Chen, Y. Liang, Structural derivative, implicit calculus equation, differential operator on fractal, and their applications, *Proceeding* of *Int. Confer. on Fractional Differentiation and its Applications*, pages 77-86, Novi Sad, Serbia, July 18 - 20, 2016

5. W. Chen, Time-space fabric underlying anomalous diffusion, *Chaos, Solitons & Fractals*, **28**(4), 923-929, 2006.
6. W. Chen, Y. Liang, X. Hei, Structural derivative based on inverse Mittag-Leffler function for modeling ultraslow diffusion, *Fract. Calc. Appl. Anal*., **19**(5), 1250-1261, 2016, DOI: 10.1515/fca-2016-0064
7. B. B. Mandelbrot, *Multifractals and 1 over f Noise: Wild Self-Affinity in Physics (1963-1976)*, Springer-Verlag, New York, 1998.