On the Navier–Stokes equations

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The millennium problem on the existence and smoothness of the Navier–Stokes equations is considered.

1. Problem description

The Navier–Stokes equations are thought to govern the motion of a fluid in $\mathbb{R}^3$, see [1]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$, $p = p(\mathbf{x}, t) \in \mathbb{R}$ be the velocity and pressure, each dependent on position $\mathbf{x} \in \mathbb{R}^3$ and time $t \geqslant 0$. We take the externally applied force to be identically zero. The fluid is assumed to be incompressible with constant viscosity $\nu > 0$ and to fill all of $\mathbb{R}^3$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \tag{3}$$

where $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^3$. In these equations $\nabla = (\frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \frac{\partial}{\partial \mathbf{x}_3})$ is the gradient operator and $\nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial \mathbf{x}_i{}^2}$ is the Laplacian operator. When $\nu = 0$, equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}_0(\mathbf{x} + e_j) = \mathbf{u}_0(\mathbf{x}) \quad \text{for} \quad 1 \leqslant j \leqslant 3 \tag{4}$$

where $e_1 = \mathbf{i} = (1, 0, 0)$, $e_2 = \mathbf{j} = (0, 1, 0)$, $e_3 = \mathbf{k} = (0, 0, 1)$. The initial condition $\mathbf{u}_0$ is a given C^∞ divergence-free vector field on $\mathbb{R}^3$. A solution of (1), (2), (3) would then be accepted to be physically reasonable if

$$\mathbf{u}(\mathbf{x} + e_j, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + e_j, t) = p(\mathbf{x}, t) \quad \text{on} \quad \mathbb{R}^3 \times [0, \infty) \quad \text{for} \quad 1 \leqslant j \leqslant 3 \tag{5}$$

and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^3 \times [0, \infty)). \tag{6}$$

I provide a proposed proof of the following statement (B), see [2].

(B) Existence and smoothness of Navier–Stokes solutions in $\mathbb{R}^3/\mathbb{Z}^3$.

Take $\nu > 0$. Let $\mathbf{u}_0$ be any smooth, divergence-free vector field satisfying (4). Then there exist smooth functions $\mathbf{u}, p$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy (1), (2), (3), (5), (6).

To prove statement (B), it is sufficient to provide a proof that rules out the possibility that there is a smooth, divergence-free $\mathbf{u}_0$ for which (1), (2), (3) have a solution with a finite blowup time, see [2].

2. Proof of statement (B)

Let the exponential series of $\mathbf{u}$, p be

$$\tilde{\mathbf{u}} = \sum_{\mathbf{L}=\mathbf{0}}^{\infty} \mathbf{a}_{\mathbf{L}} e^{k\mathbf{L}\cdot\mathbf{x}}, \tag{7}$$

$$\tilde{p} = \sum_{\mathbf{L}=\mathbf{0}}^{\infty} b_{\mathbf{L}} e^{k\mathbf{L}\cdot\mathbf{x}} \tag{8}$$

respectively. Here $\mathbf{a}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}}(t)$, $b_{\mathbf{L}} = b_{\mathbf{L}}(t)$, k is a constant, and $\sum_{\mathbf{L}=\mathbf{0}}^{\infty}$ denotes the sum over all $\mathbf{L} \in \mathbb{N}^3$. The exponential series is similar to a Taylor series. Theoretically the exponential series can recover both Taylor series and Fourier series when they converge. The initial condition is $\mathbf{u}_0 = \tilde{\mathbf{u}}|_{t=0}$ of which is convergent for all $\mathbf{x} \in \mathbb{R}^3$. Substituting $\mathbf{u} = \tilde{\mathbf{u}}$, $p = \tilde{p}$ into (1) gives

$$\sum_{\mathbf{L}=\mathbf{0}}^{\infty} \frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} e^{k\mathbf{L}\cdot\mathbf{x}} + \sum_{\mathbf{L}=\mathbf{0}}^{\infty}\sum_{\mathbf{M}=\mathbf{0}}^{\infty} (\mathbf{a}_{\mathbf{L}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} e^{k\mathbf{L}\cdot\mathbf{x}} e^{k\mathbf{M}\cdot\mathbf{x}} = \sum_{\mathbf{L}=\mathbf{0}}^{\infty} \nu k^2 |\mathbf{L}|^2 \mathbf{a}_{\mathbf{L}} e^{k\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=\mathbf{0}}^{\infty} k\mathbf{L} b_{\mathbf{L}} e^{k\mathbf{L}\cdot\mathbf{x}}. \tag{9}$$

Equating like powers of the exponentials in (9) yields

$$\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=\mathbf{0}}^{\infty} (\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} = \nu k^2 |\mathbf{L}|^2 \mathbf{a}_{\mathbf{L}} - k\mathbf{L} b_{\mathbf{L}}. \tag{10}$$

Substituting $\mathbf{u} = \tilde{\mathbf{u}}$ into (2) gives

$$\sum_{\mathbf{L}=\mathbf{0}}^{\infty} k\mathbf{L} \cdot \mathbf{a}_{\mathbf{L}} e^{k\mathbf{L}\cdot\mathbf{x}} = 0. \tag{11}$$

Equating like powers of the exponentials in (11) yields

$$\mathbf{L} \cdot \mathbf{a}_{\mathbf{L}} = 0. \tag{12}$$

Applying $\mathbf{L} \times \mathbf{L} \times$ to (10) and noting the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c} \tag{13}$$

along with (12) leads to

$$|\mathbf{L}|^2 \frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} = \sum_{\mathbf{M}=\mathbf{0}}^{\infty} \mathbf{L} \times (\mathbf{L} \times (\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}}) + \nu k^2 |\mathbf{L}|^4 \mathbf{a}_{\mathbf{L}} \tag{14}$$

which yields

$$\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} = \sum_{\mathbf{M}=\mathbf{0}}^{\infty} \hat{\mathbf{L}} \times (\hat{\mathbf{L}} \times (\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}}) + \nu k^2 |\mathbf{L}|^2 \mathbf{a}_{\mathbf{L}} \tag{15}$$

where $\mathbf{a_0} = \mathbf{a_0}(0)$ and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$. Applying $\mathbf{L}\cdot$ to (10) and noting (12) leads to

$$|\mathbf{L}|^2 b_{\mathbf{L}} = -\sum_{\mathbf{M}=\mathbf{0}}^{\infty}(\mathbf{a_{L-M}}\cdot\mathbf{L})(\mathbf{a_M}\cdot\mathbf{L}) \tag{16}$$

which yields

$$b_{\mathbf{L}} = -\sum_{\mathbf{M}=\mathbf{0}}^{\infty}(\mathbf{a_{L-M}}\cdot\hat{\mathbf{L}})(\mathbf{a_M}\cdot\hat{\mathbf{L}}) \tag{17}$$

where $b_{\mathbf{0}}$ is arbitrary. The equations for $\mathbf{a_L}$ can then be solved for $\mathbf{L} = \mathbf{0}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \ldots, \infty$. From (10) and in light of (12) it is possible to write

$$\frac{\partial \mathbf{a_L}}{\partial t}\cdot\hat{\mathbf{a}}_{\mathbf{L}} = -\sum_{\mathbf{M}=\mathbf{0}}^{\infty}(\mathbf{a_{L-M}}\cdot k\mathbf{M})\mathbf{a_M}\cdot\hat{\mathbf{a}}_{\mathbf{L}} + \nu k^2|\mathbf{L}|^2\mathbf{a_L}\cdot\hat{\mathbf{a}}_{\mathbf{L}} \tag{18}$$

where $\hat{\mathbf{a}}_{\mathbf{L}} = \mathbf{a_L}/|\mathbf{a_L}|$ is the unit vector in the direction of $\mathbf{a_L}$. Equation (18) implies

$$\frac{\partial|\mathbf{a_L}|}{\partial t} = -\sum_{\mathbf{M}=\mathbf{0}}^{\infty}(\mathbf{a_{L-M}}\cdot k\mathbf{M})\mathbf{a_M}\cdot\hat{\mathbf{a}}_{\mathbf{L}} + \nu k^2|\mathbf{L}|^2|\mathbf{a_L}|. \tag{19}$$

From (19) it is possible to write

$$\frac{\partial|\mathbf{a_L}|}{\partial t} \leqslant \sum_{\mathbf{M}=\mathbf{0}}^{\infty}|\mathbf{a_{L-M}}|k|\mathbf{M}||\mathbf{a_M}| + \nu k^2|\mathbf{L}|^2|\mathbf{a_L}| \tag{20}$$

on noting the vector identity

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta) \tag{21}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. It then follows from (20) that

$$\sum_{\mathbf{L}=\mathbf{0}}^{\infty}\frac{\partial|\mathbf{a_L}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=\mathbf{0}}^{\infty}\sum_{\mathbf{M}=\mathbf{0}}^{\infty}|\mathbf{a_{L-M}}|k|\mathbf{M}||\mathbf{a_M}|e^{k|\mathbf{L}||\mathbf{x}|} + \sum_{\mathbf{L}=\mathbf{0}}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{a_L}|e^{k|\mathbf{L}||\mathbf{x}|} \tag{22}$$

implying that

$$\sum_{\mathbf{L}=\mathbf{0}}^{\infty}\frac{\partial|\mathbf{a_L}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=\mathbf{0}}^{\infty}\sum_{\mathbf{M}=\mathbf{0}}^{\infty}|\mathbf{a_L}|k|\mathbf{M}||\mathbf{a_M}|e^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|} + \sum_{\mathbf{L}=\mathbf{0}}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{a_L}|e^{k|\mathbf{L}||\mathbf{x}|} \tag{23}$$

which yields

$$\sum_{\mathbf{L}=\mathbf{0}}^{\infty}\frac{\partial|\mathbf{a_L}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=\mathbf{0}}^{\infty}\sum_{\mathbf{M}=\mathbf{0}}^{\infty}|\mathbf{a_L}|k|\mathbf{M}||\mathbf{a_M}|e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|} + \sum_{\mathbf{L}=\mathbf{0}}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{a_L}|e^{k|\mathbf{L}||\mathbf{x}|} \tag{24}$$

on using the triangle inequality

$$|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|. \tag{25}$$

Let

$$\psi = \sum_{\mathbf{L}=\mathbf{0}}^{\infty} |\mathbf{a_L}| e^{k|\mathbf{L}|X} \tag{26}$$

where $X = |\mathbf{x}|$ and note that

$$|\tilde{\mathbf{u}}| \leqslant \psi. \tag{27}$$

Then (24) can be written as

$$\frac{\partial \psi}{\partial t} \leqslant \psi \frac{\partial \psi}{\partial X} + \nu \frac{\partial^2 \psi}{\partial X^2}. \tag{28}$$

Since $\psi \geqslant 0$, the worst case scenario is

$$\frac{\partial \psi}{\partial t} = \psi \frac{\partial \psi}{\partial X} + \nu \frac{\partial^2 \psi}{\partial X^2}. \tag{29}$$

Let

$$\psi = c \frac{\partial \phi}{\partial X} / \phi \tag{30}$$

where c is an arbitrary constant. Substituting (30) into (29) gives

$$c \frac{\partial}{\partial X}(\frac{\partial \phi}{\partial t}/\phi) = c^2 \frac{1}{2} \frac{\partial}{\partial X}((\frac{\partial \phi}{\partial X}/\phi)^2) + c\nu \frac{\partial}{\partial X}((\frac{\partial^2 \phi}{\partial X^2}\phi - (\frac{\partial \phi}{\partial X})^2)/\phi^2). \tag{31}$$

Then with $c = 2\nu$, equation (31) gives

$$\frac{\partial}{\partial X}(\frac{\partial \phi}{\partial t}/\phi) = \nu \frac{\partial}{\partial X}(\frac{\partial^2 \phi}{\partial X^2}/\phi) \tag{32}$$

which leads to

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial X^2} + h\phi \tag{33}$$

where $h = h(t)$ is arbitrary.
Let

$$\phi = \sum_{l=0}^{\infty} A_l e^{\gamma l X} \tag{34}$$

where $A_l = A_l(t)$ and γ is a constant. Substituting (34) into (33) gives

$$\sum_{l=0}^{\infty} \frac{\partial A_l}{\partial t} e^{\gamma l X} = \sum_{l=0}^{\infty} \nu \gamma^2 l^2 A_l e^{\gamma l X} + \sum_{l=0}^{\infty} A_l h e^{\gamma l X}. \tag{35}$$

Equating like powers of the exponentials in (35) yields

$$\frac{\partial A_l}{\partial t} = \nu \gamma^2 l^2 A_l + A_l h. \tag{36}$$

Equation (36) is easily solved to find

$$A_l = c_l e^{\nu \gamma^2 l^2 t + \int h \, dt} \tag{37}$$

where c_l are arbitrary constants. It then follows that

$$|\tilde{\mathbf{u}}| \leqslant \frac{c\sum_{l=0}^{\infty} c_l l\gamma e^{\nu\gamma^2 l^2 t} e^{\gamma l X}}{\sum_{l=0}^{\infty} c_l e^{\nu\gamma^2 l^2 t} e^{\gamma l X}}. \tag{38}$$

Consequently, $\tilde{\mathbf{u}}$ can only have a finite-time singularity if $\tilde{\mathbf{u}}$ has a singularity at $t = 0$. Therefore blowup is ruled out via Taylor's theorem and statement (B) is true. □

References

[1] Batchelor, G. K. 1967. An introduction to fluid dynamics. Cambridge University Press: Cambridge.
[2] Fefferman, C. L. 2000. Existence and smoothness of the Navier–Stokes equation. Clay Mathematics Institute: official problem description.