Conjecture on semiprimes \( n=pq \) related to the number of primes up to \( n \)

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Abstract. In this paper I conjecture that there exist an infinity of semiprimes \( n = p\times q \), where \( p = 30k + m_1 \) and \( q = 30h + m_2 \), \( m_1 \) and \( m_2 \) distinct, having one from the values 1, 7, 11, 17, 19, 23, 29, such that the number of primes congruent to \( m_1 \) (mod 30) up to \( n \) is equal to the number of primes congruent to \( m_2 \) (mod 30) up to \( n \). Example: for \( n = 91 = 7\times13 \), there exist 3 primes of the form \( 30k + 7 \) up to 91 (7, 37 and 67) and 3 primes of the form \( 30k + 13 \) up to 91 (13, 43 and 73).

Conjecture:

There exist an infinity of semiprimes \( n = p\times q \), where \( p = 30k + m_1 \) and \( q = 30h + m_2 \), \( m_1 \) and \( m_2 \) distinct, having one from the values 1, 7, 11, 17, 19, 23, 29, such that the number of primes congruent to \( m_1 \) (mod 30) up to \( n \) is equal to the number of primes congruent to \( m_2 \) (mod 30) up to \( n \).

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The first six semiprimes \( n = pq \) having this property:

: 77 (= 7*11), because there exist 3 primes congruent to 7 (mod 30) up to 77 (7, 37, 67) and 3 primes congruent to 11 (mod 30) up to 91 (11, 41, 71);

: 91 (= 7*13), because there exist 3 primes congruent to 7 (mod 30) up to 91 (7, 37, 67) and 3 primes congruent to 13 (mod 30) up to 91 (13, 43, 73);

: 187 (= 11*17), because there exist 5 primes congruent to 11 (mod 30) up to 187 (11, 41, 71, 101, 131) and 5 primes congruent to 17 (mod 30) up to 187 (17, 47, 107, 137, 167);

: 221 (= 13*17), because there exist 6 primes congruent to 13 (mod 30) up to 221 (13, 43, 73, 103, 163, 193) and 6 primes congruent to 17 (mod 30) up to 221 (17, 47, 107, 137, 167, 197);
299 \( (= 13 \times 23) \), because there exist 8 primes congruent to 13 \((\text{mod } 30)\) up to 299 \((13, 43, 73, 103, 163, 193, 223, 283)\) and 8 primes congruent to 23 \((\text{mod } 30)\) up to 299 \((23, 53, 83, 113, 173, 233, 263, 293)\);

391 \( (= 17 \times 23) \), because there exist 10 primes congruent to 17 \((\text{mod } 30)\) up to 391 \((17, 47, 107, 137, 167, 197, 227, 257, 317, 347)\) and 10 primes congruent to 23 \((\text{mod } 30)\) up to 391 \((23, 53, 83, 113, 173, 233, 263, 293, 353, 383)\).

**Comment:**

The number of the primes of the form \(30k + 1\), \(30k + 7\), \(30k + 11\), \(30k + 13\), \(30k + 17\), \(30k + 19\), \(30k + 23\) respectively \(30k + 29\) seem to be very close in the case of Fermat pseudoprimes to base 2 with two prime factors (the 2-Poulet numbers, see the sequence A214305 in OEIS).

**Examples:**

for \( n = 35333 \), we have:

- 462 primes of the form \(30k + 1\) up to \( n \);
- 470 primes of the form \(30k + 7\) up to \( n \);
- 476 primes of the form \(30k + 11\) up to \( n \);
- 475 primes of the form \(30k + 13\) up to \( n \);
- 474 primes of the form \(30k + 17\) up to \( n \);
- 459 primes of the form \(30k + 19\) up to \( n \);
- 470 primes of the form \(30k + 23\) up to \( n \);
- 475 primes of the form \(30k + 29\) up to \( n \).

for \( n = 164737 \), we have:

- 1874 primes of the form \(30k + 1\) up to \( n \);
- 1885 primes of the form \(30k + 7\) up to \( n \);
- 1896 primes of the form \(30k + 11\) up to \( n \);
- 1884 primes of the form \(30k + 13\) up to \( n \);
- 1885 primes of the form \(30k + 17\) up to \( n \);
- 1873 primes of the form \(30k + 19\) up to \( n \);
- 1889 primes of the form \(30k + 23\) up to \( n \);
- 1888 primes of the form \(30k + 29\) up to \( n \).

**Note:**

This property seem to be shared by other classes of numbers also (but this paper is only about semiprimes), like for instance Carmichael numbers: in the case of Hardy-Ramanujan number 1729 four such sets from eight have the same number of primes, respectively 34 \((30k + 7, 30k + 13, 30k + 17, 30k + 23)\). Open questions: are there numbers for which all eight sets have the same number of primes? Are there an infinity of such numbers?