On the Navier–Stokes equations

Daniel Thomas Hayes

April 26, 2018

The problem on the existence and smoothness of the Navier–Stokes equations is resolved.

1. Problem description

The Navier–Stokes equations are thought to govern the motion of a fluid in $\mathbb{R}^3$, see [1]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$, $p = p(\mathbf{x}, t) \in \mathbb{R}$, and $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^3$ be the velocity, pressure, and given externally applied force respectively, each dependent on position $\mathbf{x} \in \mathbb{R}^3$ and time $t \geqslant 0$. The fluid is here assumed to be incompressible with constant viscosity $\nu > 0$ and to fill all of $\mathbb{R}^3$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p + \mathbf{f}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \tag{3}$$

where $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^3$. In these equations $\nabla = (\frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \frac{\partial}{\partial \mathbf{x}_3})$ is the gradient operator and $\nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial \mathbf{x}_i{}^2}$ is the Laplacian operator. When $\nu = 0$, equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}_0(\mathbf{x} + e_j) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{f}(\mathbf{x} + e_j, t) = \mathbf{f}(\mathbf{x}, t) \;\text{ for }\; 1 \leqslant j \leqslant 3 \tag{4}$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. The initial condition $\mathbf{u}_0$ is a given C^∞ divergence-free vector field on $\mathbb{R}^3$ and

$$|\partial_{\mathbf{x}}^{\alpha} \partial_t^{\beta} \mathbf{f}| \leqslant C_{\alpha\beta\gamma}(1 + |t|)^{-\gamma} \;\text{ on }\; \mathbb{R}^3 \times [0, \infty) \;\text{ for any }\; \alpha, \beta, \gamma. \tag{5}$$

A solution of (1), (2), (3) would then be accepted to be physically reasonable if

$$\mathbf{u}(\mathbf{x} + e_j, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + e_j, t) = p(\mathbf{x}, t) \;\text{ on }\; \mathbb{R}^3 \times [0, \infty) \;\text{ for }\; 1 \leqslant j \leqslant 3 \tag{6}$$

and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^3 \times [0, \infty)). \tag{7}$$

I provide a proof of the following statement (D), see [2].

(D) Breakdown of Navier–Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$.

Take $\nu > 0$. Then there exist a smooth, divergence-free vector field $\mathbf{u}_0$ on $\mathbb{R}^3$ and a smooth $\mathbf{f}$ on $\mathbb{R}^3 \times [0, \infty)$, satisfying (4), (5), for which there exist no solutions $(\mathbf{u}, p)$ of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0, \infty)$.

2. Proof of statement (D)

Herein I take $\mathbf{f} = \mathbf{0}$. I seek an approximation of the form

$$\mathbf{u} = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \sum_{l=0}^{n} \frac{\partial^l \mathbf{u_L}}{\partial t^l}|_{t=0} \frac{t^l}{l!} e^{ik\mathbf{L}\cdot\mathbf{x}}, \tag{8}$$

$$p = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \sum_{l=0}^{n-1} \frac{\partial^l p_{\mathbf{L}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{9}$$

to the solution of (1), (2), (3), (4), (5), (6) in light of Theorem 1 and Theorem 2 in the Appendix. Here $\mathbf{u_L} = \mathbf{u_L}(t)$, $p_{\mathbf{L}} = p_{\mathbf{L}}(t)$, $i = \sqrt{-1}$, $k = 2\pi$, and $\sum_{\mathbf{L}=-\mathbf{H}}^{\mathbf{H}}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^3$ with $-H \leqslant \mathbf{L}_j \leqslant H$. Herein the smooth[1] divergence-free initial condition $\mathbf{u}_0$ on $\mathbb{R}^3$ is chosen to be

$$\mathbf{u}_0 = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \mathbf{L} \times (\mathbf{L} \times \mathbf{a_L}) \delta_{|\mathbf{L}|,\sqrt{3}} e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{10}$$

where $\delta_{i,j}$ is the Kronecker delta defined by

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{11}$$

and $\mathbf{a_L}$ are constant vectors that are chosen such that $\mathbf{u}_0 \in \mathbb{R}^3$.

Method 1

Let

$$\mathbf{u} = \sum_{l=0}^{n} \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0} \frac{t^l}{l!}, \tag{12}$$

$$p = \sum_{l=0}^{n-1} \frac{\partial^l p}{\partial t^l}|_{t=0} \frac{t^l}{l!}. \tag{13}$$

Substituting (12), (13) into (1) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^{l+1}\mathbf{u}}{\partial t^{l+1}}|_{t=0} + \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla) \frac{\partial^m \mathbf{u}}{\partial t^m}|_{t=0} \binom{l}{m} = \nu \nabla^2 \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0} - \nabla \frac{\partial^l p}{\partial t^l}|_{t=0} \tag{14}$$

where $\binom{l}{m} = \frac{l!}{m!(l-m)!}$. Substituting (12) into (2) and equating like powers of t in accordance with Theorem 1 yields

$$\nabla \cdot \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0} = 0. \tag{15}$$

[1] In this paper, smooth functions and C^∞ functions will both mean continuous functions whose derivatives and integrals are all continuous.

Applying $\nabla \times \nabla \times$ to (14) and using the identities

$$\nabla \times \nabla \times \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}, \tag{16}$$

$$\nabla \times \nabla a = \mathbf{0} \tag{17}$$

along with (15) gives

$$\nabla^2 \frac{\partial^{l+1}\mathbf{u}}{\partial t^{l+1}}|_{t=0} = \nabla \times \nabla \times \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla) \frac{\partial^m \mathbf{u}}{\partial t^m}|_{t=0} \binom{l}{m} + \nu \nabla^4 \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0}. \tag{18}$$

Applying the inverse Laplacian ∇^{-2} to (18) gives

$$\frac{\partial^{l+1}\mathbf{u}}{\partial t^{l+1}}|_{t=0} = \nabla^{-2} \nabla \times \nabla \times \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla) \frac{\partial^m \mathbf{u}}{\partial t^m}|_{t=0} \binom{l}{m} + \nu \nabla^2 \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0} + \mathbf{\Phi}_l \tag{19}$$

where $\mathbf{\Phi}_l$ must satisfy the Laplace equation

$$\nabla^2 \mathbf{\Phi}_l = \mathbf{0}. \tag{20}$$

The required solution to (20) is $\mathbf{\Phi}_l = \mathbf{0}$ in light of (4), (6). Equation (19) is then solved for $\frac{\partial^{l+1}\mathbf{u}}{\partial t^{l+1}}|_{t=0}$ where $l = 0, 1, \ldots, n-1$. Applying $\nabla \cdot$ to (14) and noting (15) yields

$$\nabla^2 \frac{\partial^l p}{\partial t^l}|_{t=0} = -\nabla \cdot \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla) \frac{\partial^m \mathbf{u}}{\partial t^m}|_{t=0} \binom{l}{m}. \tag{21}$$

Applying ∇^{-2} to (21) gives

$$\frac{\partial^l p}{\partial t^l}|_{t=0} = -\nabla^{-2} \nabla \cdot \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla) \frac{\partial^m \mathbf{u}}{\partial t^m}|_{t=0} \binom{l}{m} + \psi_l \tag{22}$$

where

$$\nabla^2 \psi_l = 0. \tag{23}$$

Arbitrary constant $\psi_l \in \mathbb{R}$ is the solution to (23) in light of (4), (6). Equation (22) is then solved for $\frac{\partial^l p}{\partial t^l}|_{t=0}$ where $l = 0, 1, \ldots, n-1$. After truncating (12), (13) in their modes, expressions for (8), (9) from Method 1 are then known in terms of given functions.
Note that for the Fourier series

$$\mathbf{g} = \sum_{\mathbf{L} \neq \mathbf{0}} \mathbf{g}_{\mathbf{L}} e^{ik\mathbf{L} \cdot \mathbf{x}} \tag{24}$$

where $\sum_{\mathbf{L} \neq \mathbf{0}}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^3$ with $\mathbf{L} \neq \mathbf{0}$, the ∇^{-2} operator is defined herein as

$$\nabla^{-2} \sum_{\mathbf{L} \neq \mathbf{0}} \mathbf{g}_{\mathbf{L}} e^{ik\mathbf{L} \cdot \mathbf{x}} = \sum_{\mathbf{L} \neq \mathbf{0}} \frac{\mathbf{g}_{\mathbf{L}} e^{ik\mathbf{L} \cdot \mathbf{x}}}{-k^2 |\mathbf{L}|^2}. \tag{25}$$

Method 2

Let

$$\mathbf{u} = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}, \tag{26}$$

$$p = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}. \tag{27}$$

Substituting (26), (27) into (1) and equating like powers of e in accordance with Theorem 2 yields

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} - ik\mathbf{L} p_{\mathbf{L}}. \tag{28}$$

Substituting (26) into (2) and equating like powers of e in accordance with Theorem 2 yields

$$\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} = 0. \tag{29}$$

Applying $\mathbf{L} \times \mathbf{L} \times$ to (28) and noting the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c} \tag{30}$$

along with (29) yields

$$|\mathbf{L}|^2 \frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = \sum_{\mathbf{M}} \mathbf{L} \times (\mathbf{L} \times (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}}) - \nu k^2 |\mathbf{L}|^4 \mathbf{u}_{\mathbf{L}}. \tag{31}$$

Equation (31) implies

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = \sum_{\mathbf{M}} \hat{\mathbf{L}} \times (\hat{\mathbf{L}} \times (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}}) - \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} \tag{32}$$

where the right hand side of (32) is $\mathbf{0}$ when $\mathbf{L} = \mathbf{0}$ and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$. Applying $\mathbf{L}\cdot$ to (28) and noting (29) gives

$$ik|\mathbf{L}|^2 p_{\mathbf{L}} = -\sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})(\mathbf{u}_{\mathbf{M}} \cdot \mathbf{L}) \tag{33}$$

implying that

$$p_{\mathbf{L}} = -\sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \tag{34}$$

where $p_{\mathbf{0}} \in \mathbb{R}$ is an arbitrary function of t. Let

$$\mathbf{u}_{\mathbf{L}} = \sum_{l=0}^{n} \frac{\partial^l \mathbf{u}_{\mathbf{L}}}{\partial t^l}|_{t=0} \frac{t^l}{l!}, \tag{35}$$

$$p_{\mathbf{L}} = \sum_{l=0}^{n-1} \frac{\partial^l p_{\mathbf{L}}}{\partial t^l}|_{t=0} \frac{t^l}{l!}. \tag{36}$$

Substituting (35) into (32) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^{l+1}\mathbf{u_L}}{\partial t^{l+1}}|_{t=0} = \sum_{m=0}^{l}\sum_{\mathbf{M}}\hat{\mathbf{L}}\times(\hat{\mathbf{L}}\times(\frac{\partial^{l-m}\mathbf{u_{L-M}}}{\partial t^{l-m}}|_{t=0}\cdot ik\mathbf{M})\frac{\partial^m\mathbf{u_M}}{\partial t^m}|_{t=0})\binom{l}{m}-\nu k^2|\mathbf{L}|^2\frac{\partial^l\mathbf{u_L}}{\partial t^l}|_{t=0}. \quad (37)$$

Equation (37) is then solved for $\frac{\partial^{l+1}\mathbf{u_L}}{\partial t^{l+1}}|_{t=0}$ where $l = 0, 1, \ldots, n-1$ and $-1 \leqslant \mathbf{L}_j \leqslant 1$. Substituting (35), (36) into (34) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^l p_{\mathbf{L}}}{\partial t^l}|_{t=0} = -\sum_{m=0}^{l}\sum_{\mathbf{M}}(\frac{\partial^{l-m}\mathbf{u_{L-M}}}{\partial t^{l-m}}|_{t=0}\cdot\hat{\mathbf{L}})(\frac{\partial^m\mathbf{u_M}}{\partial t^m}|_{t=0}\cdot\hat{\mathbf{L}})\binom{l}{m}. \quad (38)$$

Equation (38) is then solved for $\frac{\partial^l p_{\mathbf{L}}}{\partial t^l}|_{t=0}$ where $l = 0, 1, \ldots, n-1$ and $-1 \leqslant \mathbf{L}_j \leqslant 1$. Expressions for (8), (9) from Method 2 are then known in terms of given functions.
At $l = 0$ in (37) it is found that

$$\frac{\partial\mathbf{u_L}}{\partial t}|_{t=0} = \sum_{\mathbf{M}}\hat{\mathbf{L}}\times(\hat{\mathbf{L}}\times(\mathbf{u_{L-M}}|_{t=0}\cdot ik\mathbf{M})\mathbf{u_M}|_{t=0}) - \nu k^2|\mathbf{L}|^2\mathbf{u_L}|_{t=0}. \quad (39)$$

In (39) with $1 \leqslant |\mathbf{L}|^2 \leqslant 3$, $\mathbf{u_M}|_{t=0} = \mathbf{0}$ unless $|\mathbf{M}|^2 = 3$ and $\mathbf{u_{L-M}}|_{t=0} = \mathbf{0}$ unless $|\mathbf{L}-\mathbf{M}|^2 = 3$. With $|\mathbf{L}|^2 = 3$ and $|\mathbf{M}|^2 = 3$ the equation $|\mathbf{L}-\mathbf{M}|^2 = 3$ then implies $2\mathbf{L}\cdot\mathbf{M} = 3$ which is not possible as an even number can not be equal to an odd number. Likewise, with $|\mathbf{L}|^2 = 1$ and $|\mathbf{M}|^2 = 3$ the equation $|\mathbf{L}-\mathbf{M}|^2 = 3$ then implies $2\mathbf{L}\cdot\mathbf{M} = 1$ which is not possible as an even number can not be equal to an odd number. With $|\mathbf{L}|^2 = 2$ and $|\mathbf{M}|^2 = 3$ the equation $|\mathbf{L}-\mathbf{M}|^2 = 3$ then implies $\mathbf{L}\cdot\mathbf{M} = 1$ which is not possible as in this instance $|\mathbf{L}\cdot\mathbf{M}| \in \{0, 2\}$ when $-1 \leqslant \mathbf{L}_j \leqslant 1, -1 \leqslant \mathbf{M}_j \leqslant 1$. Therefore

$$\frac{\partial\mathbf{u_L}}{\partial t}|_{t=0} = -3k^2\nu\mathbf{u_L}|_{t=0}. \quad (40)$$

At $O(t)$, I find that Method 2 gives the same result for (8) as given by Method 1.
At $l = 1$ in (37) it is found that

$$\begin{aligned}\frac{\partial^2\mathbf{u_L}}{\partial t^2}|_{t=0} \quad = \quad & \sum_{\mathbf{M}}\hat{\mathbf{L}}\times(\hat{\mathbf{L}}\times((\frac{\partial\mathbf{u_{L-M}}}{\partial t}|_{t=0}\cdot ik\mathbf{M})\mathbf{u_M}|_{t=0} + (\mathbf{u_{L-M}}|_{t=0}\cdot ik\mathbf{M})\frac{\partial\mathbf{u_M}}{\partial t}|_{t=0})) \\ & -\nu k^2|\mathbf{L}|^2\frac{\partial\mathbf{u_L}}{\partial t}|_{t=0}. \end{aligned} \quad (41)$$

By a similar argument as that applied to (39) it is found in Method 2 that

$$\frac{\partial^2\mathbf{u_L}}{\partial t^2}|_{t=0} = -3k^2\nu\frac{\partial\mathbf{u_L}}{\partial t}|_{t=0} = 9k^4\nu^2\mathbf{u_L}|_{t=0}. \quad (42)$$

In fact for $l \geqslant 0$ it is found in Method 2 that

$$\frac{\partial^{l+1}\mathbf{u_L}}{\partial t^{l+1}}|_{t=0} = (-3k^2\nu)^{l+1}\mathbf{u_L}|_{t=0}. \quad (43)$$

With Method 1 for $\nu = 0$, I find that $\mathbf{u}_{tt}|_{t=0} \neq \mathbf{0}$ when truncated onto the modes with $-1 \leqslant \mathbf{L}_j \leqslant 1$. Therefore at $O(t^2)$, the approximation (8) found from Method 1 is different to the approximation (8) found from Method 2. Because of this nonuniqueness at least one of the assumptions used was invalid.

An exact solution

Herein I denote $\mathbf{u} = (u, v, w)$ and $\mathbf{x} = (x, y, z)$. Let the initial condition be

$$\mathbf{u}_0 = (\cos(k(x + y - z)), \cos(k(x - y - z)), \cos(k(x + y - z)) - \cos(k(x - y - z))) \tag{44}$$

which is consistent with (10). I used Maple to find the Maclaurin series of the solution $(\mathbf{u}, p)$ to (1), (2), (3), (4), (5), (6) using (44). The nonuniqueness of results found with Method 1 and Method 2 does occur when using (44). It appeared from the Maclaurin series of the solution $(\mathbf{u}, p)$ that

$$v = \cos(k(x - y - z))e^{\nu\lambda t}, \tag{45}$$

$$w = u - \cos(k(x - y - z))e^{\nu\lambda t}, \tag{46}$$

$$p = 0 \tag{47}$$

where $\lambda = -3k^2$. On substitution of (45), (46), (47) into (1), (2), (6), I found that u must satisfy

$$\frac{\partial u}{\partial t} + (\frac{\partial u}{\partial y} - \frac{\partial u}{\partial z})e^{\nu\lambda t} \cos(k(x - y - z)) - \nu\nabla^2 u = 0, \tag{48}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} = 0, \tag{49}$$

$$u(\mathbf{x} + e_j, t) = u(\mathbf{x}, t), \quad \text{for} \quad 1 \leqslant j \leqslant 3. \tag{50}$$

For $\nu = 0$, I used Maple to find that the exact general solution of (48) is

$$u = F(x, y + z, \frac{t\cos(k(x - y - z)) - y}{\cos(k(x - y - z))}) \tag{51}$$

where F is an arbitrary function. On matching (51) with (44) at $t = 0$, I then deduced that

$$u = \cos(2tk\cos(k(x - y - z)) - k(x + y - z)). \tag{52}$$

The solution (52) also satisfies (49), (50). The resulting $(\mathbf{u}, p)$ was then verified to be an exact solution to (1), (2), (3), (4), (5), (6) for $\nu = 0$. Integrating (52) with respect to t yields

$$\int^t u\,dt = \frac{\sin(2tk\cos(k(x - y - z)) - k(x + y - z))}{2k\cos(k(x - y - z))} \tag{53}$$

which is undefined for some values of $\mathbf{x} \in \mathbb{R}^3$ and $t \geqslant 0$.
For $\nu > 0$, it is found that for the small time $O(t)$ solution the equation (48) for u is

$$\frac{\partial u}{\partial t} + (\frac{\partial u}{\partial y} - \frac{\partial u}{\partial z})e^{\nu\lambda t} \cos(k(x - y - z)) - \nu\lambda u = 0. \tag{54}$$

Equation (54) implies

$$\frac{\partial}{\partial t}(ue^{-\nu\lambda t}) + (\frac{\partial u}{\partial y} - \frac{\partial u}{\partial z})\cos(k(x-y-z)) = 0. \tag{55}$$

Then a change of variables

$$\tau = \frac{e^{\nu\lambda t}-1}{\nu\lambda}, \tag{56}$$

$$u(\mathbf{x},t) = a(\mathbf{x},\tau)\frac{\partial\tau}{\partial t} \tag{57}$$

yields

$$\frac{\partial a}{\partial\tau} + (\frac{\partial a}{\partial y} - \frac{\partial a}{\partial z})\cos(k(x-y-z)) = 0. \tag{58}$$

Equation (49) becomes

$$\frac{\partial a}{\partial x} + \frac{\partial a}{\partial z} = 0, \tag{59}$$

the initial condition (44) implies

$$a(\mathbf{x},0) = \cos(k(x+y-z)), \tag{60}$$

and the spatially periodic boundary conditions (50) imply

$$a(\mathbf{x}+e_j,\tau) = a(\mathbf{x},\tau) \ \text{ for } \ 1 \leqslant j \leqslant 3. \tag{61}$$

Equations (58), (59), (60), (61) define an Euler problem. In light of this and (52), it is then clear that

$$u = e^{\nu\lambda t}\cos(\frac{2k}{\nu\lambda}(e^{\nu\lambda t}-1)\cos(k(x-y-z)) - k(x+y-z)) \tag{62}$$

is valid for small time when $\nu > 0$. Integrating (62) with respect to t yields

$$\int^t u\,dt = \frac{\sin(\frac{2k}{\nu\lambda}(e^{\nu\lambda t}-1)\cos(k(x-y-z)) - k(x+y-z))}{2k\cos(k(x-y-z))} \tag{63}$$

which is undefined for some values of $\mathbf{x} \in \mathbb{R}^3$ and $t \geqslant 0$.
Therefore statement (D) is true. □

Appendix

Theorem 1

Providing that the Maclaurin series

$$\breve{\mathbf{A}} = \sum_{l=0}^{n} \frac{\partial^l \mathbf{A}}{\partial t^l}\Big|_{t=0}\frac{t^l}{l!} = \sum_{l=0}^{n} \frac{\partial^l \breve{\mathbf{A}}}{\partial t^l}\Big|_{t=0}\frac{t^l}{l!} \tag{64}$$

of the exact general solution to a Q^{th} order partial differential equation

$$\frac{\partial^Q \mathbf{A}}{\partial t^Q} = \mathbf{\Psi} \tag{65}$$

exists, it will solve the coefficients of t^l for all $l = 0, 1, \ldots, n - Q$ in (65) with $\mathbf{A} = \breve{\mathbf{A}}$ provided $\mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}$ is expandable in Maclaurin series as

$$\mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}} = \sum_{l=0}^{m} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} \tag{66}$$

where $m \geqslant n$. Here all of the partial derivatives of $\mathbf{A}$ with respect to t are defined at $t = 0$.

Proof of Theorem 1

Since the Maclaurin series of $\mathbf{A}$ exists and all of the partial derivatives of $\mathbf{A}$ with respect to t are defined at $t = 0$, one can integrate (65) Q times with respect to t and then substitute the result into (64) to find

$$\breve{\mathbf{A}} = \sum_{l=0}^{n} \frac{\partial^{l-Q} \mathbf{\Psi}}{\partial t^{l-Q}}|_{t=0} \frac{t^l}{l!} = \sum_{l=0}^{n} \frac{\partial^l \int_Q \mathbf{\Psi}\, dt|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} \tag{67}$$

where $\int_Q \mathbf{\Psi}\, dt$ denotes the Q^{th} integral of $\mathbf{\Psi}$ with respect to t. Substituting $\mathbf{A} = \breve{\mathbf{A}}$ into the residual $\mathbf{r}$ of (65) then gives

$$\mathbf{r} = \sum_{l=0}^{n} \frac{\partial^{l-Q} \mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^{l-Q}}|_{t=0} \frac{t^{l-Q}}{(l-Q)!} - \sum_{l=0}^{m} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} \tag{68}$$

providing $\mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}$ is expanded in Maclaurin series as in (66). Collecting like powers of t in (68) yields

$$\mathbf{r} = \sum_{l=0}^{n-Q} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} - \sum_{l=0}^{m} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} \tag{69}$$

which shows that Theorem 1 is true. □

Theorem 2

Providing that the Fourier series

$$\tilde{\mathbf{A}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\mathbf{A}, e^{ik\mathbf{L}\cdot\mathbf{x}}) e^{ik\mathbf{L}\cdot\mathbf{x}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\tilde{\mathbf{A}}, e^{ik\mathbf{L}\cdot\mathbf{x}}) e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{70}$$

of the exact general solution to a Q^{th} order partial differential equation

$$\frac{\partial^Q \mathbf{A}}{\partial t^Q} = \mathbf{\Psi} \tag{71}$$

exists, it will solve the coefficients of $e^{ik\mathbf{L}\cdot\mathbf{x}}$ for all $-N \leqslant \mathbf{L}_j \leqslant N$ in (71) with $\mathbf{A} = \tilde{\mathbf{A}}$ provided $\boldsymbol{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}$ is expandable in Fourier series as

$$\boldsymbol{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}} = \sum_{\mathbf{L}=-\mathbf{M}}^{\mathbf{M}} P(\boldsymbol{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{72}$$

where $M \geqslant N$. Here $\mathbf{A}$ is spatially periodic and smooth for all $\mathbf{x} \in \mathbb{R}^3$, $k > 0$ is a constant, and $P(\mathbf{h}, e^{ik\mathbf{L}\cdot\mathbf{x}})$ denotes the projection of $\mathbf{h}$ onto $e^{ik\mathbf{L}\cdot\mathbf{x}}$.

Proof of Theorem 2

Since the Fourier series of $\mathbf{A}$ exists where $\mathbf{A}$ is spatially periodic and smooth for all $\mathbf{x} \in \mathbb{R}^3$, one can integrate (71) Q times with respect to t and then substitute the result into (70) to find

$$\tilde{\mathbf{A}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\int_Q \boldsymbol{\Psi}\, dt, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\int_Q \boldsymbol{\Psi}\, dt|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}}. \tag{73}$$

Substituting $\mathbf{A} = \tilde{\mathbf{A}}$ into the residual $\mathbf{r}$ of (71) then gives

$$\mathbf{r} = \frac{\partial^Q}{\partial t^Q} \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\int_Q \boldsymbol{\Psi}\, dt|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\mathbf{M}}^{\mathbf{M}} P(\boldsymbol{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{74}$$

providing $\boldsymbol{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}$ is expanded in Fourier series as in (72). Equation (74) can be written as

$$\mathbf{r} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\boldsymbol{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\mathbf{M}}^{\mathbf{M}} P(\boldsymbol{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{75}$$

which shows that Theorem 2 is true. □

References

[1] Batchelor, G. K. 1967. An introduction to fluid dynamics. Cambridge University Press: Cambridge.

[2] Fefferman, C. L. 2000. Existence and smoothness of the Navier–Stokes equation. Clay Mathematics Institute: official problem description.