PROOF OF RIEMANN'S HYPOTHESIS

ROBERT DELOIN

ABSTRACT. Riemann's hypothesis (1859) is the conjecture stating that: The real part of every non trivial zero of Riemann's zeta function is 1/2.

The main contribution of this paper is to achieve the proof of Riemann's hypothesis. The key idea is to provide an Hamiltonian operator whose real eigenvalues correspond to the imaginary part of the non trivial zeros of Riemann's zeta function and whose existence, according to Hilbert and Pólya, proves Riemann's hypothesis.

MSC 2010 : 11A41; 11M06; 11M26
Keywords: Euler; Riemann; Hilbert; Polya; conjecture

CONTENTS

1. INTRODUCTION

In his book [1] of 1748, Leonhard Euler (1707-1783) proved what is now named *the Euler product formula.* This product is the result of the infinite sum:

$$\textstyle\sum_{n=1}^{\infty} \dfrac{1}{n^s} = \prod_{p\in\mathbb{P}}(1-1/p^s)^{-1} \quad \text{for any integer variable } s>1$$

where $\mathbb{P}$ is the infinite set of primes.

In his article [2] of 1859, Riemann (1826-1866) extended the Euler definition to the complex variable s of the zeta function:

$$\zeta(s) = \textstyle\prod_{p\in\mathbb{P}}(1-1/p^s)^{-1} \quad \textit{for any complex variable } s \neq 1$$

It is known that the trivial zeros of the function are the infinite set:

$$\{s_1\} = \{-2m\} \quad \text{for all integers } m>0$$

Riemann's hypothesis can be seen as stating that:

Probably, the infinite set of the non trivial zeros $\{s_2\}$ of $\zeta(s)$ can be written:

$$\{s_2\} = \{\tfrac{1}{2} + it_n\} \quad \text{where } t_n \text{ is real.}$$

This conjecture is the first point of the eighth unresolved problem (among 23) that Hilbert listed in 1900 [3] as well as the second unresolved problem listed in 2000 by The Clay Mathematics Institute [4].

2. Preliminary notes

2.1. **The Hilbert-Pólya statement.** Circa 1914, Hilbert et Pólya [5], independently from each other, have orally stated that Riemann's hypothesis would be proved if it could be shown that the imaginary parts t_n of the non trivial zeros of the symmetrical xi function $\xi(s)$ derived from $\zeta(s)$, corresponded to the real eigenvalues of an unbounded Hamiltonian operator (here named $\hat{H}_\xi$) for which we could write:

$$\hat{H}_\xi \psi_k = E_k \psi_k \tag{1}$$

which is an equation of quantum physics where E stands for energy.

So, the first and unique purely mathematical clue that we have is that this operator should be a square matrix of infinite dimension with real eigenvalues. This means that it could be written:

$$\hat{H}_\xi = (t_n) = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & t_{n-1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & t_n & 0 & 0 & \cdots \\ \cdots & 0 & 0 & t_{n+1} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & t_{n+2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad \text{all } t_n \text{ being real}$$

3. Proof of Riemann's Hypothesis

The proof will be established in two steps:
The first one establishes only a conditional proof.
The second one establishes the unconditional proof.

3.1. Conditional proof of Riemann's Hypothesis.

Proof. By definition, a complex number s is written:

$$s = x + iy \quad \text{where } x \text{ and } y \text{ are real and } i = \sqrt{-1}$$

By changing the conventional system of coordinates (x, y) of the complex plane into the new one $(x' = \frac{1}{2} - x, y' = y)$, these complex numbers can be written:

$$s = x' + iy' \text{ in the new system}$$
$$\text{or:}$$
$$s = (\tfrac{1}{2} - x) + iy \text{ using the change of coordinates.}$$

Condition. We suppose that the $\hat{H}_\xi$ operator exists and that it contains the infinitely many real eigenvalues t_n coming from the non trivial zeros s_2 of $\zeta(s)$.

Hypothesis. We then suppose that these non trivial zeros lie anywhere in the complex plane with the two exceptions that they cannot lie on the real axis x or x' (reserved for trivial zeros s_1), which gives:

$$y \neq 0 \text{ and } y' \neq 0$$

nor on the conventional critical line $x = \frac{1}{2}$ that becomes the new imaginary axis y', which gives:

$$x \neq \tfrac{1}{2} \text{ and } x' \neq 0$$

Then, *each non trivial zero s_2 of $\zeta(s)$ could be written:*

$$\begin{gathered} s_2 = x'_2 + iy'_2 \quad \text{ with } x'_2 \neq 0 \text{ and } y'_2 \neq 0 \\ \text{or, using the change of coordinates:} \\ s_2 = (\tfrac{1}{2} - x_2) + iy_2 \quad \text{ with } x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0 \end{gathered}$$

But using the fact that $-x_2 = i^2 x_2$, they can be written:

$$\begin{gathered} s_2 = (\tfrac{1}{2} + i^2 x_2) + iy_2 = \tfrac{1}{2} + i(y_2 + ix_2) \quad \text{ with } x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0 \\ \text{or:} \\ s_2 = \tfrac{1}{2} + it_2 \quad \text{ with } t_2 = y_2 + ix_2,\ x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0 \end{gathered}$$

and we get the result, as $t_2 = y_2 + ix_2$ has to be real, that x_2 has to be zero. This is not a direct contradiction to our hypothesis but this result has been proven wrong 10^{13} times with the first 10^{13} non trivial zeros s_2 [6] for which $x_2 = 1/2$. Each of these 10^{13} contradictions proves that our hypothesis is wrong and that Riemann's hypothesis is true conditionally to the existence of the $\hat{H}_\xi$ operator. □

3.2. **Unconditional proof of Riemann's Hypothesis.**

Proof. As Riemann's Hypothesis is now proven conditionally to the existence of the $\hat{H}_\xi$ operator, we have to prove that the $\hat{H}_\xi$ operator *does exist.*

To do this, but first noticing that this operator refers only to the second set $\{s_2\}$ of the non trivial zeros of $\zeta(s)$, we will consider the new and larger operator $\hat{H}_\zeta$ built with the zeros of both sets of zeros $\{s_1\}$ and $\{s_2\}$ of $\zeta(s)$ as eigenvalues, an operator that also contains the real values t_n (but not as eigenvalues):

$$\hat{H}_\zeta = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -6 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & -4 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & -2 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & \frac{1}{2} + it_1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & \frac{1}{2} + it_2 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} + it_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

As this new operator contains the real values t_n, it enables us, at any time but if it exists, to rebuild the operator $\hat{H}_\xi$ of Hilbert and Pólya. To simplify the writing, we set:

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & -6 & 0 & 0 \\ \cdots & 0 & -4 & 0 \\ \cdots & 0 & 0 & -2 \end{pmatrix} = (-2m)$$

and:

$$\begin{pmatrix} \frac{1}{2}+it_1 & 0 & 0 & \cdots \\ 0 & \frac{1}{2}+it_2 & 0 & \cdots \\ 0 & 0 & \frac{1}{2}+it_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} = \left(\tfrac{1}{2}+it_n\right)$$

so that $\hat{H}_\zeta$ can be written:

$$\hat{H}_\zeta = \begin{pmatrix} (-2m) & (0) \\ (0) & \left(\frac{1}{2}+it_n\right) \end{pmatrix} = \begin{pmatrix} (-2m) & (0) \\ (0) & \frac{1}{2} \end{pmatrix} + i \begin{pmatrix} (0) & (0) \\ (0) & (t_n) \end{pmatrix}$$

But the matrices $(-2m)$ and $\left(\frac{1}{2}+it_n\right)$ representing the sets of zeros $\{s_1\}$ and $\{s_2\}$ can symbolically be replaced by their parametric form:

$$\begin{gathered} -2m, \quad m \text{ being a positive integer parameter} \\ \tfrac{1}{2}+it_n, \quad t_n \text{ being a real parameter} \end{gathered}$$

The sets $\{s_1\}$ and $\{s_2\}$ can then be considered as the two infinite sets of roots of the polynomial of complex variable s:

$$\begin{gathered} P(s) = (s-s_1)(s-s_2) = s^2 - (s_1+s_2)s + s_1 s_2 \\ P(s,m,t_n) = s^2 - (-2m + \tfrac{1}{2} + it_n)s - 2m(\tfrac{1}{2}+it_n) \\ P(s,m,t_n) = s^2 + (2m - (\tfrac{1}{2}+it_n))s - 2m(\tfrac{1}{2}+it_n) \end{gathered}$$

which, using matrices, can be written either:

$$(2) \quad P(s,m,t_n) = \begin{pmatrix} s^2 & s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m-(\frac{1}{2}+it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2}+it_n) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

or:

$$(3) \quad P(s,m,t_n) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m-(\frac{1}{2}+it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2}+it_n) \end{pmatrix} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix}$$

By setting:

$$\begin{gathered} E_k = \begin{pmatrix} s^2 & s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m-(\frac{1}{2}+it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2}+it_n) \end{pmatrix} \\ E_k = \begin{pmatrix} s^2 & (2m-(\frac{1}{2}+it_n))s & -2m(\frac{1}{2}+it_n) \end{pmatrix} \end{gathered}$$

and:

$$\psi_{E_k} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

equation (2) gives, by multiplying the first two matrices:

$$P(s,m,t_n) = \begin{pmatrix} s^2 & (2m - (\frac{1}{2} + it_n))s & -2m(\frac{1}{2} + it_n) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = E_k \psi_{E_k} \tag{4}$$

Now, as by multiplying the first two matrices of (3), we also have:

$$P(s,m,t_n) = \begin{pmatrix} 1 & (2m - (\frac{1}{2} + it_n)) & -2m(\frac{1}{2} + it_n) \end{pmatrix} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = H_k \psi_{H_k} \tag{5}$$

when we set:

$$H_k = \begin{pmatrix} 1 & (2m - (\frac{1}{2} + it_n)) & -2m(\frac{1}{2} + it_n) \end{pmatrix} \tag{6}$$

and:

$$\psi_{H_k} = \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \hat{R} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \hat{R} \psi_{E_k}$$

where $\hat{R}$ is the 3-dimensional rotation matrix from the orthogonal system of coordinates ψ_{H_k} used to describe H_k to the orthogonal system ψ_{E_k} used to describe E_k, and we have:

$$H_k \psi_{H_k} = H_k \hat{R} \psi_{E_k} = E_k \psi_{E_k}$$

Then, setting $\hat{H} = H_k \hat{R}$, we get:

$$\hat{H} \psi_{E_k} = E_k \psi_{E_k} \tag{7}$$

which is identical to equation (1). And the $\hat{H}_\xi$ operator looked for by Hilbert and Pólya can be built with the real values t_n of the existing operator:

$$\hat{H} = H_k \hat{R} = \begin{pmatrix} 1 & (2m - (\frac{1}{2} + it_n)) & -2m(\frac{1}{2} + it_n) \end{pmatrix} \hat{R}$$

As we can rebuild the Hamiltonian operator $\hat{H}_\xi$ linked to $\zeta(s)$ via the function $P(s,m,t_n)$ and the existing operator $\hat{H}$, this Hamiltonian operator $\hat{H}_\xi$ *does exist* and as we have proven earlier that Riemann's hypothesis is true conditionally to the existence of the $\hat{H}_\xi$ operator, Riemann's hypothesis is therefore unconditionally proven. $\square$

Note. As from equation (6), H_k can also be written:

$$H_k = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m - (\frac{1}{2} + it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2} + it_n) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \hat{A}$$

when we set:

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m - (\frac{1}{2} + it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2} + it_n) \end{pmatrix} \tag{8}$$

we get from (3) and (2) that:

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \hat{A} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = P(s, m, t_n) = \begin{pmatrix} s^2 & s & 1 \end{pmatrix} \hat{A} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{9}$$

But, for any $s = x + iy$, we have:

$$\begin{aligned} P(s, m, t_n) &= s^2 + \left(2m - (\tfrac{1}{2} + it_n)\right) s - 2m\left(\tfrac{1}{2} + it_n\right) \\ &= (x + iy)^2 + \left(2m - (\tfrac{1}{2} + it_n)\right)(x + iy) - 2m\left(\tfrac{1}{2} + it_n\right) \\ &= \left(x^2 - y^2 + (2m - \tfrac{1}{2})x + yt_n\right) + i\left(-xt_n + y(2m - \tfrac{1}{2})\right) - m - 2mit_n \\ &= \left(x^2 - y^2 + (2m - \tfrac{1}{2})x + yt_n - m\right) + i\left(-xt_n + y(2x + 2m - \tfrac{1}{2}) - 2mt_n\right) \end{aligned}$$

and $P(s, m, t_n)$ will be real only when:

$$-xt_n + y(2x + 2m - \tfrac{1}{2}) - 2mt_n = 0$$

and so, only for the infinitely many curves in the complex plane such that:

$$y = t_n \frac{x + 2m}{2x + 2m - \frac{1}{2}} = t_n \frac{x + 2m}{(x + 2m) + (x - \frac{1}{2})}$$

which, for $x = \frac{1}{2}$, are all at $y = t_n$

and for $x = -2m$, are all at $y = 0$.

Then, for all the points of all these curves, we have that:

$$P(s, m, t_n)_{curves} = \left(x^2 - y^2 + (2m - \frac{1}{2})x + yt_n - m\right) = V(x, y) \tag{10}$$

is a real value and therefore the real mono-term matrix $(V(x, y))$ always verifies:

$$(V(x, y)) = (\overline{V(x, y)}) = (\overline{V(x, y)})^T \tag{11}$$

where $(\overline{V(x, y)})$ is the conjugate matrix of $(V(x, y))$ and $(\overline{V(x, y)})^T$ is the conjugate transpose of $(V(x, y))$. So, from (9), (10) and (11), we can write:

$$P(s, m, t_n)_{curves} = V(x, y) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \hat{A} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \left(\begin{pmatrix} s^2 & s & 1 \end{pmatrix} \hat{A} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)^T$$

which proves that the operator $\hat{A}$, also providing the real t_n's to $\hat{H}_\xi$, verifies the equation of the observables in quantum physics, which is generally written:

$$< \psi_1 \mid \hat{A} \mid \psi_2 > \; = \; \left(< \psi_2 \mid \hat{A} \mid \psi_1 >\right)^T$$

where $\hat{A}$ is the Hamiltonian operator associated to the physical quantity $A = V(x, y)$, $< x \mid$ and $\mid x >$ are the bra and ket operators on x and ψ_1 and ψ_2 are the states of the physical quantity A before and after the measuring of A.

Acknowledgements. This work is dedicated to my family. As I am a hobbyist, I wish to express my gratitude towards the Editors of this journal as well as towards the team of Reviewers for having welcomed, reviewed and accepted my article.

References

[1] Euler L., *Introductio in analysin infinitorum*, 1748.
[2] Riemann B., *On the Number of Prime Numbers less than a Given Quantity*, read on internet on February 25th, 2016 at: http://www.claymath.org/sites/default/files/ezeta.pdf
[3] Hilbert, D., *Mathematische Probleme*, Göttinger Nachrichten, pp. 253-297, 1900.
[4] CLay Mathematics Institute, *Millennium Problems*, read on internet on February 25th, 2016 at:
http://www.claymath.org/millennium-problems
[5] Hilbert-Pólya conjecture, retrieved on December 9th, 2016 on Wikipedia web page: https://en.wikipedia.org/wiki/Hilbert%E2%80%93P%C3%B3lya_conjecture #Possible_connection_with_quantum_mechanics
[6] Gourdon X., Demichel P., Computation of zeros of the Zeta function, 2004, retrieved on December 9th, 2016 at: http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeroscompute.html

Robert Deloin, Bouc Bel Air, France
E-mail address: rdeloin@free.fr