Introduction to Tree Algebras in Quantum Field Theory

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Abstract

Renormalization procedures for quantum field theories are today reinterpreted using Hopf algebras and related function algebras. Integral expressions for physical numbers are typically replaced by algebraic rules, presumably arising from the unknown correct mathematical formulation of particle physics. These notes introduce the important Hopf algebras of trees, both commutative and noncommutative.

Whatever the correct mathematical language for the perturbative electroweak theory and for QCD, we now know that renormalisation procedures have a rigid algebraic structure. In principle, a physical quantity associated to a large set of Feynman diagrams may be computed using a much smaller set of diagrams, given the right physical constraints. In the first example, in scalar field theory [1], a momentum loop in a Feynman graph is mapped to a vertex of a planar rooted tree. The set of all unordered lists of planar rooted trees (forests) under admissible decompositions is a canonical example of a Hopf algebra [2][3][4]. Along with related function algebras, it was first studied in numerical analysis in [5]. In the work of Kreimer et al [6][7][8], the trees are decorated by Feynman data to give a Hopf algebra for the scalar theory.

A Hopf algebra H is first of all an algebra with a multiplication $m : H \rightarrow H$ which is associative. For example, the formal rational sums of forests (collections) of planar rooted trees is an algebra under the trivial concatenation of forests, placing one forest beside the other. The identity for this product is the empty tree, which will usually be denoted by the symbol 1. A Hopf algebra is also a *coalgebra*, which is the dual notion in a category theoretic [9] sense: the arrows in the axioms (given below) for an algebra object in a category of algebras are all reversed, so that instead of a multiplication there is a comultiplication $\Delta : H \rightarrow H \otimes H$, which is coassociative. Finally, H has an *antipode* map $S : H \rightarrow H$, which is needed to define an inverse for a function $f : H \rightarrow \mathbb{C}$ under the convolution product.

In the next section Hopf algebras are defined, and section 2 details the special example of tree algebras along with associated function algebras. On a first reading, one may skip the abstractions of section 1. In section 3 we define the important shuffle algebras and in section 4 introduce quantum groups. In section 5 we find the special integrals that appear in hadron phenomenology, and summarise the story in section 6.

1 Hopf Algebras

A Hopf algebra H is a bialgebra, meaning both an algebra and a coalgebra, with a special additional map $S : H \to H$ called the antipode, such that m and Δ are compatible. The algebra over the field $\mathbb{F}$ comes with a multiplication $m : H \otimes H \to H$ and a unit map $\eta : \mathbb{F} \to H$ such that associativity

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{m \otimes I_H} & H \otimes H \\ \downarrow{\scriptstyle I_H \otimes m} & & \downarrow{\scriptstyle m} \\ H \otimes H & \xrightarrow[m]{} & H \end{array} \tag{1}$$

holds. Here I_H denotes an identity map. The unit should satisfy $m(a, \eta) = m(\eta, a) = a$ for all $a \in H$, but in diagrams this is written

$$\begin{array}{ccc} H \simeq H \otimes \mathbb{F} & \xrightarrow{I_H} & H \\ \downarrow{\scriptstyle I_H \otimes \eta} & \nearrow{\scriptstyle m} & \\ H \otimes H & & \end{array} \tag{2}$$

for right multiplication, and similarly for the left. The coalgebra has a coproduct $\Delta : H \to H \otimes H$ and a counit $\epsilon : H \to \mathbb{F}$ which is coassociative

$$\begin{array}{ccc} H & \xrightarrow{\Delta \otimes I_H} & H \otimes H \\ \downarrow{\scriptstyle I_H \otimes \Delta} & & \downarrow{\scriptstyle \Delta} \\ H \otimes H & \xrightarrow[\Delta]{} & H \otimes H \otimes H \end{array} \tag{3}$$

The coproduct of an element $a \in H$ is often written

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)} \tag{4}$$

for some elements $a_i^{(1)}$ and $a_i^{(2)}$ in H. Compatibility of m and Δ means that

$$\Delta(xy) = \Delta(x)\Delta(y). \tag{5}$$

Example 1.1 The group algebra $\mathbb{C}G$ of sums $\sum_G a_g g$ with coefficients a_g in $\mathbb{C}$, for G a finite group. Multiplication expands the product of sums in the obvious way. The unit is the unit in G. Then $\Delta(g) = g \otimes g$ when $g \in G$, and $\epsilon(g) = 1$.

In general, a *grouplike* element $x \in H$ has

$$\Delta(x) = x \otimes x. \tag{6}$$

The grouplike elements in $\mathbb{C}G$ give the group G. Now imagine applying the product rule for derivatives to this coproduct. A *primitive* element x of H has a coproduct

$$\Delta(x) = 1 \otimes x + x \otimes 1. \tag{7}$$

Interesting Hopf algebras contain both grouplike and primitive elements, but a fundamental example with only primitives is the following.

Example 1.2 The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. Given a basis for $\mathfrak{g}$, the tensor algebra $T(\mathfrak{g})$ is the free algebra of all words in the basis elements. Multiplication is given by concatenation of words and the identity 1 is the empty word. The enveloping algebra is the quotient of $T(\mathfrak{g})$ by the ideal generated by expressions

$$x \otimes y - y \otimes x - [x, y]$$

where $[x, y]$ is the Lie bracket. Unlike $\mathfrak{g}$, its enveloping algebra is associative. Since all elements of $\mathfrak{g}$ are primitive,

$$\begin{aligned}[\Delta(x), \Delta(y)] &= [1 \otimes x + x \otimes 1, 1 \otimes y + y \otimes 1] \\ &= 1 \otimes xy + xy \otimes 1 - yx \otimes 1 - 1 \otimes yx \\ &= \Delta([x, y]).\end{aligned}$$

Given an algebra A with multiplication m_A, functions $f : H \to A$ on a Hopf algebra H have a natural *convolution* product [2] given by

$$f \star g(x) \equiv m_A(f \otimes g)\Delta(x) = \sum_{\Delta(x)} f(x^{(1)})g(x^{(2)}). \tag{8}$$

Here the terms in $\Delta(x) \in H \otimes H$ directly define $f \star g$. We are particularly interested in the cases $A = \mathbb{C}$, defining a dual Hopf structure, and $A = H$. In the latter case, the *antipode* $S : H \to H$ is the convolution inverse of the identity I_H. Following (8), its axiom is

$$\begin{array}{ccccc} H \otimes H & & \xrightarrow{S \otimes I_H} & & H \otimes H \\ \uparrow{\scriptstyle \Delta} & & & & \downarrow{\scriptstyle m} \\ H & \xrightarrow[\epsilon]{} & \mathbb{C} & \xrightarrow[\eta]{} & H \end{array} \tag{9}$$

where $\eta\epsilon$ is the identity for the $\star$ product. When $A = \mathbb{C}$, the identity for $\star$ is given by ϵ. In the simplest form of renormalization we usually require an additional *dimensional regularization* parameter δ, so elements of A are Laurent series in $\mathbb{C}[\delta^{-1}, \delta]]$.

A function f such that $f(xy) = f(x)f(y)$ is known as a *character* when $A = \mathbb{C}$. Characters clearly satisfy $f(1) = 1$ and form a group, where the antipode is used to define a $\star$ inverse

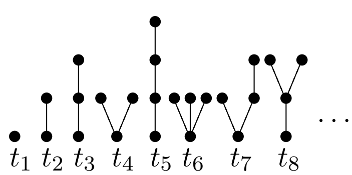
$$f^{-1}(x) \equiv fS(x). \tag{10}$$

Since the convolution is associative, any vector space of functions under $\star$ defines a Lie algebra with bracket

$$[f, g] \equiv f \star g - g \star f. \tag{11}$$

2 Commutative Trees and Forests

An undecorated planar rooted tree has a distinguished vertex, the root, drawn at the bottom of the tree. The *order* $|t|$ of a tree t is the number of vertices. A tree is *commutative* when its diagram is equivalent to any other tree obtained from it by permuting subtrees without altering the graph. So there are eight commutative trees with four or fewer vertices:



A *forest* of commutative rooted trees is an unordered list of such trees. The product of two forests is their disjoint union. Considering each distinct forest as a basis element, we can take arbitrary sums of forests over $\mathbb{Q}$, making forests into an algebra. The empty tree is the unit.

Such rooted commutative forests form a Hopf algebra. Since $\Delta(st) = \Delta(s)\Delta(t)$, where st is a forest product, it is sufficient to define the comultiplication on trees. Define a *subtree* s of t to be a connected subgraph of t containing the root. The set of edges in t that adjoin the top vertices of s has the property that no pair of edges lies on a single path from the root of t. These edges may be used to *cut* the tree t into a forest, by deleting the special edges, and the forest includes s. Let $t \backslash s$ denote the remainder of the forest besides s. The coproduct is given by

$$\Delta(t) = \sum_s t \backslash s \otimes s \tag{12}$$

where the sum is over all possible subtrees, including the empty tree 1 and the full tree t. Observe that $\Delta(1) = 1 \otimes 1$ is grouplike and

$$\Delta(\bullet) = \bullet \otimes 1 + 1 \otimes \bullet \tag{13}$$

1	$1 \otimes 1$
t_1	$1 \otimes t_1 + t_1 \otimes 1$
t_2	$1 \otimes t_2 + t_2 \otimes 1 + t_1 \otimes t_1$
t_3	$1 \otimes t_3 + t_3 \otimes 1 + t_1 \otimes t_2 + t_2 \otimes t_1$
t_4	$1 \otimes t_4 + t_4 \otimes 1 + 2t_1 \otimes t_2 + t_1 t_1 \otimes t_1$
t_5	$1 \otimes t_5 + t_5 \otimes 1 + t_1 \otimes t_3 + t_3 \otimes t_1 + t_2 \otimes t_2$
t_6	$1 \otimes t_6 + t_6 \otimes 1 + 3t_1 \otimes t_4 + 3t_1 t_1 \otimes t_2 + t_1 t_1 t_1 \otimes t_1$
t_7	$1 \otimes t_7 + t_7 \otimes 1 + t_1 \otimes t_3 + t_2 \otimes t_2 + t_1 \otimes t_4 + t_1 t_2 \otimes t_1 + t_1 t_1 \otimes t_2$
t_8	$1 \otimes t_8 + t_8 \otimes 1 + t_4 \otimes t_1 + 2t_1 \otimes t_3 + t_1 t_1 \otimes t_2$

Table 1: Coproduct $\Delta(t_i)$ on rooted trees

is primitive. The counit satisfies $\epsilon(1) = 1$ and $\epsilon(t) = 0$ for all $t \neq 1$. The value of the coproduct on the first few trees is given in Table 1.

Consider the empty tree 1. Since $\eta\epsilon(1) = 1$, the antipode axiom requires $S(1) \cdot 1 = 1$, so $S(1) = 1$. For the tree $\bullet$, the antipode is applied to (13),

$$S(\bullet) \cdot 1 + \bullet = 0,$$

giving $S(\bullet) = -\bullet$. The antipode for higher order trees follows. It looks the same as Δ (without the $\otimes$ insertions) except for a minus sign on the terms with an even number of edge cuts [10][11][12].

Given this Hopf algebra T of forests, there is a convolution product on functions $f : T \to \mathbb{C}$. After [5] we write $G_1 \subset G$ for the group of characters under $\star$ within the vector space G of all $\mathbb{C}$ linear functions on T. The other special subset of G is G_0, namely the functions f for which $f(1) = 0$, and G_0 is a sub Lie algebra.

The antipode in T defines the $\star$ inverse of $f \in G_1$ by (10). For the tree t_1,

$$f^{-1}(\bullet) + f(\bullet) = 0. \tag{14}$$

But this is only true for $f \in G_1$. In general, physical quantities depend on $f(1) = \lambda \neq 0, 1$. This scalar may be derived from some noncommutative or even nonassociative structure. So instead of assuming a coproduct based on primitive parts, we look in section 4 at Hopf algebras that are neither commutative nor cocommutative.

The Lie algebra structure for characters on the space of $\mathbb{C}$ valued linear functions on T is summarised in Table 2. We are motivated to find the simplest nontrivial bracket, starting with the asymmetry in $\Delta(t_4)$. This term in $[a, b]$ goes to zero if

$$a(t_2) = \frac{1}{2} a(t_1)^2, \tag{15}$$

known as the $c(2)$ condition in numerical analysis. The *ladder* trees (ie. t_1, t_2, t_3, t_5, $\cdots$) have left right symmetric coproducts. These cannot contribute

	()	t_1	t_2	t_4
a	1	x_1	y_1	w_1
b	1	x_2	y_2	w_2
$a \star b$	1	$x_1 + x_2$	$y_1 + y_2 + x_1 x_2$	$w_1 + w_2 + 2x_1 y_2 + {x_1}^2 x_2$
$b \star a$	1	$x_1 + x_2$	$y_1 + y_2 + x_1 x_2$	$w_1 + w_2 + 2x_2 y_1 + x_1 {x_2}^2$
$[a, b]$	0	0	0	$2(x_1 y_2 - x_2 y_1) + x_1 x_2 (x_1 - x_2)$

Table 2: Lie structure for characters

to $[a, b]$. Observe that if $a(\bullet) = b(\bullet)$ for characters a, b on T, and $[a, b] = 0$, then $a(t) = b(t)$ for all t in T.

For any forest k create a tree $B_+(k)$ with $n + 1$ vertices by gluing all trees in k above a new root vertex. That is, $B_+(t_1) = t_2$ and we take $B_+(1) = t_1$. Let T_n denote all trees with $|t| = n$. It is clear that $B_+(T_n)$ contains all possible rooted trees of order $n + 1$, and that every tree in this set is distinct. Thus B_+ respects the grading of order and the inverse B_- sends trees of order n to forests of order $n - 1$.

Our first combinatorial quantity is $\sigma(t)$, the symmetry factor of a commutative tree. This is the number of permutations of vertices that fixes the tree. For example, $\sigma(t_6) = 6$ and $\sigma(t_8) = 2$. Now let T_1, T_2, $\cdots T_k$ be a sequence of k trees. The integral *tree factorial* $t!$ is defined recursively [11] for $t = B_+(T_1, \cdots, T_k)$. Let $|t|$ be the order of t. Take $a \in G$ such that $a(\bullet) = 1$ and $a : t \mapsto 1/t!$. Then a satisfies

$$a(t)|t| = a(T_1) a(T_2) \cdots a(T_k). \tag{16}$$

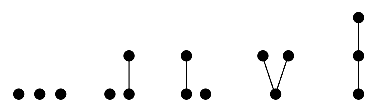
This function appears in the exact numerical solution to the following basic problem. Let $y = y(x)$ be a vector or scalar valued function of x. Consider the autonomous ODE $y' = f(y)$ given an initial value of y. Solutions for y that generalise Taylor series are sums over trees, known as *B-series* after [5]. That is, given $a : T \to \mathbb{R}$ in the convolution algebra, there is a power series in h (the step in the x direction) indexed by commutative trees,

$$B(a, hf, y) = y a(1) + \sum_{t \in T} \frac{h^{|t|}}{\sigma(t)} a(t)(f(t)(y)) \tag{17}$$

where $f(t)$ is the elementary differential associated to the tree t and the function f. It is clear that $B(\epsilon, hf, y) = y$. Let $d : T \to \mathbb{R}$ be the function such that $d(\bullet) = h^{-1}$ and $d = 0$ on all other trees including 1. Then $B(d, hf, y) = f$. In the case $a(t) = 1/t!$, as in (16), $B(a, hf, y)$ is the exact series solution for the flow. The Taylor series is recovered when the set of trees is reduced to an ordinal sequence: the corolla trees t_1, t_2, t_4, t_6, $\cdots$, represent the numbers $n \in \{1, 2, 3, 4, \cdots\}$, and for these trees, $\sigma(t) = (n-1)!$ and $t! = n$. For corollas, $\Delta(t)$ in Table 1 is given by the binomial coefficients, ignoring the factor $1 \otimes t$. A forest of corollas is precisely an integer partition.

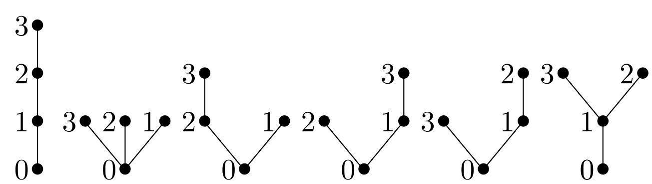
3 Shuffle and Stuffle Algebras

A function with multiple arguments probably has noncommutative arguments. For instance, the coordinates of a vector cannot be arbitrarily permuted. With trees, a noncommutative forest is what you think it is: an ordered forest of trees which are no longer invariant under permutation symmetries. As before, there is a unique empty forest and a unique forest $\bullet$. The two forests at $n = 2$ are t_2 and $\bullet\bullet$. There are five forests at $n = 3$:



The number of noncommutative forests is given by the Catalan number C_n. The usual product and coproduct give a new Hopf algebra [14], called N. As we will see, noncommutative forests are closely related to nonassociativity, whereas commutative forests are required to describe permutations.

A *decoration* for a tree is an assignment of values from a finite set S to each vertex. An important example is the mapping of trees with $n + 1$ vertices to decorations in $S = \{0, 1, \cdots, n\}$ such that each ordinal is only used once. On commutative trees, the following monotonic labelings from S make the collection of all decorated trees into a copy of the symmetric group S_n [15]. A path upwards from the root is monotonically increasing. At a given level, labels are also monotonic from right to left. For example, the six trees of S_3 are



Note the appearance of the integral symmetry factors $\sigma(t)$ from the last section, where the number of allowed numberings of a tree is

$$\alpha(t) \equiv \frac{|t|!}{t!\sigma(t)}. \tag{18}$$

As in example 1.1, there is a group algebra $\mathbb{C}S_n$ for every symmetric group S_n. We are also interested in a Hopf algebra for the union $\coprod_n \mathbb{C}S_n$ of all symmetric group algebras [2][15]. Given two arbitrary words $l_1 l_2 \cdots l_u$ and $k_1 k_2 \cdots k_v$, the *shuffle product* is defined recursively by

$$(l_1 l_2 \cdots l_u) \sqcup (k_1 k_2 \cdots k_v) = l_1(l_2 \cdots l_u \sqcup k_1 \cdots k_v) + k_1(l_1 l_2 \cdots l_u \sqcup k_2 \cdots k_v). \tag{19}$$

The empty word 1 behaves as expected: $w \sqcup 1 = 1 \sqcup w = w$ for any word w.

Example 3.1 $ab \sqcup c = abc + acb + cab$.

The simplest coproduct for shuffles is deconcatenation,

$$\Delta(l_1 l_2 \cdots l_u) = \sum_{j=0}^{u} l_{j+1} \cdots l_u \otimes l_1 \cdots l_j. \tag{20}$$

By definition, a permutation $\sigma^{-1} \in S_n$ is a (p, q) shuffle for $n = p + q$ if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p), \quad \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q). \tag{21}$$

In other words, once p out of n objects are selected, the permutation is fixed. The *shuffle sum* $\Sigma_{p,q}$ in $\mathbb{C}S_n$ is the sum of all (p, q) shuffles. The shuffle product is then written

$$l_1 \cdots l_p \sqcup l_{p+1} \cdots l_{p+q} = \sum_{\text{shuffle}} l_{\sigma(1)} \cdots l_{\sigma(p+q)}. \tag{22}$$

Let $\sigma_n \sigma_m$ denote the permutation in S_{n+m} which acts as $\sigma_n \in S_n$ on the first n objects and then as $\sigma_m \in S_m$ on the remaining ones. The Malvenuto-Reutenauer coproduct [15][16] depends on this result:

Theorem 3.2. *For any $\sigma \in S_n$ and any ordinal $p \leq n$, there exist unique permutations $\sigma_p \in S_p$ and $\sigma_q \in S_q$ such that*

$$\sigma = (\sigma_p \sigma_q) {\Sigma_{p,q}}^{-1}. \tag{23}$$

Example 3.3 $\Delta((231)) = 1 \otimes (231) + (231) \otimes 1 + (1) \otimes (32) + (12) \otimes (3)$.

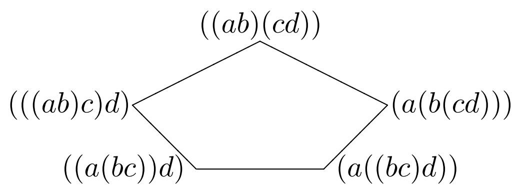
Permutations in S_n are equivalently given by planar trees with binary branchings on $n+3$ vertices, where the branchings are strictly ordered. Here each vertex defines an area between the two edges above it. A permutation is specified by numbering the n areas between leaves from 0 to n, and then re-ordering the numbers using the vertex order. In particular, the two elements of S_2 are

When the downward ordering of internal vertices on a tree is ignored, so that S_3 reduces to a set of 5 binary trees, the trees represent all possible bracketings of words, where the number of letters is given by the leaves on the binary tree. This collection of trees with $n + 1$ leaves is known as the *associahedron* in dimension $n - 1$, since each vertex below the leaves represents a bracketing of two letters. For instance, the bracketed word $(((ab)c)d)$ is one of the trees in the pentagon below.

Theorem 3.4. *The set of rooted binary associahedra trees is in bijection with the set of noncommutative forests.*

Proof: The algorithm for turning a binary tree into a forest goes as follows. Draw a vertex inside each area that is bounded by edges of the binary tree. So for S_2, there are two vertices per tree. View all internal edges on the tree as either left moving or right moving, going downwards. For each right moving edge, connect the vertices in the adjoining two areas by an edge. $\diamond$

An alternative to rooted binary trees with $n+2$ leaves/roots are *chorded polygons* with $n+2$ sides, given by the planar dual picture for the tree, assuming that a root edge is selected on the polygon. The number of chords in the diagram grades the set of trees with $n+1$ leaves, making the associahedron into a polytope: a one chord diagram defines a face of codimension 1, and a diagram with the maximal number of chords defines a vertex.



The *stuffle product* is a generalisation of the shuffle, defined when there exists an additional binary operation $(l_i, l_j) = l_k$ on the letters of the alphabet:

$$
\begin{aligned}
l_1 \cdots l_p * l_{p+1} \cdots l_{p+q} = {} & l_1(l_2 \cdots l_p * l_{p+1} \cdots l_{p+q}) + l_{p+1}(l_1 \cdots l_p * l_{p+2} \cdots l_{p+q}) \\
& + (l_1, l_{p+1})(l_2 \cdots l_p * l_{p+2} \cdots l_{p+q}).
\end{aligned}
\tag{24}
$$

Both shuffle and stuffle products give relations between multiple polylogarithms [17][18].

Example 3.5 $x_1x_2 * x_3 = x_1(x_2 * x_3) + x_3(x_1x_2) + (x_1, x_3)x_2 = x_1x_2x_3 + x_1x_3x_2 + x_3x_1x_2 + x_1(x_2, x_3) + (x_1, x_3)x_2$.

4 Quantum Algebras

Quantum Hopf algebras are neither commutative nor cocommutative. They are defined in terms of a deformation parameter, $\hbar$ or q, in such a way that there exists a limit $q \to 1$, which is a universal enveloping algebra $U(\mathfrak{g})$. Such deformations were originally introduced in the study of *associators* [19], which underlie the algebra of multiple zeta values. An associator is a non trivial arrow

$$\phi_{ABC} : A(BC) \to (AB)C \tag{25}$$

for a category with nonassociative concatenation. In the last section we saw that the list of all possible bracketings of words of length n is given equivalently by binary rooted trees or noncommutative forests.

Consider in general two functions on T that do *not* commute under pointwise multiplication. The respective values on $\bullet$ are E and F. Now even $\Delta(t_2)$ generates a term $[E, F]$ under the convolution Lie bracket. The question is, what is the most general way to define $[E, F]$ in terms of further generators and deformation parameters.

Let us use the canonical tree counit: $\epsilon(1) = 1$ and $\epsilon = 0$ for non grouplike elements. Let $f(1) = \lambda$. The condition $Sf \cdot f(1) = 1$ fixes $Sf(1) = \lambda^{-1}$. Then (14) implies that

$$Sf(\bullet) = -\lambda^{-2} f(\bullet). \tag{26}$$

This antipode no longer satisfies $S^2 = I$, indicating a Hopf algebra that is neither commutative nor cocommutative [2][4]. The canonical example is a deformation of the universal enveloping algebra of sl_2.

Recall that $U(\mathrm{sl}_2)$ has generators H, E and F. It also contains the empty word 1. The coproduct on generators is primitive. But in the deformation algebra $U_q(\mathrm{sl}_2)$ there is a grouplike element $q^{H/2}$

$$\Delta(q^{H/2}) = q^{H/2} \otimes q^{H/2} \tag{27}$$

which replaces H. The standard relations of sl_2 are

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H, \tag{28}$$

when $q = 1$. In the deformation algebra [20][21],

$$[E, F] = \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{-1/2}}. \tag{29}$$

The coproducts

$$\Delta(E) = E \otimes q^{H/2} + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + q^{-H/2} \otimes F \tag{30}$$

give the antipode rules $S(E) = -Eq^{-H/2}$ and $S(F) = -q^{H/2}F$. Compare this to (26). A traditional power series expression for $q^{H/2}$ introduces a small parameter $\hbar$

$$q^{H/2} = 1 + \frac{\hbar H}{2} + \cdots \tag{31}$$

The expression (29), as a function of H, is known as a *q-number*, since when H is replaced by $n \in \mathbb{N}$ it generalises to quantum factorials $[n]!$ and quantum binomials (also known as Gaussian polynomials). The noncommutativity of E and $q^{H/2}$ (or F and $q^{H/2}$) defines the *quantum plane*, with respect to which the quantum exponential

$$\exp_q(x) \equiv \sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}}{[n]!} x^n \tag{32}$$

obeys $\exp_q(q^{H/2}+E) = \exp_q(q^{H/2})\exp_q(E)$. Such Hopf algebras also have a *braiding* operator $R \in H \otimes H$ [19][22], which actually defines a braid diagram crossing in a category containing $H \otimes H$. Interesting categories, known as ribbon fusion categories, have both braidings and tree bracketing operations.

5 Multiple Zeta Values and Brown's Integrals

The basic argument for the special functions of QFT is a noncommutative list of ordinals n_1, n_2, $\cdots$, n_m. Such a list is represented in many ways: as a noncommutative forest of corolla trees or as a *word* in two letters a and b, where for example $3, 4, 2 \mapsto a^2ba^3bab$. Here the root of the corolla takes the label b while all leaves are assigned the letter a.

A *multiple zeta value* (MZV) is given by

$$\zeta(n_1, n_2, \cdots, n_m) \equiv \sum_{k_1 < k_2 < \cdots < k_m} \frac{1}{{k_1}^{n_1}{k_2}^{n_2}{k_3}^{n_3}\cdots {k_m}^{n_m}}, \tag{33}$$

whenever the sum is defined, or more generally by an abstract symbol $\zeta(n_1, n_2, \cdots, n_m)$ obeying certain algebraic relations. The word representation of the argument gives directly the iterated integral for an MZV. For example,

$$\zeta(3,1) = \int_0^1 \frac{\mathrm{d}z}{z} \int_0^1 \frac{\mathrm{d}z}{z} \int_0^1 \frac{\mathrm{d}z}{1-z} \int_0^1 \frac{\mathrm{d}z}{1-z}. \tag{34}$$

The number m is the *depth* of the MZV, while the *weight* equals $\sum_i n_i$. The *multiple polylogarithms* have an additional set z_1, $\cdots$, z_k of arguments,

$$L_{n_1,n_2,\cdots,n_m}(z_1, \cdots, z_m) \equiv \sum_{k_1 < k_2 < \cdots < k_m} \frac{{z_1}^{k_1}\cdots {z_m}^{k_m}}{{k_1}^{n_1}{k_2}^{n_2}{k_3}^{n_3}\cdots {k_m}^{n_m}}. \tag{35}$$

In particular, $L_1(z) = -\ln(1-z)$ and $L_{n_1,\cdots,n_m}(1) = \zeta(n_1, \cdots, n_m)$. The integral representation uses the G functions [17][18], given by

$$L_{n_1,\cdots,n_m}(z_1, \cdots, z_m) = (-1)^m G_{n_1,\cdots,n_m}(\frac{1}{z_1}, \frac{1}{z_1 z_2}, \cdots, \frac{1}{z_1 z_2 \cdots z_m}; 1). \tag{36}$$

An additional argument y replaces the 1 on the right hand side of (36). The variable y appears in our final elementary set of functions,

$$g_k(y) \equiv \frac{\ln^k y}{k!}, \tag{37}$$

which give the terms of

$$z = e^{\log z} = \sum_{k=0}^{\infty} \frac{\ln^k z}{k!}. \tag{38}$$

For $m = 1$, the classical polylogarithm obeys the shuffle algebra coproduct

$$\Delta(L_n(y)) = L_n(y) \otimes 1 + 1 \otimes L_n(y) + \sum_{k=1}^{n-1} L_{n-k}(y) \otimes g_k(y). \tag{39}$$

Multiple polylogarithms obey both a shuffle and stuffle algebra.

Example 5.1 The smallest shuffle on words in two letters, which start with a and end with b, is $ab \sqcup ab$. This gives

$$\zeta(2) \sqcup \zeta(2) = 2\zeta(2,2) + 4\zeta(3,1).$$

Example 5.2 The stuffle product of example 3.5 gives

$$L_{n_1,n_2}(x_1,x_2) * L_{n_3}(x_3) = L_{n_1,n_2,n_3}(x_1,x_2,x_3) + L_{n_1,n_3,n_2}(x_1,x_3,x_2)$$
$$+L_{n_3,n_1,n_2}(x_3,x_1,x_2) + L_{n_1,n_2+n_3}(x_1,x_2x_3) + L_{n_1+n_3,n_2}(x_1x_3,x_2).$$

A generalisation of the shuffle and stuffle products was analysed by Brown in [23]. Here the associahedra are used to tile the real points of a moduli space, and a large class of period integrals are obtained using the dihedral coordinate systems associated to the polygon chords. In particular, for a given set of chords ij, integrals are written as

$$I(a_{ij}) = \int \prod_{ij} {u_{ij}}^{a_{ij}} \omega \tag{40}$$

for integral a_{ij}, where the Laurent polynomials are in the cross ratio variables

$$u_{ij} \equiv \frac{(z_i - z_{j+1})(z_{i+1} - z_j)}{(z_i - z_j)(z_{i+1} - z_{j+1})} \tag{41}$$

and ω is a canonical differential form. Here each point i or j contributes a factor of $\mathbb{CP}^1$ to a large underlying affine space. That is, the dimension of the computation increases with particle number [24]. The integrals (40) include historically important hadronic amplitudes, which predate QCD itself.

6 Beyond Renormalization

The Connes-Kreimer Hopf algebra of decorated commutative rooted forests is equivalent to the algebra of bracketed words in an alphabet of subdivergence types. This gives the Hopf algebra for simple quantum field theories, such as ϕ^4 theory [6][7][8]. Hopf algebras of trees related to QED were introduced in [25]. Modern on-shell methods for QCD employ generalised associahedra. From this perspective, the Standard Model is not really a local gauge theory (for $SU(N)$ Lie groups) with a Lagrangian formulation

[1], because integral amplitudes typically have a simpler algebraic representation.

For a scalar field theory, Kreimer originally considers a regularization scheme applied to the antipode rule, but in general the Bogoliubov recursion is an antipode law [26][27][28]. First UV divergences are eliminated by the antipode axiom, using the zero of the tree counit. The paper [27] defines the character structure of Bogoliubov recursion using the exponential and logarithm functions for the $\star$ product. Here the failure of the product rule for an exponential, due to the noncommutativity of its arguments, is specified by the Lie algebra BCH formula. This directly gives an antipode recursion so that the physical value is a $\star$ exponential.

Physical theories may also require the $\exp_q$ function (32). Concrete examples of the braiding operator R appear in the foundations of quantum computation, using qubits and qutrits. In the braid diagrams, each strand stands for a copy of the algebra H in some category containing all relevant algebras. Since the dimension of the associahedron axiom increases with particle number, the axioms for infinite dimensional categories are probably important to the underlying theory.

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