PROOF OF RIEMANN'S HYPOTHESIS

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Abstract. Riemann's hypothesis (1859) is the conjecture stating that: The real part of every non trivial zero of Riemann's zeta function is 1/2.

The main contribution of this paper is to achieve the proof of Riemann's hypothesis. The key idea is to provide an Hamiltonian operator whose real eigenvalues correspond to the imaginary part of the non trivial zeros of Riemann's zeta function and whose existence, according to Hilbert and Pólya, proves Riemann's hypothesis.

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Contents

1. Introduction

In his book [1] of 1748, Leonhard Euler (1707-1783) proved what is now named *the Euler product formula*. This product is the result of the infinite sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathbb{P}} (1 - 1/p^s)^{-1} \quad \text{for any integer variable } s > 1$$

where $\mathbb{P}$ is the infinite set of primes.

In his article [2] of 1859, Riemann (1826-1866) extended the Euler definition to the complex variable s of the zeta function:

$$\zeta(s) = \prod_{p\in\mathbb{P}} (1 - 1/p^s)^{-1} \quad \textit{for any complex variable } s \neq 1$$

It is known that the trivial zeros of the function are the infinite set:

$$\{s_1\} = \{-2m\} \quad \text{for all integers } m > 0$$

Riemann's hypothesis can be seen as stating that:

Probably, the infinite set of the non trivial zeros $\{s_2\}$ of $\zeta(s)$ can be written:

$$\{s_2\} = \{\tfrac{1}{2} + it_n\} \quad \text{where } t_n \text{ is real.}$$

This conjecture is the first point of the eighth unresolved problem (among 23) that Hilbert listed in 1900 [3] as well as the second unresolved problem listed in 2000 by The Clay Mathematics Institute [4].

2. Preliminary notes

2.1. **The Hilbert-Pólya statement.** Circa 1914, Hilbert et Pólya [5], independently from each other, have orally stated that Riemann's hypothesis would be proved if it could be shown that the imaginary parts t_n of the non trivial zeros of the symmetrical xi function $\xi(s)$ derived from $\zeta(s)$, corresponded to the real eigenvalues of an unbounded Hamiltonian operator (here named $\hat{H}_\xi$) for which we could write:

$$\hat{H}_\xi \psi_k = E_k \psi_k \tag{1}$$

which is an equation of quantum physics where E_k stands for the kD-components in a kD-space of a physical energy, with $k = \infty$.

So, the first and unique purely mathematical clue that we have is that the operator $\hat{H}_\xi$ should be a square matrix of infinite dimension with real eigenvalues. This means that it could be written:

$$\hat{H}_\xi = (t_n) = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & t_{n-1} & 0 & 0 & 0 & \dots \\ \dots & 0 & t_n & 0 & 0 & \dots \\ \dots & 0 & 0 & t_{n+1} & 0 & \dots \\ \dots & 0 & 0 & 0 & t_{n+2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad \text{all } t_n \text{ being real}$$

3. Proof of Riemann's Hypothesis

As Hilbert-Pólya statement will be used to prove Riemann's Hypothesis, the complete proof will be established in two steps:

The first one proves that Hilbert-Pólya statement is indeed a conditional proof of Riemann's Hypothesis.

The second one establishes the unconditional proof of Riemann's Hypothesis.

3.1. **Proof of Hilbert-Pólya statement.**

Proof. By definition, a complex number s is written:

$$s = x + iy \quad \text{where } x \text{ and } y \text{ are real and } i = \sqrt{-1}$$

By changing the conventional system of coordinates (x, y) of the complex plane into the new one $(x' = \frac{1}{2} - x, y' = y)$, these complex numbers can be written:

$$s = x' + iy' \text{ in the new system}$$
or:
$$s = (\tfrac{1}{2} - x) + iy \text{ when using the change of coordinates.}$$

Condition. We suppose that the $\hat{H}_\xi$ operator exists and that it contains the infinitely many real eigenvalues t_n coming from the non trivial zeros s_2 of $\zeta(s)$.

Hypothesis. We then suppose that these non trivial zeros lie anywhere in the complex plane with the two exceptions that they cannot lie on the real axis x or x' (reserved for trivial zeros s_1), which gives:

$$y \neq 0 \text{ and } y' \neq 0$$

nor on the conventional critical line $x = \frac{1}{2}$ that becomes the new imaginary axis y', which gives:

$$x \neq \tfrac{1}{2} \text{ and } x' \neq 0$$

Then, *each non trivial zero* s_2 *of* $\zeta(s)$ could be written:

$$s_2 = x'_2 + iy'_2 \quad \text{with } x'_2 \neq 0 \text{ and } y'_2 \neq 0$$
or, using the change of coordinates:
$$s_2 = (\tfrac{1}{2} - x_2) + iy_2 \quad \text{with } x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0$$

But using the fact that $-x_2 = i^2 x_2$, they can be written:

$$s_2 = (\tfrac{1}{2} + i^2 x_2) + iy_2 = \tfrac{1}{2} + i(y_2 + ix_2) \quad \text{with } x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0$$
or:
$$s_2 = \tfrac{1}{2} + it_2 \quad \text{with } t_2 = y_2 + ix_2,\ x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0$$

and we get the result, as $t_2 = y_2 + ix_2$ has to be real, that x_2 has to be zero. This is not a direct contradiction to our hypothesis but this result has been proven wrong 10^{13} times with the first 10^{13} non trivial zeros s_2 [6] for which $x_2 = 1/2$. Each of these 10^{13} contradictions proves that our hypothesis is wrong and that Riemann's hypothesis is true conditionally to the existence of the $\hat{H}_\xi$ operator, which is exactly Hilbert-Pólya statement. □

3.2. **Preparing the unconditional proof of Riemann's Hypothesis.** As Riemann's Hypothesis is now proven conditionally to the existence of the $\hat{H}_\xi$ operator, we have to prove that the $\hat{H}_\xi$ operator *does exist*.

To do this, but first noticing that this operator refers only to the second set $\{s_2\}$ of the non trivial zeros of $\zeta(s)$, we will consider the new and larger operator $\hat{H}_\zeta$ built with the zeros of both sets $\{s_1\}$ and $\{s_2\}$ as eigenvalues, an operator that also contains the real values t_n (but not as eigenvalues):

$$\hat{H}_\zeta = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -6 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & -4 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & -2 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & \frac{1}{2}+it_1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & \frac{1}{2}+it_2 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}+it_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

As this new operator contains the real values t_n, it enables us, at any time but if it exists, to rebuild the operator $\hat{H}_\xi$ of Hilbert and Pólya. To simplify the writing, we set:

(0) for all necessary zero values on and outside the diagonal,

$$\begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & -6 & 0 & 0 \\ \dots & 0 & -4 & 0 \\ \dots & 0 & 0 & -2 \end{pmatrix} = (-2m)$$

and:

$$\begin{pmatrix} \frac{1}{2}+it_1 & 0 & 0 & \dots \\ 0 & \frac{1}{2}+it_2 & 0 & \dots \\ 0 & 0 & \frac{1}{2}+it_3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} = \left(\tfrac{1}{2}+it_n\right)$$

so that $\hat{H}_\zeta$ can be written:

$$\hat{H}_\zeta = \begin{pmatrix} (-2m) & (0) \\ (0) & \left(\frac{1}{2}+it_n\right) \end{pmatrix} = \begin{pmatrix} (-2m) & (0) \\ (0) & \left(\frac{1}{2}\right) \end{pmatrix} + i \begin{pmatrix} (0) & (0) \\ (0) & \hat{H}_\xi \end{pmatrix}$$

But the matrices $(-2m)$ and $\left(\frac{1}{2}+it_n\right)$ representing the sets of zeros $\{s_1\}$ and $\{s_2\}$ can symbolically be replaced by their parametric form:

$$-2m, \quad m>0 \text{ being an integer parameter}$$
$$\tfrac{1}{2}+it_n, \quad t_n \text{ being a real parameter}$$

The sets $\{s_1\}$ and $\{s_2\}$ can then be considered as the two infinite sets of roots of the polynomial of complex variable s:

$$P(s) = (s-s_1)(s-s_2) = s^2 - (s_1+s_2)s + s_1 s_2$$
$$P(s,m,t_n) = s^2 - (-2m + \tfrac{1}{2} + it_n)s - 2m(\tfrac{1}{2}+it_n)$$
$$P(s,m,t_n) = s^2 + (2m - (\tfrac{1}{2}+it_n))s - 2m(\tfrac{1}{2}+it_n)$$

which, using matrices, can be written either:

$$(2) \quad P(s,m,t_n) = \begin{pmatrix} s^2 & s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m-(\frac{1}{2}+it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2}+it_n) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

or:

$$(3) \quad P(s,m,t_n) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m-(\frac{1}{2}+it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2}+it_n) \end{pmatrix} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix}$$

By setting:

$$\hat{E}_k = \begin{pmatrix} s^2 & s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m - (\frac{1}{2} + it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2} + it_n) \end{pmatrix}$$
$$= \begin{pmatrix} s^2 & (2m - (\frac{1}{2} + it_n))s & -2m(\frac{1}{2} + it_n) \end{pmatrix}$$

and:

$$\psi_{E_k} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and by multiplying the first two matrices, equation (2) gives:

$$P(s, m, t_n) = \begin{pmatrix} s^2 & (2m - (\frac{1}{2} + it_n))s & -2m(\frac{1}{2} + it_n) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \hat{E}_k \psi_{E_k} \tag{4}$$

where k is now reduced to $k = 3$ so that our initial problem is also reduced to our 3D space. As by multiplying the first two matrices of (3), we also have:

$$P(s, m, t_n) = \begin{pmatrix} 1 & (2m - (\frac{1}{2} + it_n)) & -2m(\frac{1}{2} + it_n) \end{pmatrix} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \hat{H}_k \psi_{H_k} \tag{5}$$

when we set:

$$\hat{H}_k = \begin{pmatrix} 1 & (2m - (\frac{1}{2} + it_n)) & -2m(\frac{1}{2} + it_n) \end{pmatrix} \tag{6}$$

and:

$$\psi_{H_k} = \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \hat{R} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \hat{R} \psi_{E_k}$$

where $\hat{R}$ is the 3-dimensional transformation matrix from the orthogonal system of coordinates ψ_{H_k} used to describe $\hat{H}_k$ to the orthogonal system ψ_{E_k} used to describe $\hat{E}_k$, and we have:

$$\hat{H}_k \psi_{H_k} = \hat{H}_k \hat{R} \psi_{E_k} = \hat{E}_k \psi_{E_k}$$

Then, setting $\hat{H} = \hat{H}_k \hat{R}$, we get:

$$\hat{H} \psi_{E_k} = \hat{E}_k \psi_{E_k} \tag{7}$$

which is almost identical to equation (1). Here, if *almost identical* is used, it is because $\hat{E}_k$ is an operator that may not be real as requied by equation (1). That is why in the next section we look for the conditions that could make it real.

3.3. **Unconditional proof of Riemann's hypothesis.**

Proof. From equation (7) we have:

$$\hat{E}_k = \hat{H} = \hat{H}_k \hat{R}$$

and as from equation (6), $\hat{H}_k$ can also be written:

$$\hat{H}_k = \begin{pmatrix}1 & 1 & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0 \\ 0 & (2m - (\frac{1}{2} + it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2} + it_n)\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\end{pmatrix} \hat{A}$$

when we set:

$$\hat{A} = \begin{pmatrix}1 & 0 & 0 \\ 0 & (2m - (\frac{1}{2} + it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2} + it_n)\end{pmatrix} \tag{8}$$

we get from (3) and (2) that:

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix} \hat{A} \begin{pmatrix}s^2 \\ s \\ 1\end{pmatrix} = P(s, m, t_n) = \begin{pmatrix}s^2 & s & 1\end{pmatrix} \hat{A} \begin{pmatrix}1 \\ 1 \\ 1\end{pmatrix} \tag{9}$$

But, for any $s = x + iy$, we have:

$$\begin{gathered} P(s) = (s - s_1)(s - s_2) = s^2 - (s_1 + s_2)s + s_1 s_2 \\ P(s, m, t_n) = s^2 + \left(2m - (\tfrac{1}{2} + it_n)\right) s - 2m \left(\tfrac{1}{2} + it_n\right) \\ = (x + iy)^2 + \left(2m - (\tfrac{1}{2} + it_n)\right)(x + iy) - 2m\left(\tfrac{1}{2} + it_n\right) \\ = \left(x^2 - y^2 + (2m - \tfrac{1}{2})x + yt_n\right) + i\left(-xt_n + y(2m - \tfrac{1}{2})\right) - m - 2mit_n \\ = \left(x^2 - y^2 + (2m - \tfrac{1}{2})x + yt_n - m\right) + i\left(-xt_n + y(2x + 2m - \tfrac{1}{2}) - 2mt_n\right) \end{gathered}$$

and $P(s, m, t_n)$ will be real only when:

$$-xt_n + y(2x + 2m - \tfrac{1}{2}) - 2mt_n = 0$$

and so, only for the infinitely many curves in the complex plane such that:

$$y = t_n \frac{x + 2m}{2x + 2m - \frac{1}{2}} = t_n \frac{x + 2m}{(x + 2m) + (x - \frac{1}{2})}$$

which, for $x = \frac{1}{2}$, are all at $y = t_n$
and for $x = -2m$, are all at $y = 0$.

Then, for all the points of all these curves (hyperboles), we have that:

$$P(s, m, t_n)_{curves} = \left(x^2 - y^2 + (2m - \frac{1}{2})x + yt_n - m\right) = V(x, y) \tag{10}$$

is a real value and therefore the real mono-term matrix $(V(x, y))$ always verifies:

$$(V(x, y)) = (\overline{V(x, y)}) = (\overline{V(x, y)})^T \tag{11}$$

where $(\overline{V(x, y)})$ is the conjugate matrix of $(V(x, y))$ and $(\overline{V(x, y)})^T$ is the conjugate transpose of $(V(x, y))$. So, from (9), (10) and (11), we can write:

$$P(s, m, t_n)_{curves} = V(x, y) = \begin{pmatrix}1 & 1 & 1\end{pmatrix} \hat{A} \begin{pmatrix}s^2 \\ s \\ 1\end{pmatrix} = \left(\begin{pmatrix}s^2 & s & 1\end{pmatrix} \hat{A} \begin{pmatrix}1 \\ 1 \\ 1\end{pmatrix}\right)^T$$

which proves that the operator $\hat{A}$, also providing the real t_n's to $\hat{H}_\xi$, verifies the equation of the observables in quantum physics, which is generally written:

$$< \psi_1 \, |\hat{A}| \, \psi_2 > \; = \; \left(< \psi_2 \, |\hat{A}| \, \psi_1 >\right)^T$$

where $\hat{A}$ is the Hamiltonian operator associated to the physical quantity $A = V(x, y)$, $< x \mid$ and $\mid x >$ are the bra and ket operators on x, and ψ_1 and ψ_2 are the states of the physical quantity A before and after the measuring of A.

As we have:

$$\hat{E}_k = \hat{H} = \hat{H}_k \hat{R} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \hat{A} \hat{R}$$

and as $\hat{A}$ can be associated with a physical quantity $A = V(x, y)$, $\hat{E}_k = \hat{H}$ can also be associated with a physical quantity E in equation (7), this last one becoming identical to (1).

As we can rebuild the Hamiltonian operator $\hat{H}_\xi$ linked to $\zeta(s)$ via the function $P(s, m, t_n)$ and the existing operator $\hat{H}$ or $\hat{E}_k$, this Hamiltonian operator $\hat{H}_\xi$ *does exist* and as we have proven earlier that Riemann's hypothesis is true conditionally to the existence of the $\hat{H}_\xi$ operator, Riemann's hypothesis is therefore unconditionally proven. □

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References

[1] Euler L., *Introductio in analysin infinitorum*, 1748.
[2] Riemann B., *On the Number of Prime Numbers less than a Given Quantity*, read on internet on February 25th, 2016 at: http://www.claymath.org/sites/default/files/ezeta.pdf
[3] Hilbert, D., *Mathematische Probleme*, Göttinger Nachrichten, pp. 253-297, 1900.
[4] CLay Mathematics Institute, *Millennium Problems*, read on internet on February 25th, 2016 at:
http://www.claymath.org/millennium-problems
[5] Hilbert-Pólya conjecture, retrieved on December 9th, 2016 on Wikipedia web page: https://en.wikipedia.org/wiki/Hilbert%E2%80%93P%C3%B3lya_conjecture #Possible_connection_with_quantum_mechanics
[6] Gourdon X., Demichel P., Computation of zeros of the Zeta function, 2004, retrieved on December 9th, 2016 at: http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeroscompute.html

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