Surface Formulations of the Electromagnetic-Power-based Characteristic Mode Theory for Material Bodies — Part III

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***Abstract*—As a supplement to the previous Parts I and II, the Surface formulations of the ElectroMagnetic-Power-based Characteristic Mode Theory for the system constructed by Multiple Homogeneous Material bodies (Surf-MHM-EMP-CMT) are established in this Part III. The coupling phenomenon among different bodies is specifically studied, and then a new kind of power-based Characteristic Mode (CM) set, Coupling power CM (CoupCM) set, is developed for characterizing the coupling character.**

***Index Terms*—Characteristic Mode (CM), Electromagnetic Power, Interaction, Material Body, Multi-body Coupling.**

I. INTRODUCTION

THE MoM-based Characteristic Mode Theory (CMT) was established by R. F. Harrington *et al.* in the 1970s [1]-[3], such that the inherent ElectroMagnetic (EM) character of an object, especially the open object, can be efficiently extracted. Recently, the physically appropriate ideas in the MoM-based CMT are liberated from the MoM framework [4]-[5], and an ElectroMagnetic-Power-based CMT (EMP-CMT) is built [4]-[9].

It has been clearly illustrated in [4]-[9] that the EMP-CMT is an object-oriented modal theory, and its object-oriented feature is mainly demonstrated in the following two aspects.

1) The power-based Characteristic Mode (CM) sets for an EM system can be constructed by focusing on different objective EM powers. For example, the Radiated power CM (RaCM) set [4]-[5], the Stored power CM (StoCM) set [4], and the Output power CM (OutCM) set [4]-[9], etc. can be respectively constructed by optimizing the system radiated power, by optimizing the system reactively stored power, and by orthogonalizing the system output power, etc.

2) The objective EM system is regarded as a whole object, and the other EM sources (such as the impressed source and the source generated by EM environment) are uniformly treated as an external source. Then, it was proven in [4]-[5] that the various power-based CM sets are the inherent characters of the objective system, and they are completely independent of the external source.

The papers [6] and [7] mainly focus on the system constructed by a single body. The EM system considered in [4]-[5] and [8]-[9] can be constructed by multiple bodies (simply called as multi-body system), and the multi-body system was treated as a whole object in the [4]-[5] and [8]-[9], so the coupling effect among different bodies was not specifically analyzed in the [4]-[5] and [8]-[9].

Taking the system constructed by multiple homogeneous material bodies as a typical example, the coupling effect among different bodies is studied in this paper by employing the surface formulations developed in [6] and [7], and then a new power-based CM set, the Coupling power CM (CoupCM) set, is constructed for characterizing the coupling character. If the Surface formulations of the EMP-CMT for a Single Homogeneous Material body [6]-[7] can be simply denoted as Surf-SHM-EMP-CMT, the Surface formulations of the EMP-CMT for Multiple Homogeneous Material bodies developed in this paper can be similarly denoted as Surf-MHM-EMP-CMT, and the Surf-SHM-EMP-CMT and Surf-MHM-EMP-CMT are collectively referred to as the Surface formulations of the EMP-CMT for Material bodies (Surf-Mat-EMP-CMT).

In what follows, the $e^{j\omega t}$ convention is used throughout, and the EM systems constructed by double homogeneous material bodies are focused on, and the systems constructed by any number of homogeneous material bodies can be similarly discussed. In addition, the discussions in this paper can also be easily generalized to the systems constructed by multiple metal bodies, the systems constructed by multiple inhomogeneous material bodies, and the metal-material combined systems.

II. SOURCE-FIELD RELATIONSHIPS

For an objective double-body system, the regions occupied by two bodies are respectively denoted as V_1 and V_2, and the region occupied by whole system is denoted as $V = V_1 \cup V_2$. When an external field $\bar{F}^{inc}$ incidents on the system, the scattering sources will be excited on every V_i, and then the scattering field $\bar{F}_i^{sca}$ is generated as illustrated in Fig. 1, here $i = 1,2$. The summation of $\bar{F}_1^{sca}$ and $\bar{F}_2^{sca}$ is the total scattering field generated by whole system, and it is denoted as $\bar{F}^{sca}$, i.e., $\bar{F}^{sca} = \bar{F}_1^{sca} + \bar{F}_2^{sca}$, here $F = E,H$. The summation of $\bar{F}^{inc}$ and

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$\bar{F}^{sca}$ is the total field, and it is denoted as $\bar{F}^{tot}$, i.e., $\bar{F}^{tot} = \bar{F}^{inc} + \bar{F}^{sca}$. Based on the superposition principle [10], the $\bar{F}_i^{sca}$ can be viewed as the scattering field due to the excitation $\bar{F}^{inc} + \bar{F}_j^{sca}$, here $(i,j) = (1,2),(2,1)$. To simplify the symbolic system of this paper, the $\bar{F}^{inc} + \bar{F}_j^{sca}$ is denoted as $\bar{f}_i^{inc}$, i.e., $\bar{f}_i^{inc} \triangleq \bar{F}^{inc} + \bar{F}_j^{sca}$, here $(F,f) = (E,e),(H,h)$. In addition, it is obvious that $\bar{f}_i^{inc} = \bar{F}^{tot} - \bar{F}_i^{sca}$.

The physical scattering currents on V_i include the volume ohmic electric current $\bar{J}_i^{vo}$, the volume polarized electric current $\bar{J}_i^{vp}$, and the volume magnetized magnetic current $\bar{M}_i^{vm}$. In addition, the summation of $\bar{J}_i^{vo}$ and $\bar{J}_i^{vp}$ is denoted as $\bar{J}_i^{vop}$, i.e., $\bar{J}_i^{vop} \triangleq \bar{J}_i^{vo} + \bar{J}_i^{vp}$. For the body V_i, the surface equivalent currents on its boundary ∂V_i are defined as follows [6]-[7]

$$\bar{J}_{i;\pm}^{SE}(\bar{r}) \triangleq \left[\hat{n}_{i;\pm}(\bar{r}) \times \bar{H}^{tot}(\bar{r}_{i;\pm})\right]_{\bar{r}_{i;\pm}\to\bar{r}} \quad , \quad (\bar{r} \in \partial V_i) \tag{1}$$

$$\bar{M}_{i;\pm}^{SE}(\bar{r}) \triangleq \left[\bar{E}^{tot}(\bar{r}_{i;\pm}) \times \hat{n}_{i;\pm}(\bar{r})\right]_{\bar{r}_{i;\pm}\to\bar{r}} \quad , \quad (\bar{r} \in \partial V_i) \tag{2}$$

here $\hat{n}_{i;+}$ and $\hat{n}_{i;-}$ are respectively the external and internal normal directions of ∂V_i; $\bar{r}_{i;+} \in \text{ext}\,V_i \triangleq \mathbb{R}^3 \setminus \text{cl}\,V_i$, and $\bar{r}_{i;-} \in \text{int}\,V_i$; the $\mathbb{R}^3$ is three-dimensional Euclidean space; the symbols $\text{cl}\,V_i$, $\text{int}\,V_i$, and $\text{ext}\,V_i$ are respectively the closer, interior, and exterior of set V_i. In this paper, it is restricted that $V_i = \text{cl}\,V_i$, and then $V_i = \text{int}\,V_i \cup \partial V_i$ [11]. In fact, the $\bar{C}_{i;\pm}^{SE}$ can be uniformly denoted as follows [6]-[7]

$$\bar{C}_i^{SE}(\bar{r}) \triangleq \bar{C}_{i;-}^{SE}(\bar{r}) = -\bar{C}_{i;+}^{SE}(\bar{r}) \quad , \quad (\bar{r} \in \partial V_i) \tag{3}$$

here $C = J, M$.

Based on the method given in [6]-[7], the various fields and physical scattering currents corresponding to body V_i can be expressed as the functions of the surface equivalent currents on ∂V_i as follows

$$\begin{aligned} \bar{F}_i^{sca}(\bar{r}) &= \mathcal{F}_i^{sca}(\bar{J}_i^{SE}, \bar{M}_i^{SE}) \\ &= \begin{cases} \mathcal{F}_{i;-}^{sca}(\bar{J}_i^{SE}, \bar{M}_i^{SE}) \quad , \quad (\bar{r} \in \text{int}\,V_i) \\ \mathcal{F}_{i;+}^{sca}(\bar{J}_i^{SE}, \bar{M}_i^{SE}) \quad , \quad (\bar{r} \in \text{ext}\,V_i) \end{cases} \end{aligned} \tag{4}$$

$$\bar{F}_{i;-}^{tot}(\bar{r}) = \mathcal{F}_{i;-}^{tot}(\bar{J}_i^{SE}, \bar{M}_i^{SE}) \quad , \quad (\bar{r} \in \text{int}\,V_i) \tag{5}$$

and

$$\begin{aligned} \bar{f}_{i;-}^{inc}(\bar{r}) &= f_{i;-}^{inc}(\bar{J}_i^{SE}, \bar{M}_i^{SE}) \\ &= \bar{F}_{i;-}^{tot}(\bar{r}) - \bar{F}_i^{sca}(\bar{r}) \\ &= \bar{F}_{i;-}^{inc}(\bar{r}) + \bar{F}_j^{sca}(\bar{r}) \quad , \quad (\bar{r} \in \text{int}\,V_i) \end{aligned} \tag{6.1}$$

$$\begin{aligned} \bar{F}_{i;-}^{inc}(\bar{r}) &= \mathcal{F}_{i;-}^{inc}(\bar{J}_i^{SE}, \bar{M}_i^{SE}, \bar{J}_j^{SE}, \bar{M}_j^{SE}) \\ &= \bar{F}_{i;-}^{tot}(\bar{r}) - \bar{F}^{sca}(\bar{r}) \\ &= \bar{f}_{i;-}^{inc}(\bar{r}) - \bar{F}_j^{sca}(\bar{r}) \quad , \quad (\bar{r} \in \text{int}\,V_i) \end{aligned} \tag{6.2}$$

and

$$\bar{J}_i^{vo}(\bar{r}) = \mathcal{J}_i^{vo}(\bar{J}_i^{SE}, \bar{M}_i^{SE}) \quad , \quad (\bar{r} \in \text{int}\,V_i) \tag{7.1}$$

$$\bar{J}_i^{vp}(\bar{r}) = \mathcal{J}_i^{vp}(\bar{J}_i^{SE}, \bar{M}_i^{SE}) \quad , \quad (\bar{r} \in \text{int}\,V_i) \tag{7.2}$$

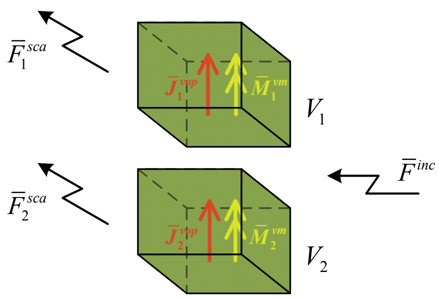

Fig. 1. Various fields and scattering currents.

$$\bar{J}_i^{vop}(\bar{r}) = \mathcal{J}_i^{vop}(\bar{J}_i^{SE}, \bar{M}_i^{SE}) \quad , \quad (\bar{r} \in \text{int}\,V_i) \tag{7.3}$$

$$\bar{M}_i^{vm}(\bar{r}) = \mathcal{M}_i^{vm}(\bar{J}_i^{SE}, \bar{M}_i^{SE}) \quad , \quad (\bar{r} \in \text{int}\,V_i) \tag{8}$$

here $i = 1,2$; $F = E,H$, and correspondingly $\mathcal{F} = \mathcal{E},\mathcal{H}$; $f = e,h$, and correspondingly $f = e,h$; to utilize the subscript "$-$" in $\bar{F}_{i;-}^{tot}$, $\bar{f}_{i;-}^{inc}$, and $\bar{F}_{i;-}^{inc}$ is to emphasize that the $\bar{F}_{i;-}^{tot}$, $\bar{f}_{i;-}^{inc}$, and $\bar{F}_{i;-}^{inc}$ are the corresponding fields in the interior of V_i; to utilize the subscript "i" in the operators in (4)-(8) is to emphasize that the mathematical expressions of these operators are different for the case $i = 1$ and the case $i = 2$. In addition, it must be emphasized that the operator $\mathcal{F}_{i;-}^{inc}$ is also dependent on the body V_j besides the body V_i itself, because of the coupling between two bodies, and this is just the reason why the arguments of operator $\mathcal{F}_{i;-}^{inc}$ include both $\{\bar{J}_i^{SE}, \bar{M}_i^{SE}\}$ and $\{\bar{J}_j^{SE}, \bar{M}_j^{SE}\}$.

In the above (4)-(8), the operators $\mathcal{F}_i^{sca}$ and $\mathcal{F}_{i;-}^{tot}$ are essentially the same as the operators (10) and (11) in paper [6]; the operator $f_{i;-}^{inc}$ is essentially the same as the operator (12) in [6]; the operators (7.3) and (8) are essentially the same as the operators (13.1) and (13.2) in [6]. Then, the specific mathematical expressions of these operators are not repeated here. However, it must be clearly emphasized that: although the operator $\mathcal{F}_{i;-}^{inc}$ in above (6.2) has a similar symbolic representation to the operator (12) in [6], they are essentially different from each other, because there doesn't exist the coupling between different bodies in [6], whereas there exists the coupling between different bodies as illustrated in the third equalities of the above (6.1) and (6.2).

III. BASIC VARIABLES

In [6]-[7], it has been pointed out that the surface equivalent electric and magnetic currents are not independent, and it is necessary for Surf-SHM-EMP-CMT to establish the relation between them. In fact, it is also indispensable for Surf-MHM-EMP-CMT to establish the relation between $\bar{J}_i^{SE}$ and $\bar{M}_i^{SE}$, and it is just the main topic of this section.

In the following parts of this section, the surface equivalent currents are decomposed into some parts at first, to do the preparation for constructing the *basic variables* in Sec. III-C. Secondly, the mapping from modal space to expansion vector space is established. Thirdly, the method *to unify variables* (i.e., to construct basic variables and to express all EM quantities in terms of basic variables [7]) for every single body in a double-body system is provided in expansion vector space. At last, the method to unify variables for whole double-body system is given.

A. *The decompositions for* ∂V_i *and* $\bar{C}_i^{SE}$

In this subsection, the boundary of every single body is decomposed into some parts, and then the decompositions for surface equivalent currents are provided.

1) The decomposition for ∂V_i

Obviously, the ∂V_1 is a closed set in $\mathbb{R}^3$, so $\text{cl}(\partial V_1 \setminus \partial V_2) \subseteq \partial V_1$.

If $\text{cl}(\partial V_1 \setminus \partial V_2) \neq \partial V_1$, the set $\partial V_1 \setminus \text{cl}(\partial V_1 \setminus \partial V_2)$ is constructed by some surfaces, and it is denoted as S_{12}, i.e.,

$$\begin{aligned} S_{12} &\triangleq \partial V_1 \setminus \text{cl}(\partial V_1 \setminus \partial V_2) \\ &= \partial V_2 \setminus \text{cl}(\partial V_2 \setminus \partial V_1) \end{aligned} \tag{9}$$

and two typical examples are illustrated in the Figs. 2 (d) and 2 (e). The second equality in (9) can be easily proven by using the language of point set topology [11], and the proof is not specifically provided here. In addition, it is obvious that $S_{12} \subseteq \partial V_i$ for any $i = 1,2$, so

$$(\partial V_i \setminus S_{12}) \cap S_{12} = \varnothing \tag{10.1}$$

$$(\partial V_i \setminus S_{12}) \cup S_{12} = \partial V_i \tag{10.2}$$

for any $i = 1,2$.

If $\text{cl}(\partial V_1 \setminus \partial V_2) = \partial V_1$, the S_{12} can also be defined as (9), and $S_{12} = \varnothing$ for this case. Three typical examples are illustrated in the Figs. 2 (a), 2 (b), and 2 (c).

2) The decomposition for $\bar{C}_i^{SE}$

It is easy to prove that $\text{cl} S_{12} = \partial V_i$, if and only if $S_{12} = \partial V_i$. This implies that the set $\partial V_i \setminus S_{12}$ includes some surfaces, if and only if $S_{12} \neq \partial V_i$. Based on this observation and the (10), the $\bar{C}_i^{SE}$ can be decomposed as follows

$$\bar{C}_i^{SE}(\bar{r}) = \bar{C}_{i0}^{SE}(\bar{r}) + \bar{C}_{ij}^{SE}(\bar{r}) \quad , \quad (\bar{r} \in \partial V_i) \tag{11}$$

here $C = J, M$, and

$$\bar{C}_{i0}^{SE}(\bar{r}) \triangleq \begin{cases} \bar{C}_i^{SE}(\bar{r}) & , \quad (\bar{r} \in \partial V_i \setminus S_{12}) \\ 0 & , \quad (\bar{r} \in S_{12}) \end{cases} \tag{12.1}$$

$$\bar{C}_{ij}^{SE}(\bar{r}) \triangleq \begin{cases} 0 & , \quad (\bar{r} \in \partial V_i \setminus S_{12}) \\ \bar{C}_i^{SE}(\bar{r}) & , \quad (\bar{r} \in S_{12}) \end{cases} \tag{12.2}$$

In (11)-(12), $(i,j) = (1,2),(2,1)$. In addition, it is obvious that

$$\bar{C}_{12}^{SE}(\bar{r}) = -\bar{C}_{21}^{SE}(\bar{r}) \tag{13}$$

because of the (3).

In particular, $\bar{C}_{i0}^{SE} = \bar{C}_i^{SE}$, when $S_{12} = \varnothing$; $\bar{C}_{ij}^{SE} = \bar{C}_i^{SE}$, when $S_{12} = \partial V_i$. In fact, the case $S_{12} = \varnothing$ corresponds to that there doesn't exist any interface between V_1 and V_2, i.e., the set $\partial V_1 \cap \partial V_2$ only may include some lines or points, as illustrated in Figs. 2 (a), 2 (b), and 2 (c); the case $S_{12} \neq \varnothing$ corresponds to that there exist some interfaces between bodies V_1 and V_2, as illustrated in Figs. 2 (d) and 2 (e); the case $S_{12} = \partial V_i$ corresponds to that the body V_i is entirely surrounded by body V_j, as illustrated in Fig. 2 (e). This paper only focuses on the case $S_{12} \neq \varnothing, \partial V_1, \partial V_2$, because the discussions for the cases $S_{12} = \varnothing$ and $S_{12} = \partial V_i$ can be easily finished as a similar way.

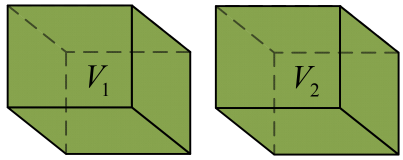

Fig. 2 (a). The V_1 and V_2 don't contact with each other, i.e., $\partial V_1 \cap \partial V_2 = \varnothing$, and this paper doesn't focus on this case.

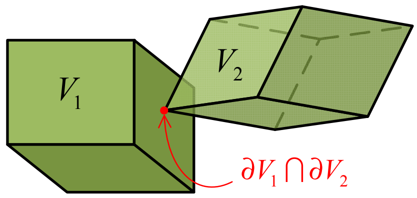

Fig. 2 (b). The intersection between ∂V_1 and ∂V_2 is a point, and this paper doesn't focus on this case.

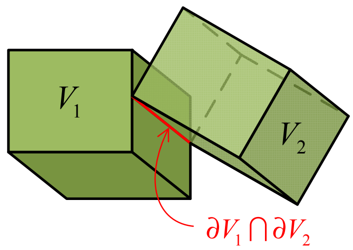

Fig. 2 (c). The intersection between ∂V_1 and ∂V_2 is a line, and this paper doesn't focus on this case.

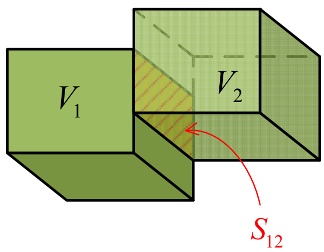

Fig. 2 (d). The intersection between ∂V_1 and ∂V_2 is a surface, and this case is specifically studied in this paper.

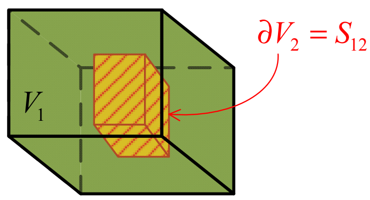

Fig. 2 (e). The body V_2 is entirely surrounded by body V_1, and this paper doesn't focus on this case.

B. *From modal space to expansion vector space*

The $\bar{C}_{i0}^{SE}$ and $\bar{C}_{ij}^{SE}$ are respectively expanded in terms of the basis function sets $\{\bar{b}_\xi^{C_{i0}}\}_{\xi=1}^{\Xi^{C_{i0}}}$ and $\{\bar{b}_\xi^{C_{12}} = -\bar{b}_\xi^{C_{21}}\}_{\xi=1}^{\Xi^{C_{12}}=\Xi^{C_{21}}}$ as follows

$$\bar{C}_{i0}^{SE}(\bar{r}) = \sum_{\xi=1}^{\Xi^{C_{i0}}} a_\xi^{C_{i0}} \bar{b}_\xi^{C_{i0}}(\bar{r}) = \bar{B}^{C_{i0}} \cdot \bar{a}^{C_{i0}} \quad , \quad (\bar{r} \in \partial V_i \setminus S_{12}) \tag{14}$$

and

$$\bar{C}_{12}^{SE}(\bar{r}) = \sum_{\xi=1}^{\Xi^{C_{12}}} a_\xi^{C_{12}} \bar{b}_\xi^{C_{12}}(\bar{r}) = \bar{B}^{C_{12}} \cdot \bar{a}^{C_{12}} \quad , \quad (\bar{r} \in S_{12}) \tag{15.1}$$

$$\begin{aligned} \bar{C}_{21}^{SE}(\bar{r}) &= \sum_{\xi=1}^{\Xi^{C_{21}}} a_\xi^{C_{21}} \bar{b}_\xi^{C_{21}}(\bar{r}) = \bar{B}^{C_{21}} \cdot \bar{a}^{C_{21}} \\ &= \sum_{\xi=1}^{\Xi^{C_{12}}} a_\xi^{C_{12}} \left[-\bar{b}_\xi^{C_{12}}(\bar{r})\right] = \bar{B}^{C_{21}} \cdot \bar{a}^{C_{12}} \quad , \quad (\bar{r} \in S_{12}) \end{aligned} \tag{15.2}$$

here $i = 1,2$, and $C = J, M$, and the third equality in (15.2) is due to (13). In (14)-(15),

$$\bar{B}^{C_{i0}} = \left[\bar{b}_1^{C_{i0}}(\bar{r}) \; , \; \bar{b}_2^{C_{i0}}(\bar{r}) \; , \; \cdots \; , \; \bar{b}_{\Xi^{C_{i0}}}^{C_{i0}}(\bar{r})\right] \tag{16.1}$$

$$\bar{a}^{C_{i0}} = \left[\; a_1^{C_{i0}} \; , \; a_2^{C_{i0}} \; , \; \cdots \; , \; a_{\Xi^{C_{i0}}}^{C_{i0}} \;\right]^T \tag{16.2}$$

and

$$\bar{B}^{C_{12}} = \left[\bar{b}_1^{C_{12}}(\bar{r}) \; , \; \bar{b}_2^{C_{12}}(\bar{r}) \; , \; \cdots \; , \; \bar{b}_{\Xi^{C_{12}}}^{C_{12}}(\bar{r})\right] \tag{17.1}$$

$$\bar{a}^{C_{12}} = \left[\; a_1^{C_{12}} \; , \; a_2^{C_{12}} \; , \; \cdots \; , \; a_{\Xi^{C_{12}}}^{C_{12}} \;\right]^T \tag{17.2}$$

and

$$\bar{B}^{C_{21}} = -\bar{B}^{C_{12}} \tag{17.1'}$$

$$\bar{a}^{C_{21}} = \bar{a}^{C_{12}} \tag{17.2'}$$

here the superscript "T" represents the transpose of matrix.

For the convince of following discussions, the basis function sets $\{\bar{b}_\xi^{C_1}\}_{\xi=1}^{\Xi^{C_1}}$ and $\{\bar{b}_\xi^{C_2}\}_{\xi=1}^{\Xi^{C_2}}$ are defined as follows

$$\left\{\bar{b}_\xi^{C_1}\right\}_{\xi=1}^{\Xi^{C_1}} \triangleq \left\{ \bar{b}_1^{C_{10}} , \bar{b}_2^{C_{10}} , \cdots , \bar{b}_{\Xi^{C_{10}}}^{C_{10}} , \bar{b}_1^{C_{12}} , \bar{b}_2^{C_{12}} , \cdots , \bar{b}_{\Xi^{C_{12}}}^{C_{12}} \right\} \tag{18.1}$$

$$\left\{\bar{b}_\xi^{C_2}\right\}_{\xi=1}^{\Xi^{C_2}} \triangleq \left\{ -\bar{b}_1^{C_{12}} , -\bar{b}_2^{C_{12}} , \cdots , -\bar{b}_{\Xi^{C_{12}}}^{C_{12}} , \bar{b}_1^{C_{20}} , \bar{b}_2^{C_{20}} , \cdots , \bar{b}_{\Xi^{C_{20}}}^{C_{20}} \right\} \tag{18.2}$$

here $\Xi^{C_i} = \Xi^{C_{i0}} + \Xi^{C_{12}}$. Then, the following vectors can be introduced.

$$\bar{B}^{C_1} = \left[\bar{B}^{C_{10}} \quad \bar{B}^{C_{12}}\right] \tag{19.1}$$

$$\bar{a}^{C_1} = \begin{bmatrix} \bar{a}^{C_{10}} \\ \bar{a}^{C_{12}} \end{bmatrix} \tag{19.2}$$

and

$$\bar{B}^{C_2} = \left[\bar{B}^{C_{21}} \quad \bar{B}^{C_{20}}\right] = \left[-\bar{B}^{C_{12}} \quad \bar{B}^{C_{20}}\right] \tag{20.1}$$

$$\bar{a}^{C_2} = \begin{bmatrix} \bar{a}^{C_{21}} \\ \bar{a}^{C_{20}} \end{bmatrix} = \begin{bmatrix} \bar{a}^{C_{12}} \\ \bar{a}^{C_{20}} \end{bmatrix} \tag{20.2}$$

Correspondingly,

$$\bar{C}_i^{SE}(\bar{r}) = \sum_{\xi=1}^{\Xi^{C_i}} a_\xi^{C_i} \bar{b}_\xi^{C_i}(\bar{r}) = \bar{B}^{C_i} \cdot \bar{a}^{C_i} \quad , \quad (\bar{r} \in \partial V_i) \tag{21}$$

C. *Variable unification*

The methods to unify variables for a single body have been carefully discussed in [6] and [7], and the $\bar{a}^{J_i}$ and $\bar{a}^{M_i}$ can be related to each other as follows

$$\bar{a}^{\Psi_i} = \bar{\bar{T}}^{\Phi_i \to \Psi_i} \cdot \bar{a}^{\Phi_i} \tag{22}$$

here $(\Phi, \Psi) = (J, M)$, if the surface equivalent electric current is selected as basic variable; $(\Phi, \Psi) = (M, J)$, if the surface equivalent magnetic current is selected as basic variable. If the transformation matrix $\bar{\bar{T}}^{\Phi_i \to \Psi_i}$ is partitioned according to the partition ways of the vectors in (19.2) and (20.2), the relation (22) can be equivalently rewritten as the following portioned version.

$$\begin{bmatrix} \bar{a}^{\Psi_{10}} \\ \bar{a}^{\Psi_{12}} \end{bmatrix} = \begin{bmatrix} \bar{\bar{T}}^{\Phi_{10} \to \Psi_{10}} & \bar{\bar{T}}^{\Phi_{12} \to \Psi_{10}} \\ \bar{\bar{T}}^{\Phi_{10} \to \Psi_{12}} & \bar{\bar{T}}^{\Phi_{12} \to \Psi_{12}} \end{bmatrix} \cdot \begin{bmatrix} \bar{a}^{\Phi_{10}} \\ \bar{a}^{\Phi_{12}} \end{bmatrix} \tag{23.1}$$

$$\begin{bmatrix} \bar{a}^{\Psi_{12}} \\ \bar{a}^{\Psi_{20}} \end{bmatrix} = \begin{bmatrix} \bar{\bar{T}}^{\Phi_{12} \to \Psi_{12}} & \bar{\bar{T}}^{\Phi_{20} \to \Psi_{12}} \\ \bar{\bar{T}}^{\Phi_{12} \to \Psi_{20}} & \bar{\bar{T}}^{\Phi_{20} \to \Psi_{20}} \end{bmatrix} \cdot \begin{bmatrix} \bar{a}^{\Phi_{12}} \\ \bar{a}^{\Phi_{20}} \end{bmatrix} \tag{23.2}$$

Based on the (23), the following relation can be derived

$$\bar{a}^{\Psi} = \bar{\bar{T}}^{\Phi \to \Psi} \cdot \bar{a}^{\Phi} \tag{24}$$

here

$$\begin{aligned} \bar{\bar{T}}^{\Phi \to \Psi} &= \begin{bmatrix} \bar{\bar{T}}^{\Phi_{10} \to \Psi_{10}} & \bar{\bar{T}}^{\Phi_{12} \to \Psi_{10}} & 0 \\ \bar{\bar{T}}^{\Phi_{10} \to \Psi_{12}} & \bar{\bar{T}}^{\Phi_{12} \to \Psi_{12}} & 0 \\ 0 & \bar{\bar{T}}^{\Phi_{12} \to \Psi_{20}} & \bar{\bar{T}}^{\Phi_{20} \to \Psi_{20}} \end{bmatrix} \\ &= \begin{bmatrix} \bar{\bar{T}}^{\Phi_{10} \to \Psi_{10}} & \bar{\bar{T}}^{\Phi_{12} \to \Psi_{10}} & 0 \\ 0 & \bar{\bar{T}}^{\Phi_{12} \to \Psi_{12}} & \bar{\bar{T}}^{\Phi_{20} \to \Psi_{12}} \\ 0 & \bar{\bar{T}}^{\Phi_{12} \to \Psi_{20}} & \bar{\bar{T}}^{\Phi_{20} \to \Psi_{20}} \end{bmatrix} \end{aligned} \tag{25}$$

and

$$\bar{a}^{\Phi} = \begin{bmatrix} \bar{a}^{\Phi_{10}} \\ \bar{a}^{\Phi_{12}} \\ \bar{a}^{\Phi_{20}} \end{bmatrix} \quad , \quad \bar{a}^{\Psi} = \begin{bmatrix} \bar{a}^{\Psi_{10}} \\ \bar{a}^{\Psi_{12}} \\ \bar{a}^{\Psi_{20}} \end{bmatrix} \tag{26}$$

IV. OUTPUT POWER, COUPLING POWER, AND THEIR MATRIX FORMS

In [6]-[7], some different surface formulations for the output power of a single body have been established, and several typical surface formulations for the output power of a double-body system are given in this section. In addition, the surface formulations of the coupling power between the two bodies in double-body system are also provided here.

A. The surface formulations for the powers of a double-body system

The input power P^{inp} from incident field to system and the output power P^{out} of system are as follows

$$\begin{aligned} P^{inp} = P^{out} &= \sum_{i=1,2}\Big[(1/2)\big\langle \bar{J}_i^{vop},\bar{E}^{inc}\big\rangle_{V_i} + (1/2)\big\langle \bar{H}^{inc},\bar{M}_i^{vm}\big\rangle_{V_i}\Big] \\ &= \sum_{i=1,2}\Big\{\Big[(1/2)\big\langle \bar{J}_i^{vop},\bar{E}^{tot}\big\rangle_{V_i} + (1/2)\big\langle \bar{H}^{tot},\bar{M}_i^{vm}\big\rangle_{V_i}\Big] \\ &\quad -\Big[(1/2)\big\langle \bar{J}_i^{vop},\bar{E}^{sca}\big\rangle_{V_i} + (1/2)\big\langle \bar{H}^{sca},\bar{M}_i^{vm}\big\rangle_{V_i}\Big]\Big\} \\ &= \sum_{i=1,2}\big(P_i^{out} + P_{ji}^{coup}\big) \end{aligned} \tag{27}$$

here P_i^{out} is the output power of V_i, and P_{ji}^{coup} corresponds to the coupling between V_i and V_j, and [7]

$$\begin{aligned} P_i^{out} &= (1/2)\big\langle \bar{J}_i^{vop},\bar{e}_i^{inc}\big\rangle_{V_i} + (1/2)\big\langle \bar{h}_i^{inc},\bar{M}_i^{vm}\big\rangle_{V_i} \\ &= \frac{\mu_i\Delta\varepsilon_{ic}^*}{\mu_0\varepsilon_0-\mu_i\varepsilon_{ic}^*}\left[\frac{1}{2}\big\langle \bar{J}_i^{SE},\bar{e}_{i;-}^{inc}\big\rangle_{\partial V_i} + \frac{\mu_0}{\mu_i}\frac{1}{2}\big\langle \bar{h}_{i;-}^{inc},\bar{M}_i^{SE}\big\rangle_{\partial V_i}^*\right] \\ &\quad + \frac{\varepsilon_{ic}\Delta\mu_i}{\varepsilon_0\mu_0-\varepsilon_{ic}\mu_i}\left[\frac{\varepsilon_0}{\varepsilon_{ic}}\frac{1}{2}\big\langle \bar{J}_i^{SE},\bar{e}_{i;-}^{inc}\big\rangle_{\partial V_i}^* + \frac{1}{2}\big\langle \bar{h}_{i;-}^{inc},\bar{M}_i^{SE}\big\rangle_{\partial V_i}\right] \end{aligned} \tag{28}$$

$$\begin{aligned} P_{ji}^{coup} &= -\Big[(1/2)\big\langle \bar{J}_i^{vop},\bar{E}_j^{sca}\big\rangle_{V_i} + (1/2)\big\langle \bar{H}_j^{sca},\bar{M}_i^{vm}\big\rangle_{V_i}\Big] \\ &= \frac{\mu_i\Delta\varepsilon_{ic}^*}{\mu_0\varepsilon_0-\mu_i\varepsilon_{ic}^*}\left[-\frac{1}{2}\big\langle \bar{J}_i^{SE},\bar{E}_j^{sca}\big\rangle_{\partial V_i} - \frac{\mu_0}{\mu_i}\frac{1}{2}\big\langle \bar{H}_j^{sca},\bar{M}_i^{SE}\big\rangle_{\partial V_i}^*\right] \\ &\quad + \frac{\varepsilon_{ic}\Delta\mu_i}{\varepsilon_0\mu_0-\varepsilon_{ic}\mu_i}\left[-\frac{\varepsilon_0}{\varepsilon_{ic}}\frac{1}{2}\big\langle \bar{J}_i^{SE},\bar{E}_j^{sca}\big\rangle_{\partial V_i}^* - \frac{1}{2}\big\langle \bar{H}_j^{sca},\bar{M}_i^{SE}\big\rangle_{\partial V_i}\right] \end{aligned} \tag{29}$$

here μ_i and ε_{ic} are respectively the permeability and complex permittivity of body V_i, and $\Delta\mu_i = \mu_i - \mu_0$, and $\Delta\varepsilon_{ic} = \varepsilon_{ic} - \varepsilon_0$; $\Delta\varepsilon_{ic}^* \triangleq (\Delta\varepsilon_{ic})^*$; the superscript "$*$" represents the complex conjugate of relevant quantity; the inner product is defined as $<\bar{f},\bar{g}>_\Omega \triangleq \int_\Omega \bar{f}^*\cdot\bar{g}\,d\Omega$. The derivation for the second equality of (29) is based on the Maxwell's equations of $\bar{F}_j^{sca}$ and $\bar{F}^{tot}$, and its detailed procedure will not be given here, because of its simplicity.

Based on the conclusions given in [5]-[7], the various powers related to the formulation (27) can be summarized in the (30). The first line in (30) corresponds to the second equality in (27), and they are based on the first equality of the formulation (12) in paper [5]; the second line in (30) corresponds to the third equality in (27), and they are based on the second equality of the formulation (12) in paper [5]; the third line in (30) corresponds to the fourth equality in (27), and they are based on the second equality of the formulation (6.1) in this paper.

In (30), the specific mathematical expressions of the powers $jP_i^{tot,react,mat}$, $P_i^{tot,loss}$, and $P_i^{sca,vac}$ are as follows [5]-[7]

$$\begin{aligned} jP_i^{tot,react,mat} &= (1/2)\big\langle \bar{J}_i^{vp},\bar{E}^{tot}\big\rangle_{V_i} + (1/2)\big\langle \bar{H}^{tot},\bar{M}_i^{vm}\big\rangle_{V_i} \\ &= j\,2\omega\left[\frac{1}{4}\big\langle \bar{H}^{tot},\Delta\mu\bar{H}^{tot}\big\rangle_{V_i} - \frac{1}{4}\big\langle \Delta\varepsilon\bar{E}^{tot},\bar{E}^{tot}\big\rangle_{V_i}\right] \end{aligned} \tag{31.1}$$

$$\begin{aligned} P_i^{tot,loss} &= (1/2)\big\langle \bar{J}_i^{vo},\bar{E}^{tot}\big\rangle_{V_i} \\ &= (1/2)\big\langle \sigma\bar{E}^{tot},\bar{E}^{tot}\big\rangle_{V_i} \end{aligned} \tag{31.2}$$

$$\begin{aligned} P_i^{sca,vac} &= -\Big[(1/2)\big\langle \bar{J}_i^{vop},\bar{E}_i^{sca}\big\rangle_{V_i} + (1/2)\big\langle \bar{H}_i^{sca},\bar{M}_i^{vm}\big\rangle_{V_i}\Big] \\ &= P_i^{sca,rad} + j\,P_i^{sca,react,vac} \end{aligned} \tag{31.3}$$

here the "j" appearing in (31.1) and (31.3) is the imaginary unity, and

$$P_i^{sca,rad} = \frac{1}{2}\oint\!\!\!\oint_{S_\infty}\Big[\bar{E}_i^{sca}\times\big(\bar{H}_i^{sca}\big)^*\Big]\cdot d\bar{S} \tag{32.1}$$

$$P_i^{sca,react,vac} = 2\omega\left[\frac{1}{4}\big\langle \bar{H}_i^{sca},\mu_0\bar{H}_i^{sca}\big\rangle_{\mathbb{R}^3} - \frac{1}{4}\big\langle \varepsilon_0\bar{E}_i^{sca},\bar{E}_i^{sca}\big\rangle_{\mathbb{R}^3}\right] \tag{32.2}$$

The powers $P^{inc\to V_i}$, $P^{tot\to V_i}$, and $P^{sca\to V_i}$ are respectively the powers done by $\bar{F}^{inc}$, $\bar{F}^{tot}$, and $\bar{F}^{sca}$ on V_i, and their mathematical expressions are as follows

$$P^{inc\to V_i} = (1/2)\big\langle \bar{J}_i^{vop},\bar{E}^{inc}\big\rangle_{V_i} + (1/2)\big\langle \bar{H}^{inc},\bar{M}_i^{vm}\big\rangle_{V_i} \tag{33.1}$$

$$P^{tot\to V_i} = (1/2)\big\langle \bar{J}_i^{vop},\bar{E}^{tot}\big\rangle_{V_i} + (1/2)\big\langle \bar{H}^{tot},\bar{M}_i^{vm}\big\rangle_{V_i} \tag{33.2}$$

$$P^{sca\to V_i} = (1/2)\big\langle \bar{J}_i^{vop},\bar{E}^{sca}\big\rangle_{V_i} + (1/2)\big\langle \bar{H}^{sca},\bar{M}_i^{vm}\big\rangle_{V_i} \tag{33.3}$$

The powers $P^{sca,vac}$ and P^{coup} are as follows

$$P^{sca,vac} = P^{sca,rad} + j\,P^{sca,react,vac} \tag{34}$$

$$P^{coup} = P^{coup,rad} + j\,P^{coup,react,vac} \tag{35}$$

here

$$P^{sca,rad} = \frac{1}{2}\oint\!\!\!\oint_{S_\infty}\Big[\bar{E}^{sca}\times\big(\bar{H}^{sca}\big)^*\Big]\cdot d\bar{S} \tag{36.1}$$

$$P^{sca,react,vac} = 2\omega\left[\frac{1}{4}\big\langle \bar{H}^{sca},\mu_0\bar{H}^{sca}\big\rangle_{\mathbb{R}^3} - \frac{1}{4}\big\langle \varepsilon_0\bar{E}^{sca},\bar{E}^{sca}\big\rangle_{\mathbb{R}^3}\right] \tag{36.2}$$

and

$$\begin{aligned} P^{inp} = P^{out} &= \overbrace{j\,P_1^{tot,react,mat} + P_1^{tot,loss} + P_1^{sca,vac} + P_{21}^{coup}}^{P^{inc\to V_1}} + \overbrace{P_{12}^{coup} + P_2^{sca,vac} + P_2^{tot,loss} + j\,P_2^{tot,react,mat}}^{P^{inc\to V_2}} \\ &= \overbrace{j\,P_1^{tot,react,mat} + P_1^{tot,loss}}^{P^{tot\to V_1}} + \underbrace{\overbrace{P_1^{sca,vac} + P_{21}^{coup}}^{-P^{sca\to V_1}} + \overbrace{P_{12}^{coup} + P_2^{sca,vac}}^{-P^{sca\to V_2}}}_{-P^{sca\to V} = P^{sca,vac}} + \overbrace{P_2^{tot,loss} + j\,P_2^{tot,react,mat}}^{P^{tot\to V_2}} \\ &= \underbrace{j\,P_1^{tot,react,mat} + P_1^{tot,loss} + P_1^{sca,vac}}_{P_1^{out}} + \underbrace{P_{21}^{coup} + P_{12}^{coup}}_{P^{coup}} + \underbrace{P_2^{sca,vac} + P_2^{tot,loss} + j\,P_2^{tot,react,mat}}_{P_2^{out}} \end{aligned} \tag{30}$$

$$P^{coup,rad} = \frac{1}{2}\oiint_{S_\infty}\left[\bar{E}_j^{sca}\times\left(\bar{H}_i^{sca}\right)^* + \bar{E}_i^{sca}\times\left(\bar{H}_j^{sca}\right)^*\right]\cdot d\bar{S} \quad (37.1)$$

$$\begin{aligned} P^{coup,react,vac} = & \ 2\omega\left[\frac{1}{4}\left\langle\bar{H}_i^{sca},\mu_0\bar{H}_j^{sca}\right\rangle_{\mathbb{R}^3} - \frac{1}{4}\left\langle\varepsilon_0\bar{E}_i^{sca},\bar{E}_j^{sca}\right\rangle_{\mathbb{R}^3}\right] \\ & + 2\omega\left[\frac{1}{4}\left\langle\bar{H}_j^{sca},\mu_0\bar{H}_i^{sca}\right\rangle_{\mathbb{R}^3} - \frac{1}{4}\left\langle\varepsilon_0\bar{E}_j^{sca},\bar{E}_i^{sca}\right\rangle_{\mathbb{R}^3}\right] \end{aligned} \quad (37.2)$$

B. The matrix forms for the powers of a double-body system: The relation (24) is not utilized.

Inserting the (4)-(8) and (21) into the (28)-(29), the P_i^{out} and P_{ji}^{coup} can be written as the following matrix forms.

$$P_i^{out} = \bar{a}^H\cdot\bar{\bar{P}}_i\cdot\bar{a} \quad (38)$$

$$P_{ji}^{coup} = \bar{a}^H\cdot\bar{\bar{P}}_{ji}\cdot\bar{a} \quad (39)$$

and then the (27) can be written as the following matrix form

$$P^{inp} = P^{out} = \bar{a}^H\cdot\bar{\bar{P}}\cdot\bar{a} \quad (40)$$

here

$$\bar{\bar{P}} = \bar{\bar{P}}_1 + \bar{\bar{P}}_{21} + \bar{\bar{P}}_{12} + \bar{\bar{P}}_2 \quad (41)$$

and

$$\bar{a} = \begin{bmatrix}\bar{a}^J\\ \bar{a}^M\end{bmatrix} \quad (42)$$

To derive the matrices $\bar{\bar{P}}_1$ and $\bar{\bar{P}}_2$ based on the first equality in (28)

If the first equality in (28) is utilized, the matrices $\bar{\bar{P}}_1$ and $\bar{\bar{P}}_2$ are as follows

$$\bar{\bar{P}}_1 = \begin{bmatrix} \bar{\bar{P}}_1^{J_{10}J_{10}} & \bar{\bar{P}}_1^{J_{10}J_{12}} & 0 & \bar{\bar{P}}_1^{J_{10}M_{10}} & \bar{\bar{P}}_1^{J_{10}M_{12}} & 0 \\ \bar{\bar{P}}_1^{J_{12}J_{10}} & \bar{\bar{P}}_1^{J_{12}J_{12}} & 0 & \bar{\bar{P}}_1^{J_{12}M_{10}} & \bar{\bar{P}}_1^{J_{12}M_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{\bar{P}}_1^{M_{10}J_{10}} & \bar{\bar{P}}_1^{M_{10}J_{12}} & 0 & \bar{\bar{P}}_1^{M_{10}M_{10}} & \bar{\bar{P}}_1^{M_{10}M_{12}} & 0 \\ \bar{\bar{P}}_1^{M_{12}J_{10}} & \bar{\bar{P}}_1^{M_{12}J_{12}} & 0 & \bar{\bar{P}}_1^{M_{12}M_{10}} & \bar{\bar{P}}_1^{M_{12}M_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (43.1)$$

$$\bar{\bar{P}}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\bar{P}}_2^{J_{12}J_{12}} & \bar{\bar{P}}_2^{J_{12}J_{20}} & 0 & \bar{\bar{P}}_2^{J_{12}M_{12}} & \bar{\bar{P}}_2^{J_{12}M_{20}} \\ 0 & \bar{\bar{P}}_2^{J_{20}J_{12}} & \bar{\bar{P}}_2^{J_{20}J_{20}} & 0 & \bar{\bar{P}}_2^{J_{20}M_{12}} & \bar{\bar{P}}_2^{J_{20}M_{20}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\bar{P}}_2^{M_{12}J_{12}} & \bar{\bar{P}}_2^{M_{12}J_{20}} & 0 & \bar{\bar{P}}_2^{M_{12}M_{12}} & \bar{\bar{P}}_2^{M_{12}M_{20}} \\ 0 & \bar{\bar{P}}_2^{M_{20}J_{12}} & \bar{\bar{P}}_2^{M_{20}J_{20}} & 0 & \bar{\bar{P}}_2^{M_{20}M_{12}} & \bar{\bar{P}}_2^{M_{20}M_{20}} \end{bmatrix} \quad (43.2)$$

here the partition ways of $\bar{\bar{P}}_1$ and $\bar{\bar{P}}_2$ are based on the partition ways of (26) and (42), and

$$\bar{\bar{P}}_i^{C'_{i0}C''_{i0}} = \left[p_{i;\xi\zeta}^{C'_{i0}C''_{i0}}\right]_{\Xi^{C'_{i0}}\times\Xi^{C''_{i0}}} \quad (44.1)$$

$$\bar{\bar{P}}_i^{C'_{i0}C''_{12}} = \left[p_{i;\xi\zeta}^{C'_{i0}C''_{12}}\right]_{\Xi^{C'_{i0}}\times\Xi^{C''_{12}}} \quad (44.2)$$

$$\bar{\bar{P}}_i^{C''_{12}C'_{i0}} = \left[p_{i;\xi\zeta}^{C''_{12}C'_{i0}}\right]_{\Xi^{C''_{12}}\times\Xi^{C'_{i0}}} \quad (44.3)$$

$$\bar{\bar{P}}_i^{C'_{12}C''_{12}} = \left[p_{i;\xi\zeta}^{C'_{12}C''_{12}}\right]_{\Xi^{C'_{12}}\times\Xi^{C''_{12}}} \quad (44.4)$$

here

$$\begin{aligned} p_{i;\xi\zeta}^{C'_{i0}C''_{i0}} = & \ (1/2)\left\langle\mathcal{J}_i^{vop}\left(\bar{b}_\xi^{C'_{i0}}\right),\mathcal{e}_{i;-}^{inc}\left(\bar{b}_\zeta^{C''_{i0}}\right)\right\rangle_{\text{int}V_i} \\ & + (1/2)\left\langle\mathcal{h}_{i;-}^{inc}\left(\bar{b}_\xi^{C'_{i0}}\right),\mathcal{M}_i^{vm}\left(\bar{b}_\zeta^{C''_{i0}}\right)\right\rangle_{\text{int}V_i} \end{aligned} \quad (45.1)$$

$$\begin{aligned} p_{i;\xi\zeta}^{C'_{i0}C''_{12}} = & \ \gamma_i\Big[(1/2)\left\langle\mathcal{J}_i^{vop}\left(\bar{b}_\xi^{C'_{i0}}\right),\mathcal{e}_{i;-}^{inc}\left(\bar{b}_\zeta^{C''_{12}}\right)\right\rangle_{\text{int}V_i} \\ & + (1/2)\left\langle\mathcal{h}_{i;-}^{inc}\left(\bar{b}_\xi^{C'_{i0}}\right),\mathcal{M}_i^{vm}\left(\bar{b}_\zeta^{C''_{12}}\right)\right\rangle_{\text{int}V_i}\Big] \end{aligned} \quad (45.2)$$

$$\begin{aligned} p_{i;\xi\zeta}^{C''_{12}C'_{i0}} = & \ \gamma_i\Big[(1/2)\left\langle\mathcal{J}_i^{vop}\left(\bar{b}_\zeta^{C''_{12}}\right),\mathcal{e}_{i;-}^{inc}\left(\bar{b}_\xi^{C'_{i0}}\right)\right\rangle_{\text{int}V_i} \\ & + (1/2)\left\langle\mathcal{h}_{i;-}^{inc}\left(\bar{b}_\zeta^{C''_{12}}\right),\mathcal{M}_i^{vm}\left(\bar{b}_\xi^{C'_{i0}}\right)\right\rangle_{\text{int}V_i}\Big] \end{aligned} \quad (45.3)$$

$$\begin{aligned} p_{i;\xi\zeta}^{C'_{12}C''_{12}} = & \ (1/2)\left\langle\mathcal{J}_i^{vop}\left(\bar{b}_\xi^{C'_{12}}\right),\mathcal{e}_{i;-}^{inc}\left(\bar{b}_\zeta^{C''_{12}}\right)\right\rangle_{\text{int}V_i} \\ & + (1/2)\left\langle\mathcal{h}_{i;-}^{inc}\left(\bar{b}_\xi^{C'_{12}}\right),\mathcal{M}_i^{vm}\left(\bar{b}_\zeta^{C''_{12}}\right)\right\rangle_{\text{int}V_i} \end{aligned} \quad (45.4)$$

In (44)-(45), $i=1,2$; $C',C''=J,M$; $\gamma_1=1$, and $\gamma_2=-1$; $\mathcal{J}_i^{vop}\left(\bar{b}_\xi^{J_{i0}}\right)\triangleq\mathcal{J}_i^{vop}\left(\bar{b}_\xi^{J_{i0}},0\right)$, and $\mathcal{J}_i^{vop}\left(\bar{b}_\xi^{M_{i0}}\right)\triangleq\mathcal{J}_i^{vop}\left(0,\bar{b}_\xi^{M_{i0}}\right)$, and the other operators can be similarly explained.

To derive the matrices $\bar{\bar{P}}_1$ and $\bar{\bar{P}}_2$ based on the second equality in (28)

If the second equality in (28) is utilized, the matrices $\bar{\bar{P}}_1$ and $\bar{\bar{P}}_2$ are as follows

$$\begin{aligned} \bar{\bar{P}}_i = & \ \frac{\mu_i\Delta\varepsilon_{ic}^*}{\mu_0\varepsilon_0-\mu_i\varepsilon_{ic}^*}\left(\bar{\bar{P}}_{i;JE}+\frac{\mu_0}{\mu_i}\bar{\bar{P}}_{i;HM}^H\right) \\ & + \frac{\varepsilon_{ic}\Delta\mu_i}{\varepsilon_0\mu_0-\varepsilon_{ic}\mu_i}\left(\frac{\varepsilon_0}{\varepsilon_{ic}}\bar{\bar{P}}_{i;JE}^H+\bar{\bar{P}}_{i;HM}\right) \end{aligned} \quad (46)$$

here $i=1,2$. The above matrices $\bar{\bar{P}}_{i;JE}$ and $\bar{\bar{P}}_{i;HM}$ are as follows

$$\bar{\bar{P}}_{1;JE} = \begin{bmatrix} \bar{\bar{P}}_{1;JE}^{J_{10}J_{10}} & \bar{\bar{P}}_{1;JE}^{J_{10}J_{12}} & 0 & \bar{\bar{P}}_{1;JE}^{J_{10}M_{10}} & \bar{\bar{P}}_{1;JE}^{J_{10}M_{12}} & 0 \\ \bar{\bar{P}}_{1;JE}^{J_{12}J_{10}} & \bar{\bar{P}}_{1;JE}^{J_{12}J_{12}} & 0 & \bar{\bar{P}}_{1;JE}^{J_{12}M_{10}} & \bar{\bar{P}}_{1;JE}^{J_{12}M_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (47.1)$$

$$\bar{\bar{P}}_{2;JE} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\bar{P}}_{2;JE}^{J_{12}J_{12}} & \bar{\bar{P}}_{2;JE}^{J_{12}J_{20}} & 0 & \bar{\bar{P}}_{2;JE}^{J_{12}M_{12}} & \bar{\bar{P}}_{2;JE}^{J_{12}M_{20}} \\ 0 & \bar{\bar{P}}_{2;JE}^{J_{20}J_{12}} & \bar{\bar{P}}_{2;JE}^{J_{20}J_{20}} & 0 & \bar{\bar{P}}_{2;JE}^{J_{20}M_{12}} & \bar{\bar{P}}_{2;JE}^{J_{20}M_{20}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (47.2)$$

and

$$\overline{\overline{P}}_{1;HM} = \begin{bmatrix} 0 & 0 & 0 & \overline{\overline{P}}_{1;HM}^{J_{10}M_{10}} & \overline{\overline{P}}_{1;HM}^{J_{10}M_{12}} & 0 \\ 0 & 0 & 0 & \overline{\overline{P}}_{1;HM}^{J_{12}M_{10}} & \overline{\overline{P}}_{1;HM}^{J_{12}M_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\overline{P}}_{1;HM}^{M_{10}M_{10}} & \overline{\overline{P}}_{1;HM}^{M_{10}M_{12}} & 0 \\ 0 & 0 & 0 & \overline{\overline{P}}_{1;HM}^{M_{12}M_{10}} & \overline{\overline{P}}_{1;HM}^{M_{12}M_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{48.1}$$

$$\overline{\overline{P}}_{2;HM} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{\overline{P}}_{2;HM}^{J_{12}M_{12}} & \overline{\overline{P}}_{2;HM}^{J_{12}M_{20}} \\ 0 & 0 & 0 & 0 & \overline{\overline{P}}_{2;HM}^{J_{20}M_{12}} & \overline{\overline{P}}_{2;HM}^{J_{20}M_{20}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{\overline{P}}_{2;HM}^{M_{12}M_{12}} & \overline{\overline{P}}_{2;HM}^{M_{12}M_{20}} \\ 0 & 0 & 0 & 0 & \overline{\overline{P}}_{2;HM}^{M_{20}M_{12}} & \overline{\overline{P}}_{2;HM}^{M_{20}M_{20}} \end{bmatrix} \tag{48.2}$$

in which

$$\overline{\overline{P}}_{i;JE}^{J_{i0}C'_{i0}} = \left[p_{i;JE;\xi\zeta}^{J_{i0}C'_{i0}} \right]_{\Xi^{J_{i0}} \times \Xi^{C'_{i0}}} \tag{49.1}$$

$$\overline{\overline{P}}_{i;JE}^{J_{i0}C'_{12}} = \left[p_{i;JE;\xi\zeta}^{J_{i0}C'_{12}} \right]_{\Xi^{J_{i0}} \times \Xi^{C'_{12}}} \tag{49.2}$$

$$\overline{\overline{P}}_{i;JE}^{J_{12}C'_{i0}} = \left[p_{i;JE;\xi\zeta}^{J_{12}C'_{i0}} \right]_{\Xi^{J_{12}} \times \Xi^{C'_{i0}}} \tag{49.3}$$

$$\overline{\overline{P}}_{i;JE}^{J_{12}C'_{12}} = \left[p_{i;JE;\xi\zeta}^{J_{12}C'_{12}} \right]_{\Xi^{J_{12}} \times \Xi^{C'_{12}}} \tag{49.4}$$

and

$$\overline{\overline{P}}_{i;HM}^{C'_{i0}M_{i0}} = \left[p_{i;HM;\xi\zeta}^{C'_{i0}M_{i0}} \right]_{\Xi^{C'_{i0}} \times \Xi^{M_{i0}}} \tag{50.1}$$

$$\overline{\overline{P}}_{i;HM}^{C'_{12}M_{i0}} = \left[p_{i;HM;\xi\zeta}^{C'_{12}M_{i0}} \right]_{\Xi^{C'_{12}} \times \Xi^{M_{i0}}} \tag{50.2}$$

$$\overline{\overline{P}}_{i;HM}^{C'_{i0}M_{12}} = \left[p_{i;HM;\xi\zeta}^{C'_{i0}M_{12}} \right]_{\Xi^{C'_{i0}} \times \Xi^{M_{12}}} \tag{50.3}$$

$$\overline{\overline{P}}_{i;HM}^{C'_{12}M_{12}} = \left[p_{i;HM;\xi\zeta}^{C'_{12}M_{12}} \right]_{\Xi^{C'_{12}} \times \Xi^{M_{12}}} \tag{50.4}$$

here

$$p_{i;JE;\xi\zeta}^{J_{i0}C'_{i0}} = (1/2)\left\langle \overline{b}_{\xi}^{J_{i0}}, e_{i;-}^{inc}\left(\overline{b}_{\zeta}^{C'_{i0}}\right)\right\rangle_{\partial V_i} \tag{51.1}$$

$$p_{i;JE;\xi\zeta}^{J_{i0}C'_{12}} = \gamma_i (1/2)\left\langle \overline{b}_{\xi}^{J_{i0}}, e_{i;-}^{inc}\left(\overline{b}_{\zeta}^{C'_{12}}\right)\right\rangle_{\partial V_i} \tag{51.2}$$

$$p_{i;JE;\xi\zeta}^{J_{12}C'_{i0}} = \gamma_i (1/2)\left\langle \overline{b}_{\xi}^{J_{12}}, e_{i;-}^{inc}\left(\overline{b}_{\zeta}^{C'_{i0}}\right)\right\rangle_{\partial V_i} \tag{51.3}$$

$$p_{i;JE;\xi\zeta}^{J_{12}C'_{12}} = (1/2)\left\langle \overline{b}_{\xi}^{J_{12}}, e_{i;-}^{inc}\left(\overline{b}_{\zeta}^{C'_{12}}\right)\right\rangle_{\partial V_i} \tag{51.4}$$

and

$$p_{i;HM;\xi\zeta}^{C'_{i0}M_{i0}} = (1/2)\left\langle h_{i;-}^{inc}\left(\overline{b}_{\xi}^{C'_{i0}}\right), \overline{b}_{\zeta}^{M_{i0}}\right\rangle_{\partial V_i} \tag{52.1}$$

$$p_{i;HM;\xi\zeta}^{C'_{12}M_{i0}} = \gamma_i (1/2)\left\langle h_{i;-}^{inc}\left(\overline{b}_{\xi}^{C'_{12}}\right), \overline{b}_{\zeta}^{M_{i0}}\right\rangle_{\partial V_i} \tag{52.2}$$

$$p_{i;HM;\xi\zeta}^{C'_{i0}M_{12}} = \gamma_i (1/2)\left\langle h_{i;-}^{inc}\left(\overline{b}_{\xi}^{C'_{i0}}\right), \overline{b}_{\zeta}^{M_{12}}\right\rangle_{\partial V_i} \tag{52.3}$$

$$p_{i;HM;\xi\zeta}^{C'_{12}M_{12}} = (1/2)\left\langle h_{i;-}^{inc}\left(\overline{b}_{\xi}^{C'_{12}}\right), \overline{b}_{\zeta}^{M_{12}}\right\rangle_{\partial V_i} \tag{52.4}$$

In (49)-(52), $i = 1,2$; $C' = J, M$.

To derive the matrix $\overline{\overline{P}}_{ji}$ based on the first equality in (29)

If the first equality in (29) is utilized, the matrix $\overline{\overline{P}}_{ji}$ in (39) is as follows

$$\overline{\overline{P}}_{ji} = \begin{bmatrix} 0 & \overline{\overline{P}}_{ji}^{J_{10}J_{12}} & \overline{\overline{P}}_{ji}^{J_{10}J_{20}} & 0 & \overline{\overline{P}}_{ji}^{J_{10}M_{12}} & \overline{\overline{P}}_{ji}^{J_{10}M_{20}} \\ \overline{\overline{P}}_{ji}^{J_{12}J_{10}} & \overline{\overline{P}}_{ji}^{J_{12}J_{12}} & \overline{\overline{P}}_{ji}^{J_{12}J_{20}} & \overline{\overline{P}}_{ji}^{J_{12}M_{10}} & \overline{\overline{P}}_{ji}^{J_{12}M_{12}} & \overline{\overline{P}}_{ji}^{J_{12}M_{20}} \\ \overline{\overline{P}}_{ji}^{J_{20}J_{10}} & \overline{\overline{P}}_{ji}^{J_{20}J_{12}} & 0 & \overline{\overline{P}}_{ji}^{J_{20}M_{10}} & \overline{\overline{P}}_{ji}^{J_{20}M_{12}} & 0 \\ 0 & \overline{\overline{P}}_{ji}^{M_{10}J_{12}} & \overline{\overline{P}}_{ji}^{M_{10}J_{20}} & 0 & \overline{\overline{P}}_{ji}^{M_{10}M_{12}} & \overline{\overline{P}}_{ji}^{M_{10}M_{20}} \\ \overline{\overline{P}}_{ji}^{M_{12}J_{10}} & \overline{\overline{P}}_{ji}^{M_{12}J_{12}} & \overline{\overline{P}}_{ji}^{M_{12}J_{20}} & \overline{\overline{P}}_{ji}^{M_{12}M_{10}} & \overline{\overline{P}}_{ji}^{M_{12}M_{12}} & \overline{\overline{P}}_{ji}^{M_{12}M_{20}} \\ \overline{\overline{P}}_{ji}^{M_{20}J_{10}} & \overline{\overline{P}}_{ji}^{M_{20}J_{12}} & 0 & \overline{\overline{P}}_{ji}^{M_{20}M_{10}} & \overline{\overline{P}}_{ji}^{M_{20}M_{12}} & 0 \end{bmatrix} \tag{53}$$

The submatrices in (53) are as follows

$$\overline{\overline{P}}_{ji}^{C'_{j0}C''_{12}} = \left[p_{ji;\xi\zeta}^{C'_{j0}C''_{12}} \right]_{\Xi^{C'_{j0}} \times \Xi^{C''_{12}}} \tag{54.1}$$

$$\overline{\overline{P}}_{ji}^{C''_{12}C'_{j0}} = \left[p_{ji;\zeta\xi}^{C''_{12}C'_{j0}} \right]_{\Xi^{C''_{12}} \times \Xi^{C'_{j0}}} \tag{54.2}$$

and

$$\overline{\overline{P}}_{ji}^{C'_{i0}C''_{12}} = \left[p_{ji;\xi\zeta}^{C'_{i0}C''_{12}} \right]_{\Xi^{C'_{i0}} \times \Xi^{C''_{12}}} \tag{54.3}$$

$$\overline{\overline{P}}_{ji}^{C''_{12}C'_{i0}} = \left[p_{ji;\zeta\xi}^{C''_{12}C'_{i0}} \right]_{\Xi^{C''_{12}} \times \Xi^{C'_{i0}}} \tag{54.4}$$

and

$$\overline{\overline{P}}_{ji}^{C_{12}C_{12}} = \left[p_{ji;\xi\zeta}^{C_{12}C_{12}} \right]_{\Xi^{C_{12}} \times \Xi^{C_{12}}} \tag{54.5}$$

and

$$\overline{\overline{P}}_{ji}^{J_{12}M_{12}} = \left[p_{ji;\xi\zeta}^{J_{12}M_{12}} \right]_{\Xi^{J_{12}} \times \Xi^{M_{12}}} \tag{54.6}$$

$$\overline{\overline{P}}_{ji}^{M_{12}J_{12}} = \left[p_{ji;\zeta\xi}^{M_{12}J_{12}} \right]_{\Xi^{M_{12}} \times \Xi^{J_{12}}} \tag{54.7}$$

and

$$\overline{\overline{P}}_{ji}^{C'_{10}C''_{20}} = \left[p_{ji;\xi\zeta}^{C'_{10}C''_{20}} \right]_{\Xi^{C'_{10}} \times \Xi^{C''_{20}}} \tag{54.8}$$

$$\overline{\overline{P}}_{ji}^{C''_{20}C'_{10}} = \left[p_{ji;\zeta\xi}^{C''_{20}C'_{10}} \right]_{\Xi^{C''_{20}} \times \Xi^{C'_{10}}} \tag{54.9}$$

In (54.1) and (54.2),

$$p_{ji;\xi\zeta}^{C'_{j0}C''_{12}} = -\gamma_i (1/2)\left\langle \mathcal{H}_{j;+}^{sca}\left(\overline{b}_{\xi}^{C'_{j0}}\right), \mathcal{M}_i^{vm}\left(\overline{b}_{\zeta}^{C''_{12}}\right)\right\rangle_{\text{int}\, V_i} \tag{55.1}$$

$$p_{ji;\zeta\xi}^{C''_{12}C'_{j0}} = -\gamma_i (1/2)\left\langle \mathcal{J}_i^{vop}\left(\overline{b}_{\zeta}^{C''_{12}}\right), \mathcal{E}_{j;+}^{sca}\left(\overline{b}_{\xi}^{C'_{j0}}\right)\right\rangle_{\text{int}\, V_i} \tag{55.2}$$

In (54.3) and (54.4),

$$p_{ji;\xi\zeta}^{C'_{i0}C''_{12}} = -\gamma_j (1/2)\left\langle \mathcal{J}_i^{vop}\left(\overline{b}_{\xi}^{C'_{i0}}\right), \mathcal{E}_{j;+}^{sca}\left(\overline{b}_{\zeta}^{C''_{12}}\right)\right\rangle_{\text{int}\, V_i} \tag{55.3}$$

$$p_{ji;\zeta\xi}^{C''_{12}C'_{i0}} = -\gamma_j (1/2)\left\langle \mathcal{H}_{j;+}^{sca}\left(\overline{b}_{\zeta}^{C''_{12}}\right), \mathcal{M}_i^{vm}\left(\overline{b}_{\xi}^{C'_{i0}}\right)\right\rangle_{\text{int}\, V_i} \tag{55.4}$$

In (54.5),

$$\begin{aligned} p_{ji;\xi\zeta}^{C_{12}C_{12}} &= (1/2)\left\langle \mathcal{J}_i^{vop}\left(\bar{b}_\xi^{C_{12}}\right), \mathcal{E}_{j;+}^{sca}\left(\bar{b}_\zeta^{C_{12}}\right)\right\rangle_{\operatorname{int}V_i} \\ &\quad + (1/2)\left\langle \mathcal{H}_{j;+}^{sca}\left(\bar{b}_\xi^{C_{12}}\right), \mathcal{M}_i^{vm}\left(\bar{b}_\zeta^{C_{12}}\right)\right\rangle_{\operatorname{int}V_i} \end{aligned} \tag{55.5}$$

In (54.6) and (54.7),

$$\begin{aligned} p_{ji;\xi\zeta}^{J_{12}M_{12}} &= (1/2)\left\langle \mathcal{J}_i^{vop}\left(\bar{b}_\xi^{J_{12}}\right), \mathcal{E}_{j;+}^{sca}\left(\bar{b}_\zeta^{M_{12}}\right)\right\rangle_{\operatorname{int}V_i} \\ &\quad + (1/2)\left\langle \mathcal{H}_{j;+}^{sca}\left(\bar{b}_\xi^{J_{12}}\right), \mathcal{M}_i^{vm}\left(\bar{b}_\zeta^{M_{12}}\right)\right\rangle_{\operatorname{int}V_i} \end{aligned} \tag{55.6}$$

$$\begin{aligned} p_{ji;\zeta\xi}^{M_{12}J_{12}} &= (1/2)\left\langle \mathcal{J}_i^{vop}\left(\bar{b}_\zeta^{M_{12}}\right), \mathcal{E}_{j;+}^{sca}\left(\bar{b}_\xi^{J_{12}}\right)\right\rangle_{\operatorname{int}V_i} \\ &\quad + (1/2)\left\langle \mathcal{H}_{j;+}^{sca}\left(\bar{b}_\zeta^{M_{12}}\right), \mathcal{M}_i^{vm}\left(\bar{b}_\xi^{J_{12}}\right)\right\rangle_{\operatorname{int}V_i} \end{aligned} \tag{55.7}$$

In (54.8) and (54.9),

$$\begin{aligned} p_{ji;\xi\zeta}^{C'_{10}C''_{20}} &= -\Big[\delta_{2j}(1/2)\left\langle \mathcal{J}_1^{vop}\left(\bar{b}_\xi^{C'_{10}}\right), \mathcal{E}_{2;+}^{sca}\left(\bar{b}_\zeta^{C''_{20}}\right)\right\rangle_{\operatorname{int}V_1} \\ &\quad + \delta_{1j}(1/2)\left\langle \mathcal{H}_{1;+}^{sca}\left(\bar{b}_\xi^{C'_{10}}\right), \mathcal{M}_2^{vm}\left(\bar{b}_\zeta^{C''_{20}}\right)\right\rangle_{\operatorname{int}V_2}\Big] \end{aligned} \tag{55.8}$$

$$\begin{aligned} p_{ji;\zeta\xi}^{C''_{20}C'_{10}} &= -\Big[\delta_{1j}(1/2)\left\langle \mathcal{J}_2^{vop}\left(\bar{b}_\zeta^{C''_{20}}\right), \mathcal{E}_{1;+}^{sca}\left(\bar{b}_\xi^{C'_{10}}\right)\right\rangle_{\operatorname{int}V_2} \\ &\quad + \delta_{2j}(1/2)\left\langle \mathcal{H}_{2;+}^{sca}\left(\bar{b}_\zeta^{C''_{20}}\right), \mathcal{M}_1^{vm}\left(\bar{b}_\xi^{C'_{10}}\right)\right\rangle_{\operatorname{int}V_1}\Big] \end{aligned} \tag{55.9}$$

In the above (53)-(55), $(j,i)=(2,1),(1,2)$; $C',C''=J,M$, and $C=J,M$; δ_{mn} is the Kronecker delta symbol.

To derive the matrix $\overline{\overline{P}}_{ji}$ based on the second equality in (29)

If the second equality in (29) is utilized, the matrix $\overline{\overline{P}}_{ji}$ in (39) is as follows

$$\begin{aligned} \overline{\overline{P}}_{ji} &= \frac{\mu_i \Delta\varepsilon_{ic}^*}{\mu_0\varepsilon_0 - \mu_i\varepsilon_{ic}^*}\left(\overline{\overline{P}}_{ji;JE} + \frac{\mu_0}{\mu_i}\overline{\overline{P}}_{ji;HM}^H\right) \\ &\quad + \frac{\varepsilon_{ic}\Delta\mu_i}{\varepsilon_0\mu_0 - \varepsilon_{ic}\mu_i}\left(\frac{\varepsilon_0}{\varepsilon_{ic}}\overline{\overline{P}}_{ji;JE}^H + \overline{\overline{P}}_{ji;HM}\right) \end{aligned} \tag{56}$$

The above matrices $\overline{\overline{P}}_{ji;JE}$ and $\overline{\overline{P}}_{ji;HM}$ are as follows

$$\overline{\overline{P}}_{21;JE} = \begin{bmatrix} 0 & \overline{\overline{P}}_{21;JE}^{J_{10}J_{12}} & \overline{\overline{P}}_{21;JE}^{J_{10}J_{20}} & 0 & \overline{\overline{P}}_{21;JE}^{J_{10}M_{12}} & \overline{\overline{P}}_{21;JE}^{J_{10}M_{20}} \\ 0 & \overline{\overline{P}}_{21;JE}^{J_{12}J_{12}} & \overline{\overline{P}}_{21;JE}^{J_{12}J_{20}} & 0 & \overline{\overline{P}}_{21;JE}^{J_{12}M_{12}} & \overline{\overline{P}}_{21;JE}^{J_{12}M_{20}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{57.1}$$

$$\overline{\overline{P}}_{12;JE} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \overline{\overline{P}}_{12;JE}^{J_{12}J_{10}} & \overline{\overline{P}}_{12;JE}^{J_{12}J_{12}} & 0 & \overline{\overline{P}}_{12;JE}^{J_{12}M_{10}} & \overline{\overline{P}}_{12;JE}^{J_{12}M_{12}} & 0 \\ \overline{\overline{P}}_{12;JE}^{J_{20}J_{10}} & \overline{\overline{P}}_{12;JE}^{J_{20}J_{12}} & 0 & \overline{\overline{P}}_{12;JE}^{J_{20}M_{10}} & \overline{\overline{P}}_{12;JE}^{J_{20}M_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{57.2}$$

and

$$\overline{\overline{P}}_{21;HM} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\overline{P}}_{21;HM}^{J_{12}M_{10}} & \overline{\overline{P}}_{21;HM}^{J_{12}M_{12}} & 0 \\ 0 & 0 & 0 & \overline{\overline{P}}_{21;HM}^{J_{20}M_{10}} & \overline{\overline{P}}_{21;HM}^{J_{20}M_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\overline{P}}_{21;HM}^{M_{12}M_{10}} & \overline{\overline{P}}_{21;HM}^{M_{12}M_{12}} & 0 \\ 0 & 0 & 0 & \overline{\overline{P}}_{21;HM}^{M_{20}M_{10}} & \overline{\overline{P}}_{21;HM}^{M_{20}M_{12}} & 0 \end{bmatrix} \tag{58.1}$$

$$\overline{\overline{P}}_{12;HM} = \begin{bmatrix} 0 & 0 & 0 & 0 & \overline{\overline{P}}_{12;HM}^{J_{10}M_{12}} & \overline{\overline{P}}_{12;HM}^{J_{10}M_{20}} \\ 0 & 0 & 0 & 0 & \overline{\overline{P}}_{12;HM}^{J_{12}M_{12}} & \overline{\overline{P}}_{12;HM}^{J_{12}M_{20}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{\overline{P}}_{12;HM}^{M_{10}M_{12}} & \overline{\overline{P}}_{12;HM}^{M_{10}M_{20}} \\ 0 & 0 & 0 & 0 & \overline{\overline{P}}_{12;HM}^{M_{12}M_{12}} & \overline{\overline{P}}_{12;HM}^{M_{12}M_{20}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{58.2}$$

in which

$$\overline{\overline{P}}_{ji;JE}^{J_{i0}C'_{j0}} = \left[p_{ji;JE;\xi\zeta}^{J_{i0}C'_{j0}}\right]_{\Xi^{J_{i0}}\times\Xi^{C'_{j0}}} \tag{59.1}$$

$$\overline{\overline{P}}_{ji;JE}^{J_{i0}C'_{12}} = \left[p_{ji;JE;\xi\zeta}^{J_{i0}C'_{12}}\right]_{\Xi^{J_{i0}}\times\Xi^{C'_{12}}} \tag{59.2}$$

$$\overline{\overline{P}}_{ji;JE}^{J_{12}C'_{j0}} = \left[p_{ji;JE;\xi\zeta}^{J_{12}C'_{j0}}\right]_{\Xi^{J_{12}}\times\Xi^{C'_{j0}}} \tag{59.3}$$

$$\overline{\overline{P}}_{ji;JE}^{J_{12}C'_{12}} = \left[p_{ji;JE;\xi\zeta}^{J_{12}C'_{12}}\right]_{\Xi^{J_{12}}\times\Xi^{C'_{12}}} \tag{59.4}$$

and

$$\overline{\overline{P}}_{ji;HM}^{C'_{j0}M_{i0}} = \left[p_{ji;HM;\xi\zeta}^{C'_{j0}M_{i0}}\right]_{\Xi^{C'_{j0}}\times\Xi^{M_{i0}}} \tag{60.1}$$

$$\overline{\overline{P}}_{ji;HM}^{C'_{12}M_{i0}} = \left[p_{ji;HM;\xi\zeta}^{C'_{12}M_{i0}}\right]_{\Xi^{C'_{12}}\times\Xi^{M_{i0}}} \tag{60.2}$$

$$\overline{\overline{P}}_{ji;HM}^{C'_{j0}M_{12}} = \left[p_{ji;HM;\xi\zeta}^{C'_{j0}M_{12}}\right]_{\Xi^{C'_{j0}}\times\Xi^{M_{12}}} \tag{60.3}$$

$$\overline{\overline{P}}_{ji;HM}^{C'_{12}M_{12}} = \left[p_{ji;HM;\xi\zeta}^{C'_{12}M_{12}}\right]_{\Xi^{C'_{12}}\times\Xi^{M_{12}}} \tag{60.4}$$

here

$$p_{ji;JE;\xi\zeta}^{J_{i0}C'_{j0}} = -\ (1/2)\left\langle \bar{b}_\xi^{J_{i0}}, \mathcal{E}_{j;+}^{sca}\left(\bar{b}_\zeta^{C'_{j0}}\right)\right\rangle_{\partial V_i} \tag{61.1}$$

$$p_{ji;JE;\xi\zeta}^{J_{i0}C'_{12}} = -\gamma_j(1/2)\left\langle \bar{b}_\xi^{J_{i0}}, \mathcal{E}_{j;+}^{sca}\left(\bar{b}_\zeta^{C'_{12}}\right)\right\rangle_{\partial V_i} \tag{61.2}$$

$$p_{ji;JE;\xi\zeta}^{J_{12}C'_{j0}} = -\gamma_i(1/2)\left\langle \bar{b}_\xi^{J_{12}}, \mathcal{E}_{j;+}^{sca}\left(\bar{b}_\zeta^{C'_{j0}}\right)\right\rangle_{\partial V_i} \tag{61.3}$$

$$p_{ji;JE;\xi\zeta}^{J_{12}C'_{12}} = (1/2)\left\langle \bar{b}_\xi^{J_{12}}, \mathcal{E}_{j;+}^{sca}\left(\bar{b}_\zeta^{C'_{12}}\right)\right\rangle_{\partial V_i} \tag{61.4}$$

and

$$p_{ji;HM;\xi\zeta}^{C'_{j0}M_{i0}} = -\ (1/2)\left\langle \mathcal{H}_{j;+}^{sca}\left(\bar{b}_\xi^{C'_{j0}}\right), \bar{b}_\zeta^{M_{i0}}\right\rangle_{\partial V_i} \tag{62.1}$$

$$p_{ji;HM;\xi\zeta}^{C'_{12}M_{i0}} = -\gamma_j(1/2)\left\langle \mathcal{H}_{j;+}^{sca}\left(\bar{b}_\xi^{C'_{12}}\right), \bar{b}_\zeta^{M_{i0}}\right\rangle_{\partial V_i} \tag{62.2}$$

$$p_{ji;HM;\xi\zeta}^{C'_{j0}M_{12}} = -\gamma_i(1/2)\left\langle \mathcal{H}_{j;+}^{sca}\left(\bar{b}_\xi^{C'_{j0}}\right), \bar{b}_\zeta^{M_{12}}\right\rangle_{\partial V_i} \tag{62.3}$$

$$p_{ji;HM;\xi\zeta}^{C'_{12}M_{12}} = (1/2)\left\langle \mathcal{H}_{j;+}^{sca}\left(\bar{b}_\xi^{C'_{12}}\right), \bar{b}_\zeta^{M_{12}}\right\rangle_{\partial V_i} \tag{62.4}$$

In (56)-(62), $(j,i)=(2,1),(1,2)$; $C'=J,M$.

C. The matrix forms for the powers of a double-body system: The relation (24) is utilized.

Inserting the (24) and (42) into (38)-(40), the (38)-(40) can be rewritten as follows

$$P_i^{out} = \left(\bar{a}^{\Phi}\right)^H \cdot \bar{\bar{P}}_i^{\Phi} \cdot \bar{a}^{\Phi} \quad (63)$$

$$P_{ji}^{coup} = \left(\bar{a}^{\Phi}\right)^H \cdot \bar{\bar{P}}_{ji}^{\Phi} \cdot \bar{a}^{\Phi} \quad (64)$$

$$P^{inp/out} = \left(\bar{a}^{\Phi}\right)^H \cdot \bar{\bar{P}}^{\Phi} \cdot \bar{a}^{\Phi} \quad (65)$$

here $\Phi=J$, if the surface equivalent electric current is selected as basic variable; $\Phi=M$, if the surface equivalent magnetic current is selected as basic variable. In (63)-(65),

$$\bar{\bar{P}}_X^{\Phi} = \begin{bmatrix} \bar{\bar{I}} \\ \bar{\bar{T}}^{J\to M} \end{bmatrix}^H \cdot \bar{\bar{P}}_X \cdot \begin{bmatrix} \bar{\bar{I}} \\ \bar{\bar{T}}^{J\to M} \end{bmatrix} \quad (66.1)$$

for the case $\Phi=J$, and

$$\bar{\bar{P}}_X^{\Phi} = \begin{bmatrix} \bar{\bar{T}}^{M\to J} \\ \bar{\bar{I}} \end{bmatrix}^H \cdot \bar{\bar{P}}_X \cdot \begin{bmatrix} \bar{\bar{T}}^{M\to J} \\ \bar{\bar{I}} \end{bmatrix} \quad (66.2)$$

for the case $\Phi=M$. In (66), $\bar{\bar{P}}_X=\bar{\bar{P}}_i,\bar{\bar{P}}_{ji},\bar{\bar{P}}$, and correspondingly $\bar{\bar{P}}_X^{\Phi}=\bar{\bar{P}}_i^{\Phi},\bar{\bar{P}}_{ji}^{\Phi},\bar{\bar{P}}^{\Phi}$; the $\bar{\bar{I}}$ is identity matrix.

The matrix form for the coupling power P^{coup} is as follows

$$P^{coup} = \left(\bar{a}^{\Phi}\right)^H \cdot \bar{\bar{P}}^{coup;\Phi} \cdot \bar{a}^{\Phi} \quad (67)$$

here

$$\bar{\bar{P}}^{coup;\Phi} = \bar{\bar{P}}_{21}^{\Phi} + \bar{\bar{P}}_{12}^{\Phi} \quad (68)$$

V. COUPLING POWER CHARACTERISTIC MODE (COUPCM) SET

As illustrated in [4]-[9], the EMP-CMT is an object-oriented modal theory, and the power-based CM sets being similar to the ones discussed in [4]-[9] are not repeated here. However, the CM set focusing on the coupling power between two bodies is specifically researched in this section, because the coupling phenomenon between the elements in a whole system has not been analyzed in [4]-[9].

A. The CM set which can orthogonalize the active part of power P^{coup} (the time-average of coupling power)

If the matrix $\bar{\bar{P}}^{coup;\Phi}$ is decomposed as follows

$$\bar{\bar{P}}^{coup;\Phi} = \bar{\bar{P}}_{+}^{coup;\Phi} + j\,\bar{\bar{P}}_{-}^{coup;\Phi} \quad (69)$$

it can be concluded that

$$P^{coup,act} = \mathrm{Re}\left\{P^{coup}\right\} = \left(\bar{a}^{\Phi}\right)^H \cdot \bar{\bar{P}}_{+}^{coup;\Phi} \cdot \bar{a}^{\Phi} \quad (70.1)$$

$$P^{coup,react} = \mathrm{Im}\left\{P^{coup}\right\} = \left(\bar{a}^{\Phi}\right)^H \cdot \bar{\bar{P}}_{-}^{coup;\Phi} \cdot \bar{a}^{\Phi} \quad (70.2)$$

here the $P^{coup,act}$ and $P^{coup,react}$ are respectively the active and reactive parts of P^{coup}. In (69),

$$\bar{\bar{P}}_{+}^{coup;\Phi} = \frac{1}{2}\left[\bar{\bar{P}}^{coup;\Phi} + \left(\bar{\bar{P}}^{coup;\Phi}\right)^H\right] \quad (71.1)$$

$$\bar{\bar{P}}_{-}^{coup;\Phi} = \frac{1}{2j}\left[\bar{\bar{P}}^{coup;\Phi} - \left(\bar{\bar{P}}^{coup;\Phi}\right)^H\right] \quad (71.2)$$

The "j" appearing in (69) and (71.2) is the imaginary unity.

Because the matrix $\bar{\bar{P}}_{+}^{coup;\Phi}$ is Hermitian, there exists an independent and complete expansion vector set $\{\bar{a}_{\xi}^{coup;\Phi}\}_{\xi=1}^{\Xi^{\Phi}}$, such that [12]

$$\left(\bar{a}_{\xi}^{coup;\Phi}\right)^H \cdot \bar{\bar{P}}_{+}^{coup;\Phi} \cdot \bar{a}_{\zeta}^{coup;\Phi} = P_{\xi}^{coup,act}\delta_{\xi\zeta} \quad (72)$$

for any $\xi,\zeta=1,2,\cdots,\Xi^{\Phi}$, here $\Xi^{\Phi}=\Xi^{\Phi_{10}}+\Xi^{\Phi_{12}}+\Xi^{\Phi_{20}}$. If the zero characteristic value exists, and $P_{\xi}^{coup,act}=0$ for $\xi=n_1,n_2,\cdots,n_N$, all modes in the space spanned by set $\{\bar{a}_{n_1}^{coup;\Phi},\bar{a}_{n_2}^{coup;\Phi},\cdots,\bar{a}_{n_N}^{coup;\Phi}\}$ have zero active coupling powers.

B. The CM set which can orthogonalize the active part of power P_{ji}^{coup} (the time-average of one-way coupling power)

Sometimes, a one-way decoupling between V_1 and V_2 is more desired than the complete decoupling discussed in above subsection. Just like the decomposition for matrix $\bar{\bar{P}}^{coup;\Phi}$, the matrix $\bar{\bar{P}}_{ji}^{coup;\Phi}$ can be similarly decomposed as follows

$$\bar{\bar{P}}_{ji}^{coup;\Phi} = \bar{\bar{P}}_{ji;+}^{coup;\Phi} + j\,\bar{\bar{P}}_{ji;-}^{coup;\Phi} \quad (73)$$

here

$$\bar{\bar{P}}_{ji;+}^{coup;\Phi} = \frac{1}{2}\left[\bar{\bar{P}}_{ji}^{coup;\Phi} + \left(\bar{\bar{P}}_{ji}^{coup;\Phi}\right)^H\right] \quad (74.1)$$

$$\bar{\bar{P}}_{ji;-}^{coup;\Phi} = \frac{1}{2j}\left[\bar{\bar{P}}_{ji}^{coup;\Phi} - \left(\bar{\bar{P}}_{ji}^{coup;\Phi}\right)^H\right] \quad (74.2)$$

Then,

$$P_{ji}^{coup,act} = \mathrm{Re}\left\{P_{ji}^{coup}\right\} = \left(\bar{a}^{\Phi}\right)^H \cdot \bar{\bar{P}}_{ji;+}^{coup;\Phi} \cdot \bar{a}^{\Phi} \quad (75.1)$$

$$P_{ji}^{coup,react} = \mathrm{Im}\left\{P_{ji}^{coup}\right\} = \left(\bar{a}^{\Phi}\right)^H \cdot \bar{\bar{P}}_{ji;-}^{coup;\Phi} \cdot \bar{a}^{\Phi} \quad (75.2)$$

Because the matrix $\bar{\bar{P}}_{ji;+}^{coup;\Phi}$ is Hermitian, there exists an independent and complete expansion vector set $\{\bar{a}_{ji;\xi}^{coup;\Phi}\}_{\xi=1}^{\Xi^{\Phi}}$, such that [12]

$$\left(\bar{a}_{ji;\xi}^{coup;\Phi}\right)^H \cdot \bar{\bar{P}}_{ji;+}^{coup;\Phi} \cdot \bar{a}_{ji;\zeta}^{coup;\Phi} = P_{ji;\xi}^{coup,act}\delta_{\xi\zeta} \quad (76)$$

for any $\xi,\zeta=1,2,\cdots,\Xi^{\Phi}$. If the zero characteristic value exists, and $P_{ji;\xi}^{coup,act}=0$ for $\xi=n_1,n_2,\cdots,n_N$, all modes in the space spanned by set $\{\bar{a}_{ji;n_1}^{coup;\Phi},\bar{a}_{ji;n_2}^{coup;\Phi},\cdots,\bar{a}_{ji;n_N}^{coup;\Phi}\}$ have zero one-way active coupling powers, i.e., their time-average powers done by modal scattering field $\bar{F}_{j;\xi}^{sca}$ on modal scattering currents $\{\bar{J}_{i;\xi}^{vop},\bar{M}_{i;\xi}^{vm}\}$ are zeros.

VI. Conclusions

As a supplement to the previous Surf-SHM-EMP-CMT established in the Parts I and II, the Surf-MHM-EMP-CMT is provided in this Part III. Some surface formulations for the coupling powers among different bodies are provided, and then a new kind of power-based CM set, CoupCM set, is developed for depicting the inherent coupling character among different bodies. It is found out that the zero space of the power quadratic matrix corresponding to the active part of coupling power is valuable for the decoupling applications.

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This work is dedicated to my mother.

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