Further validation to the variational method to obtain flow relations for generalized Newtonian fluids

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Abstract

We continue our investigation to the use of the variational method to derive flow relations for generalized Newtonian fluids in confined geometries. While in the previous investigations we used the straight circular tube geometry with eight fluid rheological models to demonstrate and establish the variational method, the focus here is on the plane long thin slit geometry using those eight rheological models, namely: Newtonian, power law, Ree-Eyring, Carreau, Cross, Casson, Bingham and Herschel-Bulkley. We demonstrate how the variational principle based on minimizing the total stress in the flow conduit can be used to derive analytical expressions, which are previously derived by other methods, or used in conjunction with numerical procedures to obtain numerical solutions which are virtually identical to the solutions obtained previously from well established methods of fluid dynamics. In this regard, we use the method of Weissenberg-Rabinowitsch-Mooney-Schofield (WRMS), with our adaptation from the circular pipe geometry to the long thin slit geometry, to derive analytical formulae for the eight types of fluid where these derived formulae are used for comparison and validation of the variational formulae and numerical solutions. Although some examples may be of little value, the optimization principle which the variational method is based upon has a significant theoretical value as it reveals the tendency of the flow system to assume a configuration that minimizes the total stress. Our proposal also offers a new methodology to tackle common problems in fluid dynamics and rheology.

Keywords: Euler-Lagrange variational principle; fluid mechanics; rheology; generalized Newtonian fluid; slit flow; pressure-flow rate relation; Newtonian; power law; Ree-Eyring; Carreau; Cross; Casson; Bingham; Herschel-Bulkley; Weissenberg-Rabinowitsch-Mooney-Schofield method.

1 Introduction

The flow of Newtonian and non-Newtonian fluids in various confined geometries, such as tubes and slits, is commonplace in many natural and technological systems. Hence, many methods have been proposed and developed to solve the flow problems in such geometries applying different physical principles and employing a diverse collection of analytical, empirical and numerical techniques. These methods range from employing the first principles of fluid dynamics which are based on the rules of classical mechanics to more specialized techniques such as the use of Weissenberg-Rabinowitsch-Mooney-Schofield relation or one of the Navier-Stokes adaptations [1, 2].

One of the elegant mathematical branches that is regularly employed in the physical sciences is the calculus of variation which is based on optimizing functionals that describe certain physical phenomena. The variational method is widely used in many disciplines of theoretical and applied sciences, such as quantum mechanics and statistical physics, as well as many fields of engineering. Apart from its mathematical beauty, the method has a big advantage over many other competing methods by giving an insight into the investigated phenomena. The method does not only solve the problem formally and hence provides a mathematical solution but it also reveals the Nature habits and its inclination to economize or lavish on one of the involved physical attributes or the other such as time, speed, entropy and energy. Some of the well known examples that are based on the variational principle or derived from the variational method are the Fermat principle of least time and the curve of fastest descent (brachistochrone). These examples, among many other variational examples, have played a significant role in the development of the modern natural sciences and mathematical methods.

In reference [3] we made an attempt to exploit the variational method to obtain analytic or numeric relations for the flow of generalized Newtonian fluids in confined geometries where we postulated that the flow profile in a flow conduit will adjust itself to minimize the total stress. This attempt was later [4] extended to include more types of non-Newtonian fluids. In the above references, the flow of eight fluid models (Newtonian, power law, Bingham, Herschel-Bulkley, Carreau, Cross, Ree-Eyring and Casson) in straight cylindrical tubes was investigated analytically and/or numerically with some of these models confirming the stated variational hypothesis while others, due to mathematical difficulties or limitation of the underlying principle, demonstrated behavioral trends that are consistent with the variational hypothesis.

No mathematically rigorous proof was gives in [3, 4] to establish the proposed variational method that is based on minimizing the total stress in its generality. Furthermore, we do not make any attempt here to present such a proof. However, in the present paper we make an attempt to consolidate our previous proposal and findings by giving more examples, this time from the slit geometry rather than the tube geometry, to validate the use of the variational principle in deriving flow relations in confined geometries for generalized Newtonian fluids.

The plan for this paper is as follow. In the next section § 2 we present the general formulation of the variational method as applied to the long thin slits and derive the main variational equation that will be used in obtaining the flow relations for the generalized Newtonian fluids. In section § 3 we apply and validate the variational method for five types of non-viscoplastic fluids, namely: Newtonian, power law, Ree-Eyring, Carreau and Cross; while in section § 4 we apply and validate the method for three types of viscoplastic fluids, namely: Casson, Bingham and Herschel-Bulkley. We separate the viscoplastic from the non-viscoplastic because the variational method strictly applies only to non-viscoplastic fluids, and hence its use with viscoplastic fluids is an approximation which is valid and good only when the yield stress value of these fluids is low and hence the departure from fluidity is minor. In the validation of both non-viscoplastic and viscoplastic types we use the aforementioned WRMS method where we compare the variational solutions to the analytical solutions obtained from the WRMS method where analytical formulae for the eight types of fluid are derived in the Appendix (§ 7). The paper in ended in section § 5 where general discussion and conclusions about the paper, its objectives and achievements are presented.

2 Method

The rheological behavior of generalized Newtonian fluids in one dimensional shear flow is described by the following constitutive relation

$$\tau = \mu\gamma \tag{1}$$

where τ is the shear stress, γ is the rate of shear strain, and μ is the shear viscosity which is normally a function of the contemporary rate of shear strain but not of the deformation history although it may also be a function of other physical parameters such as temperature and pressure. The latter parameters are not considered in the present investigation as we assume a static physical setting (i.e. isothermal, isobaric, etc.) apart from the purely kinematical aspects of the deformation process that is necessary to initiate and sustain the flow.

In the following we use the slit geometry, depicted in Figure 1, as our flow apparatus where $2B$ is the slit thickness, L is the length of the slit across which a pressure drop Δp is imposed, and W is the part of the slit width that is under consideration although for the purpose of eliminating lateral edge effects we assume that the total width of the slit is much larger than the considered part W. We choose our coordinates system so that the slit smallest dimension is being positioned symmetrically with respect to the plane $z = 0$.

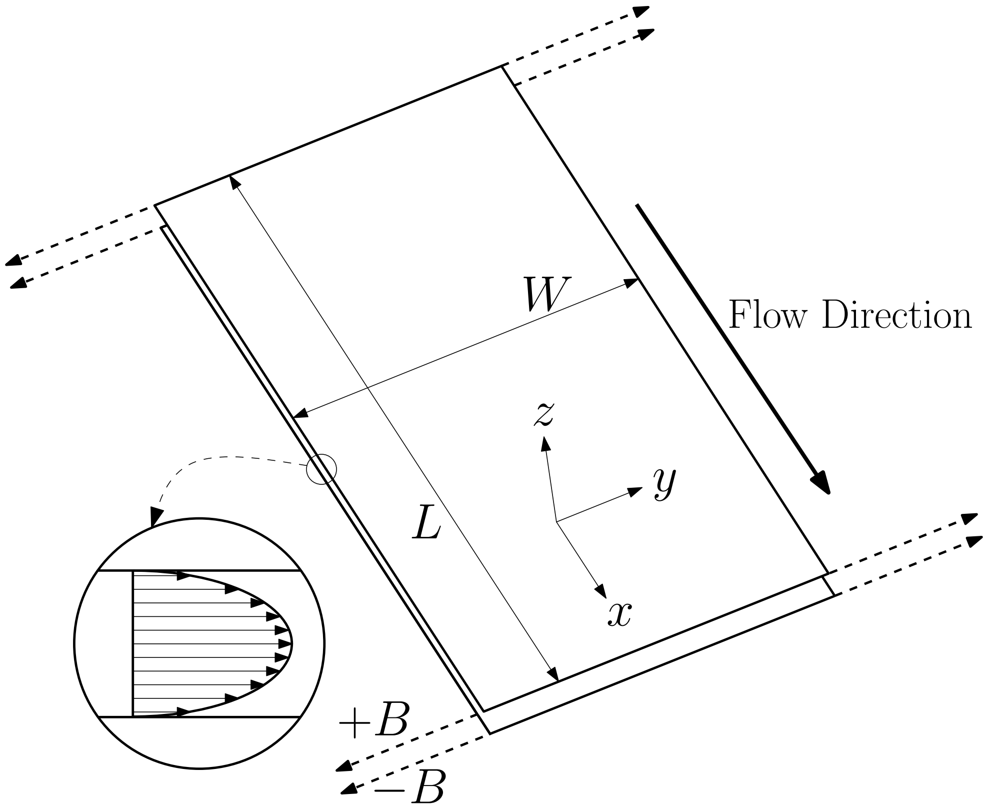


Figure 1: Schematic drawing of the slit geometry which is used in the present investigation.

For the slit geometry of Figure 1 the total stress is given by

$$\tau_t = \int_{\tau_{-B}}^{\tau_{+B}} d\tau = \int_{-B}^{+B} \frac{d\tau}{dz} dz = \int_{-B}^{+B} \frac{d}{dz} (\mu\gamma)\, dz = \int_{-B}^{+B} \left(\gamma \frac{d\mu}{dz} + \mu \frac{d\gamma}{dz} \right) dz \qquad (2)$$

where τ_t is the total stress, and $\tau_{\pm B}$ is the shear stress at the slit walls corresponding to $z = \pm B$.

The total stress, as given by Equation 2, can be minimized by applying the

Euler-Lagrange variational principle which, in one of its forms, is given by

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial x} = 0 \tag{3}$$

where the symbols corresponding to our problem statement are defined as

$$x \equiv z, \quad y \equiv \gamma, \quad f \equiv \gamma\frac{d\mu}{dz} + \mu\frac{d\gamma}{dz}, \quad \text{and} \quad \frac{\partial f}{\partial y'} \equiv \frac{\partial}{\partial \gamma'}\left(\gamma\frac{d\mu}{dz} + \mu\frac{d\gamma}{dz}\right) = \mu \tag{4}$$

On substituting these symbols into Equation 3 the following equation is obtained

$$\frac{d}{dz}\left(\gamma\frac{d\mu}{dz} + \mu\frac{d\gamma}{dz} - \mu\frac{d\gamma}{dz}\right) - \frac{\partial}{\partial z}\left(\gamma\frac{d\mu}{dz} + \mu\frac{d\gamma}{dz}\right) = 0 \tag{5}$$

Considering the fact that for the considered flow systems

$$\gamma\frac{d\mu}{dz} + \mu\frac{d\gamma}{dz} = G \tag{6}$$

where G is a constant, it can be shown that Equation 5 can be reduced to two independent variational forms

$$\frac{d}{dz}\left(\gamma\frac{d\mu}{dz}\right) = 0 \tag{7}$$

and

$$\frac{d}{dz}\left(\mu\frac{d\gamma}{dz}\right) = 0 \tag{8}$$

In the following two sections we use the second form where we outline the application of the variational method, as summarized in Equation 8, to validate and demonstrate the use of the variational method to obtain flow relations correlating the volumetric flow rate Q through the slit to the pressure drop Δp across the slit length L for generalized Newtonian fluids. The plan is that we derive fully analytical expressions when this is viable, as in the case of Newtonian and power law fluids, and partly analytical solutions when the former is not viable, e.g. for Ree-Eyring and Herschel-Bulkley fluids. In the latter case, the solution is obtained numerically in its final stages, following a variationally based derivation, by using numerical integration and simple numerical solvers.

In this investigation we assume a laminar, incompressible, time-independent, fully-developed, isothermal flow where entry and exit edge effects are negligible. We also assume negligible body forces and a blunt flow speed profile with a no-shear stationary region at the profile center plane which is consistent with the considered type of fluids and flow conditions, i.e. viscous generalized Newtonian fluids in a pressure-driven laminar flow. As for the plane slit geometry, we assume, following what is stated in the literature, a long thin slit with $B \ll W$ and $B \ll L$ although we believe that some of these conditions are redundant according to our own statement and problem settings. We also assume that the slit is rigid and uniform in shape and size, that is its walls are not made of deformable materials, such as elastic or viscoelastic, and the slit does not experience an abrupt or gradual change in B.

3 Non-Viscoplastic Fluids

For non-viscoplastic fluids, the variational principle strictly applies. In this section we apply the variational method to five non-viscoplastic fluids and compare the variational solutions to the analytical solutions obtained from the WRMS method. These fluids are: Newtonian, power law, Ree-Eyring, Carreau and Cross. All these models have analytical solutions that can be obtained from various traditional methods of fluid dynamics which are not based on the variational principle. Hence the agreement between the solutions obtained from the traditional methods with the solutions obtained from the variational method will validate and vindicate the variational approach.

As indicated earlier, to derive non-variational analytical relations we use a method similar to the one ascribed to Weissenberg, Rabinowitsch, Mooney, and Schofield [1] for the flow of generalized Newtonian fluids in uniform tubes with circular cross sections, where we adapt and apply the procedure to long thin slits, and hence we label this method with WRMS to abbreviate the names of its originators. The WRMS method is fully explained and applied in the Appendix (§ 7) to derive analytical relations to all the eight fluid models that are used in the present paper. For some of these fluids full analytical solutions from the variational principle are obtained and hence a direct comparison between the analytical expressions obtained from the two methods can be made, while for other fluids a mixed analytical and numerical procedure is employed to obtain numerical solutions from the variational principle and hence a representative sample of numerical solutions from both methods is presented for comparison and validation, as will be clarified and demonstrated in the following subsections.

3.1 Newtonian

The viscosity of Newtonian fluids is constant, that is

$$\mu = \mu_o \tag{9}$$

and therefore Equation 8 becomes

$$\frac{d}{dz}\left(\mu_o \frac{d\gamma}{dz}\right) = 0 \tag{10}$$

On performing the outer integration we obtain

$$\mu_o \frac{d\gamma}{dz} = A \tag{11}$$

where A is the constant of integration. On performing the inner integration we obtain

$$\gamma = \frac{A}{\mu_o} z + D \tag{12}$$

where D is a second constant of integration. Now from the no-shear condition at the slit center plane $z = 0$, D can be determined, that is

$$\gamma\,(z = 0) = 0 \qquad \Rightarrow \qquad D = 0 \tag{13}$$

Similarly, from the no-slip boundary condition [5] at $z = \pm B$ which controls the wall shear stress we determine A, i.e.

$$\tau_{\pm B} = \frac{F_\perp}{\sigma_\parallel} = \frac{2BW\Delta p}{2WL} = \frac{B\Delta p}{L} \tag{14}$$

where $\tau_{\pm B}$ is the shear stress at the slit walls, $F_\perp$ is the flow driving force which is normal to the slit cross section in the flow direction, and $\sigma_\parallel$ is the area of the slit walls which is tangential to the flow direction. Hence

$$\gamma\,(z = \pm B) = \frac{\tau_{\pm B}}{\mu_o} = \frac{B\Delta p}{\mu_o L} = \frac{AB}{\mu_o} \qquad \Rightarrow \qquad A = \frac{\Delta p}{L} \tag{15}$$

Therefore

$$\gamma\,(z) = \frac{\Delta p}{\mu_o L} z \tag{16}$$

On integrating the rate of shear strain with respect to z, the standard parabolic speed profile is obtained, that is

$$v(z) = \int dv = \int \frac{dv}{dz} dz = \int -\gamma dz = -\int \frac{\Delta p}{\mu_o L} z dz = -\frac{\Delta p}{2\mu_o L} z^2 + E \tag{17}$$

where $v(z)$ is the fluid speed at z in the x direction and E is another constant of integration which can be determined from the no-slip at the wall boundary condition, that is

$$v(z = \pm B) = 0 \qquad \Rightarrow E = \frac{\Delta p}{2\mu_o L} B^2 \tag{18}$$

i.e.

$$v(z) = \frac{\Delta p}{2\mu_o L} \left(B^2 - z^2\right) \tag{19}$$

The volumetric flow rate is then obtained by integrating the flow speed profile over the slit cross sectional area in the z direction, that is

$$Q = \int_{-B}^{+B} v W dz = \frac{2W\Delta p}{2\mu_o L} \int_0^B \left(B^2 - z^2\right) dz = \frac{W\Delta p}{\mu_o L} \left[B^2 z - \frac{z^3}{3}\right]_0^B \tag{20}$$

that is

$$Q = \frac{2WB^3\Delta p}{3\mu_o L} \tag{21}$$

which is the well known volumetric flow rate formula for the flow of Newtonian fluids in a plane long thin slit as obtained by other methods which are not based on the variational principle. This formula is derived in the Appendix (§ 7, Equation 80) using the WRMS method. It also can be found in several classic textbooks of fluid mechanics, e.g. Bird *et al.* [2] Table 4.5-14 where $\mu_o \equiv \mu$ and $\Delta p \equiv P_0 - P_L$.

3.2 Power Law

The shear dependent viscosity of power law fluids is given by [1, 2, 6]

$$\mu = k\gamma^{n-1} \tag{22}$$

where k is the power law viscosity consistency coefficient and n is the flow behavior index. On applying the Euler-Lagrange variational principle (Equation 8) we obtain

$$\frac{d}{dz}\left(k\gamma^{n-1}\frac{d\gamma}{dz}\right)=0 \tag{23}$$

On performing the outer integral we obtain

$$k\gamma^{n-1}\frac{d\gamma}{dz}=A \tag{24}$$

On separating the two variables in the last equation and integrating both sides we obtain

$$\gamma=\sqrt[n]{\frac{n}{k}\left(Az+D\right)} \tag{25}$$

where A and D are the constants of integration which can be determined from the two limiting conditions, that is

$$\gamma\left(z=0\right)=0 \qquad \Rightarrow \qquad D=0 \tag{26}$$

and

$$\gamma\left(z=B\right)=\sqrt[n]{\frac{\tau_B}{k}}=\sqrt[n]{\frac{B\Delta p}{Lk}}=\sqrt[n]{\frac{n}{k}AB} \qquad \Rightarrow \qquad A=\frac{\Delta p}{nL} \tag{27}$$

where the first step in the last equation is obtained from the constitutive relation of power law fluids, i.e.

$$\tau=k\gamma^{n} \tag{28}$$

with the substitution $z=B$ in Equation 25. Hence, from Equation 25 we obtain

$$\gamma=\sqrt[n]{\frac{\Delta p}{kL}}z^{1/n} \tag{29}$$

On integrating the rate of shear strain with respect to z, the flow speed profile is determined, i.e.

$$v\left(z\right)=\int dv=\int\frac{dv}{dz}dz=-\int\gamma dz=-\int\sqrt[n]{\frac{\Delta p}{kL}}z^{1/n}dz=-\frac{n}{n+1}\sqrt[n]{\frac{\Delta p}{kL}}z^{1+1/n}+E \tag{30}$$

where E is another constant of integration which can be determined from the no-slip at the wall condition, that is

$$v\left(z=B\right)=0 \qquad \Rightarrow \qquad E=\frac{n}{n+1}\sqrt[n]{\frac{\Delta p}{kL}}B^{1+1/n} \tag{31}$$

i.e.

$$v\left(z\right)=\frac{n}{n+1}\sqrt[n]{\frac{\Delta p}{kL}}\left(B^{1+1/n}-z^{1+1/n}\right) \tag{32}$$

The volumetric flow rate can then be obtained by integrating the flow speed profile with respect to the cross sectional area in the z direction, that is

$$Q=\int_{-B}^{+B}v\,W dz=\frac{2Wn}{n+1}\sqrt[n]{\frac{\Delta p}{kL}}\int_{0}^{B}\left(B^{1+1/n}-z^{1+1/n}\right)dz \tag{33}$$

$$=\frac{2Wn}{n+1}\sqrt[n]{\frac{\Delta p}{kL}}\left[B^{1+1/n}z-\frac{z^{2+1/n}}{2+1/n}\right]_{0}^{B} \tag{34}$$

$$=\frac{2Wn}{n+1}\sqrt[n]{\frac{\Delta p}{kL}}\left[B^{2+1/n}-\frac{B^{2+1/n}}{2+1/n}\right] \tag{35}$$

i.e.

$$Q=\frac{2WB^{2}n}{2n+1}\sqrt[n]{\frac{B\Delta p}{kL}} \tag{36}$$

which is the well known volumetric flow rate relation for the flow of power law fluids in a long thin slit as obtained by other non-variational methods. This formula is derived in the Appendix (§ 7, Equation 84) using the WRMS method. It can also be found in textbooks of fluid mechanics such as Bird *et al.* [2] Table 4.2-1 where $k\equiv m$ and $\Delta p\equiv P_0-P_L$.

3.3 Ree-Eyring

For Ree-Eyring fluids, the constitutive relation between shear stress and rate of strain is given by [2]

$$\tau=\tau_{c}\,\mathrm{asinh}\left(\frac{\mu_{r}\gamma}{\tau_{c}}\right) \tag{37}$$

where τ_c is a characteristic shear stress and μ_r is the viscosity at vanishing rate of strain. Hence, the generalized Newtonian viscosity is given by

$$\mu = \frac{\tau}{\gamma} = \frac{\tau_c \operatorname{asinh}\left(\frac{\mu_r \gamma}{\tau_c}\right)}{\gamma} \tag{38}$$

On substituting μ from the last relation into Equation 8 we obtain

$$\frac{d}{dz}\left(\frac{\tau_c \operatorname{asinh}\left(\frac{\mu_r \gamma}{\tau_c}\right)}{\gamma}\frac{d\gamma}{dz}\right) = 0 \tag{39}$$

On integrating once we get

$$\frac{\tau_c \operatorname{asinh}\left(\frac{\mu_r \gamma}{\tau_c}\right)}{\gamma}\frac{d\gamma}{dz} = A \tag{40}$$

where A is a constant. On separating the two variables and integrating again we obtain

$$\int \frac{\tau_c \operatorname{asinh}\left(\frac{\mu_r \gamma}{\tau_c}\right)}{\gamma} d\gamma = Az \tag{41}$$

where the constant of integration D is absorbed within the integral on the left hand side. The integral on the left hand side of Equation 41 when evaluated analytically produces an expression that involves logarithmic and polylogarithmic functions which when computed produce significant errors especially in the neighborhood of $z = 0$. To solve this problem we used a numerical integration procedure to evaluate this integral, and hence obtain A, numerically using the boundary condition at the slit wall, that is

$$\gamma\left(z = B\right) \equiv \gamma_B = \frac{\tau_c}{\mu_r}\sinh\left(\frac{\tau_B}{\tau_c}\right) \tag{42}$$

where τ_B is given by Equation 14. This was then followed by obtaining γ as a function of z using a bisection numerical solver in conjunction with a numerical integration procedure based on Equation 41. Due to the fact that the constant of integration, D, is absorbed in the left hand side and a numerical integration procedure was used rather than an analytical evaluation of the integral on the left hand side of Equation 41, there was no need for an analytical evaluation of this constant using the boundary condition at the slit center plane, i.e.

$$\gamma\left(z = 0\right) = 0 \tag{43}$$

The numerically obtained γ was then integrated numerically with respect to z to obtain the flow speed as a function of z where the no-slip boundary condition at the wall is used to have an initial value $v = 0$ that is incremented on moving inward from the wall toward the center plane. The flow speed profile was then integrated numerically with respect to the cross sectional area to obtain the volumetric flow rate.

To test the validity of the variational method we made extensive comparisons between the solutions obtained from the variational method to those obtained from the WRMS method using widely varying ranges of fluid and slit parameters. In Figure 2 we compare the WRMS analytical solutions as derived in the Appendix (§ 7, Equation 88) with the variational solutions for two sample cases. As seen, the two methods agree very well which is typical in all the investigated cases. The minor differences between the solutions of the two methods can be easily explained by the accumulated errors arising from repetitive use of numerical integration and numerical solvers in the variational method. The errors as estimated by the percentage relative difference are typically less than 0.5% when using reliable numerical integration schemes with reasonably fine discretization mesh and tiny error margin for the convergence condition of the numerical solver. This is also true in general for the other types of fluid that will follow in this section.

3.4 Carreau

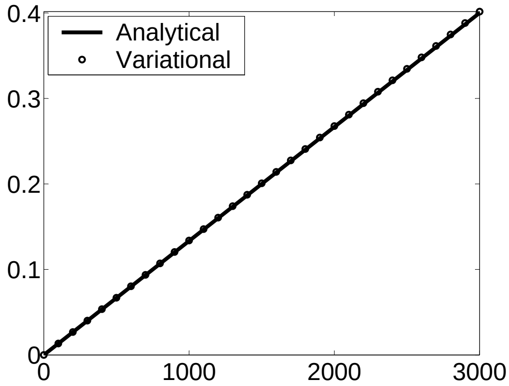
For Carreau fluids, the viscosity is given by [2, 6–8]

$$\mu = \mu_i + (\mu_0 - \mu_i)\left[1 + \lambda^2\gamma^2\right]^{(n-1)/2} \tag{44}$$

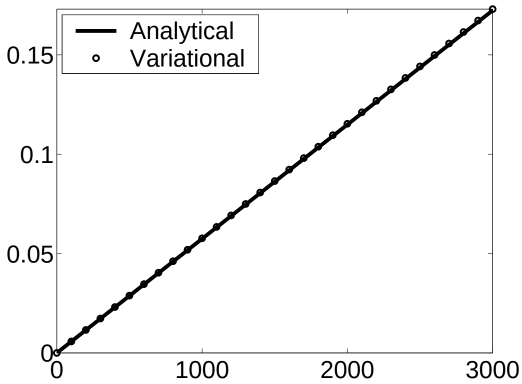
where μ_0 is the zero-shear viscosity, μ_i is the infinite-shear viscosity, λ is a characteristic time constant, and n is the flow behavior index. On applying the Euler-Lagrange variational principle (Equation 8) and following the derivation, as outlined in the previous subsections, we obtain

$$\mu_i\gamma + (\mu_0 - \mu_i)\,\gamma\,{}_2F_1\left(\frac{1}{2}, \frac{1-n}{2}; \frac{3}{2}; -\lambda^2\gamma^2\right) = Az + D \tag{45}$$

where ${}_2F_1$ is the hypergeometric function of the given argument with its real part being used in the computation, and A and D are the constants of integration. From the two boundary conditions at $z = 0$ and $z = B$, A and D can be determined, that is



(a)



(b)

Figure 2: Comparing the WRMS analytical solutions, as given by Equation 88, to the variational solutions for the flow of Ree-Eyring fluids in long thin slits with (a) $\mu_r = 0.005$ Pa.s, $\tau_c = 600$ Pa, $B = 0.01$ m, $W = 1.0$ m and $L = 1.0$ m; and (b) $\mu_r = 0.015$ Pa.s, $\tau_c = 400$ Pa, $B = 0.013$ m, $W = 1.0$ m and $L = 1.7$ m. In both sub-figures, the vertical axis represents the volumetric flow rate, Q, in m^3.s^{-1} while the horizontal axis represents the pressure drop, Δp, in Pa. The average percentage relative difference between the WRMS analytical solutions and the variational solutions for these cases are about 0.38% and 0.39% respectively.

$$\gamma(z = 0) = 0 \qquad \Rightarrow \qquad D = 0 \tag{46}$$

and

$$\gamma(z = B) = \gamma_B \qquad \Rightarrow \qquad \mu_i \gamma_B + (\mu_0 - \mu_i) \gamma_B \, {}_2F_1\left(\frac{1}{2}, \frac{1-n}{2}; \frac{3}{2}; -\lambda^2 \gamma_B^2\right) = AB \tag{47}$$

where γ_B is the shear rate at the slit wall. Now, by definition we have

$$\mu_B \gamma_B = \tau_B \tag{48}$$

that is

$$\left[\mu_i + (\mu_0 - \mu_i)\left[1 + \lambda^2 \gamma_B^2\right]^{(n-1)/2}\right] \gamma_B = \frac{B \Delta p}{L} \tag{49}$$

From the last equation, γ_B can be obtained numerically by a numerical solver, based for example on a bisection method, and hence from Equation 47 A is obtained. Equation 45 can then be solved numerically to obtain the shear rate γ as a function of z. This will be followed by integrating γ numerically with respect to z to obtain the speed profile, $v(r)$, where the no-slip boundary condition at the wall is used to have an initial value $v(z = \pm B) = 0$ that is incremented on moving inward toward the center plane. The speed profile, in its turn, will be integrated numerically with respect to the slit cross sectional area to obtain the volumetric flow rate Q.

In Figure 3 we present two sample cases for the flow of Carreau fluids in thin slits where the WRMS analytical solutions, as given by Equation 96, are compared to the variational solutions. Good agreement can be seen in these plots which are typical of the investigated cases. The main reason for the difference between the WRMS and variational solutions is the persistent use of numerical solvers and numerical integration in the implementation of the variational method as well as the use of the hypergeometric function in both methods. The numerical implementation of this function can cause instability and failure to converge satisfactorily in some cases.

3.5 Cross

For Cross fluids, the viscosity is given by [6, 9]

$$\mu = \mu_i + \frac{\mu_0 - \mu_i}{1 + \lambda^m \gamma^m} \tag{50}$$

where μ_0 is the zero-shear viscosity, μ_i is the infinite-shear viscosity, λ is a characteristic time constant, and m is an indicial parameter. Following a similar derivation method to that outlined in Carreau, we obtain

$$\mu_i \gamma + (\mu_0 - \mu_i)\, \gamma \,{}_2F_1\left(1, \frac{1}{m}; \frac{m+1}{m}; -\lambda^m \gamma^m\right) = Az \tag{51}$$

where

$$A = \frac{\mu_i \gamma_B + (\mu_0 - \mu_i)\, \gamma_B \,{}_2F_1\left(1, \frac{1}{m}; \frac{m+1}{m}; -\lambda^m \gamma_B^m\right)}{B} \tag{52}$$

with γ_B being obtained numerically as outlined in Carreau. Equation 51 can then be solved numerically to obtain the shear rate γ as a function of z. This is followed by obtaining v from γ and Q from v by using numerical integration, as before.

In Figure 4 we present two sample cases for the flow of Cross fluids in thin slits where we compare the WRMS analytical solutions, as given by Equation 104, to the variational solutions. As seen in these plots, the agreement is good with the main reason for the departure between the two methods is the persistent use of numerical solvers and numerical integration in the variational method as well as the use of

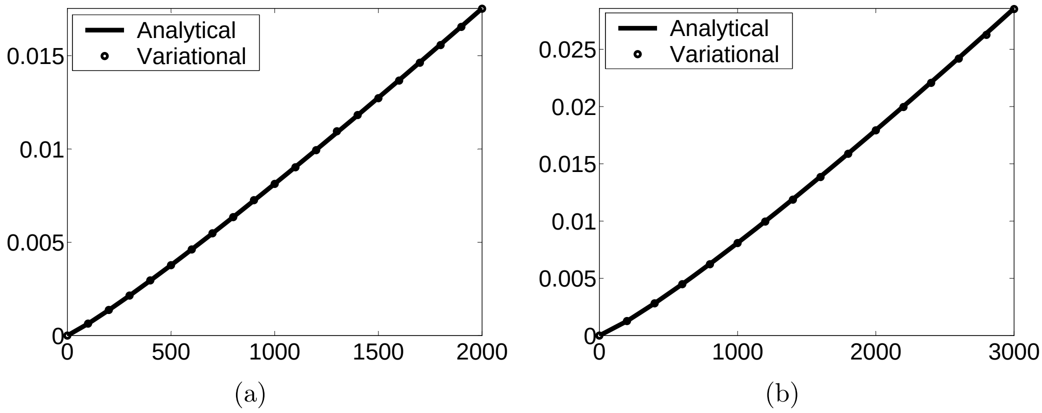


Figure 3: Comparing the WRMS analytical solutions, as given by Equation 96, to the variational solutions for the flow of Carreau fluids in long thin slits with (a) $n = 0.9$, $\mu_0 = 0.13$ Pa.s, $\mu_i = 0.002$ Pa.s, $\lambda = 0.8$ s, $B = 0.011$ m, $W = 1.0$ m and $L = 1.25$ m; and (b) $n = 0.85$, $\mu_0 = 0.15$ Pa.s, $\mu_i = 0.01$ Pa.s, $\lambda = 1.65$ s, $B = 0.011$ m, $W = 1.0$ m and $L = 1.4$ m. The axes are as in Figure 2, while the differences are about 0.21% and 0.25% respectively.

the hypergeometric function in both methods, as explained in the case of Carreau.

4 Viscoplastic Fluids

The yield stress fluids are not strictly subject to the variational principle due to the existence of a solid non-yield region at the center which does not obey the stress optimization condition and hence the Euler-Lagrange variational method is not strictly applicable to these fluids. However, the method provides a good approximation when the value of the yield stress is low so that the effect of the non-yield region at and around the center plane of the slit on the flow profile is minor. In the following subsections we apply the variational method to three yield stress fluids and obtain some solutions from sample cases which are representative of the many cases that were investigated.

4.1 Casson

For Casson fluids, the constitutive relation is given by [2, 6]

$$\tau = \left[(K\gamma)^{1/2} + \tau_o^{1/2} \right]^2 \tag{53}$$

where K is the viscosity consistency coefficient, and τ_o is the yield stress. Hence

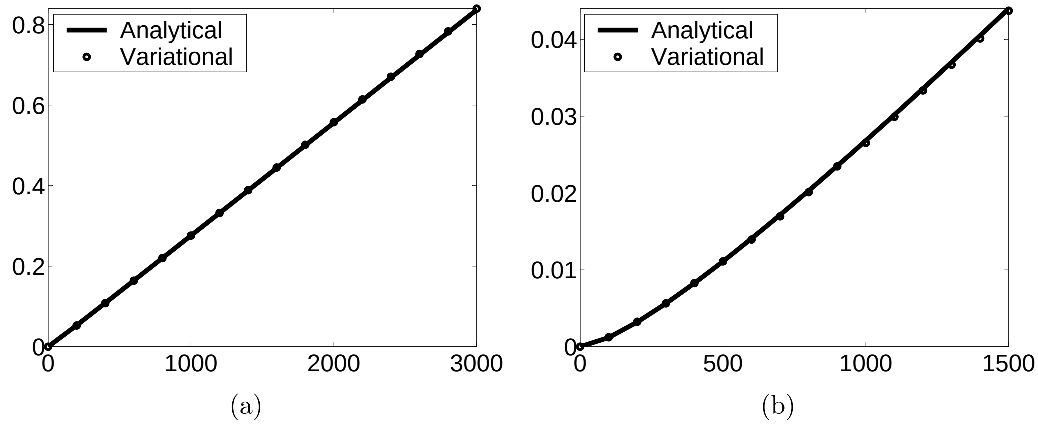


Figure 4: Comparing the WRMS analytical solutions, as given by Equation 104, to the variational solutions for the flow of Cross fluids in long thin slits with (a) $m = 0.65$, $\mu_0 = 0.032$ Pa.s, $\mu_i = 0.004$ Pa.s, $\lambda = 4.5$ s, $B = 0.013$ m, $W = 1.0$ m and $L = 1.3$ m; and (b) $m = 0.5$, $\mu_0 = 0.075$ Pa.s, $\mu_i = 0.004$ Pa.s, $\lambda = 0.8$ s, $B = 0.006$ m, $W = 1.0$ m and $L = 0.8$ m. The axes are as in Figure 2, while the differences are about 0.29% and 0.88% respectively.

$$\mu = \frac{\tau}{\gamma} = \frac{\left[(K\gamma)^{1/2} + \tau_o^{1/2}\right]^2}{\gamma} \tag{54}$$

On substituting μ from the last equation into the main variational relation, as given by Equation 8, we obtain

$$\frac{d}{dz}\left(\frac{\left[(K\gamma)^{1/2} + \tau_o^{1/2}\right]^2}{\gamma}\frac{d\gamma}{dz}\right) = 0 \tag{55}$$

On integrating twice with respect to z we obtain

$$K\gamma + 4\left(K\tau_o\gamma\right)^{1/2} + \tau_o \ln\left(\gamma\right) = Az + D \tag{56}$$

where A and D are constants. Now, from the boundary condition at the slit center plane we have

$$\gamma\left(z = 0\right) = 0 \tag{57}$$

so we set $D = 0$ to constrain the solution at $z = 0$. As for the second boundary condition at the slit wall, $z = B$, we have

$$K\gamma_B + 4\left(K\tau_o\gamma_B\right)^{1/2} + \tau_o \ln\left(\gamma_B\right) = AB \tag{58}$$

where the rate of shear strain at the slit wall, γ_B, is obtained from applying the constitutive relation at the wall and hence is given by

$$\gamma_B = \frac{\left[\sqrt{\tau_B} - \tau_o^{1/2}\right]^2}{K} \tag{59}$$

with the shear stress at the slit wall, τ_B, being given by Equation 14. Hence

$$A = \frac{K\gamma_B + 4\left(K\tau_o\gamma_B\right)^{1/2} + \tau_o \ln\left(\gamma_B\right)}{B} \tag{60}$$

Equation 56 defines γ implicitly in terms of z and hence it is solved numerically using, for instance, a numerical bisection method to find γ as a function of z with avoidance of the very immediate neighborhood of $z = 0$ which, as explained earlier, is not subject to the variational method. This is equivalent to integrating between τ_o and τ_B, rather than between 0 and τ_B, in the WRMS method, as employed in the Appendix (§ 7), for the case of Casson, Bingham and Herschel-Bulkley fluids. Although the value of z that defines the start of the yield region near the center is not known exactly, we already assumed that the use of the variational method is only legitimate when τ_o is small and hence the non-yield plug region is small and hence its effect is minor, so any error from an ambiguity in the exact limit of the integral near the $z = 0$ should be negligible especially at high flow rates where this region shrinks and hence using a lower limit of the integral at the immediate neighborhood of $z = 0$ will give a more exact definition of the yield region. On obtaining γ numerically, v and Q can be obtained successively by numerical integration, as before.

In Figure 5 we present two sample cases for the flow of Casson fluids in thin slits where we compare the WRMS analytical solutions, as given by Equation 110, with the variational solutions. As seen, the agreement is reasonably good considering that the variational method is just an approximation and hence it is not supposed to produce identical results to the analytically derived solutions. The two plots also indicate that the approximation is worsened as the value of the yield stress increases, resulting in the increase of the effect of the non-yield region at the center plane of the slit which is not subject to the variational principle, and hence more deviation between the two methods is observed.

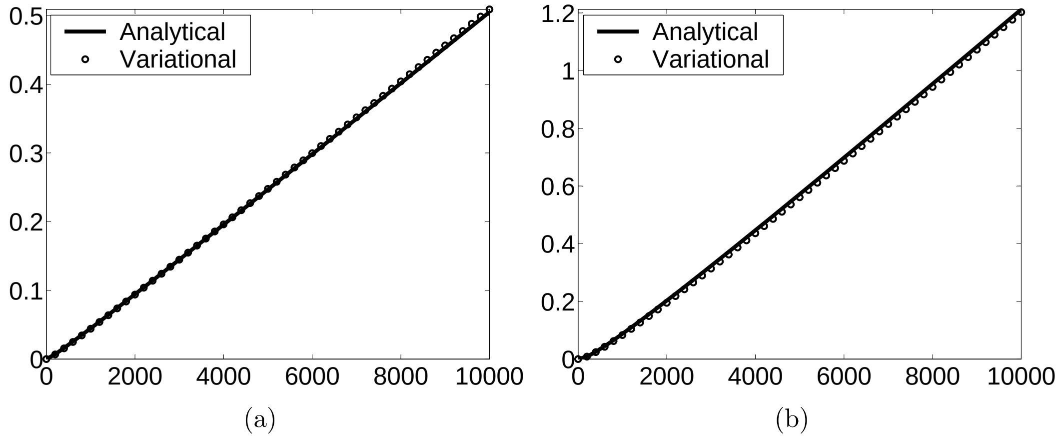


Figure 5: Comparing the WRMS analytical solutions, as given by Equation 110, to the variational solutions for the flow of Casson fluids in long thin slits with (a) $K = 0.025$ Pa.s, $\tau_o = 0.1$ Pa, $B = 0.01$ m, $W = 1.0$ m and $L = 0.5$ m; and (b) $K = 0.05$ Pa.s, $\tau_o = 0.5$ Pa, $B = 0.025$ m, $W = 1.0$ m and $L = 1.5$ m. The axes are as in Figure 2, while the differences are about 0.57% and 2.84% respectively.

4.2 Bingham

For Bingham fluids, the viscosity is given by [1, 2, 6]

$$\mu = \frac{\tau_o}{\gamma} + C' \tag{61}$$

where τ_o is the yield stress and C' is the viscosity consistency coefficient. On applying the variational principle, as formulated by Equation 8, and following the previously outlined method we obtain

$$\tau_o \ln \gamma + C'\gamma = Az + D \tag{62}$$

where A and D are the constants of integration. Using the boundary conditions at the center plane and at the slit wall and following a similar procedure to that of Casson, we obtain

$$D = 0 \qquad \text{and} \qquad A = \frac{\tau_o}{B} \ln\left(\frac{B\Delta p}{LC'} - \frac{\tau_o}{C'}\right) + \left(\frac{\Delta p}{L} - \frac{\tau_o}{B}\right) \tag{63}$$

The strain rate is then obtained numerically from Equation 62, and thereby v and Q are computed successively, as explained before.

In Figure 6 two sample cases of the flow of Bingham fluids in thin slits are presented. As seen, the agreement between the WRMS solutions, as obtained from Equation 113, and the variational solutions are rather good despite the fact that the variational method is an approximation when applied to viscoplastic fluids.

4.3 Herschel-Bulkley

The viscosity of Herschel-Bulkley fluids is given by [1, 2, 6]

$$\mu = \frac{\tau_o}{\gamma} + C\gamma^{n-1} \tag{64}$$

where τ_o is the yield stress, C is the viscosity consistency coefficient and n is the flow behavior index. On following a procedure similar to the procedure of Bingham model with the application of the γ two boundary conditions, we get the following equation

$$\tau_o \ln\gamma + \frac{C}{n}\gamma^n = Az \tag{65}$$

where

$$A = \frac{\tau_o}{B}\ln\left[\left(\frac{B\Delta p}{LC} - \frac{\tau_o}{C}\right)^{1/n}\right] + \frac{1}{n}\left(\frac{\Delta p}{L} - \frac{\tau_o}{B}\right) \tag{66}$$

On solving Equation 65 numerically, γ as a function of z is obtained, followed by obtaining v and Q, as explained previously.

In Figure 7 we compare the WRMS analytical solutions of Equation 117 to the variational solutions for two sample Herschel-Bulkley fluids, one shear thinning and one shear thickening, both with yield stress. As seen, the agreement between the solutions of the two methods is good as in the previous cases of Casson and Bingham fluids.

5 Conclusions

In this paper we provided further evidence for the validity of the variational method which is based on minimizing the total stress in the flow conduit to obtain flow relations for the generalized Newtonian fluids in confined geometries. Our investigation in the present paper, which is related to the plane long thin slit geometry, confirms our previous findings which were established using the straight circular uniform tube geometry. Eight fluid types are used in the present investigation: Newtonian, power law, Ree-Eyring, Carreau, Cross, Casson, Bingham and Herschel-Bulkley. This effort, added to the previous investigations, should be

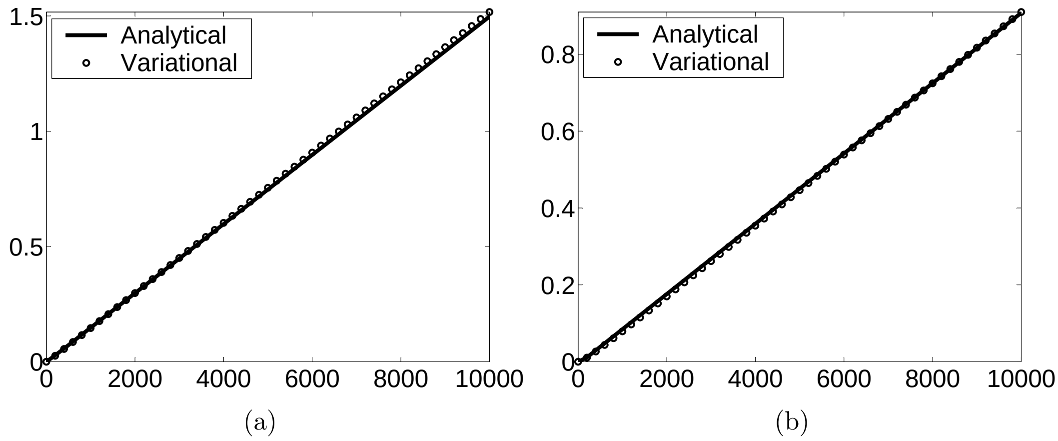


Figure 6: Comparing the WRMS analytical solutions, as given by Equation 113, to the variational solutions for the flow of Bingham fluids in long thin slits with (a) $C = 0.02$ Pa.s, $\tau_o = 0.25$ Pa, $B = 0.015$ m, $W = 1.0$ m and $L = 0.75$ m; and (b) $C = 0.034$ Pa.s, $\tau_o = 0.75$ Pa, $B = 0.018$ m, $W = 1.0$ m and $L = 1.25$ m. The axes are as in Figure 2, while the differences are about 1.12% and 1.96% respectively.

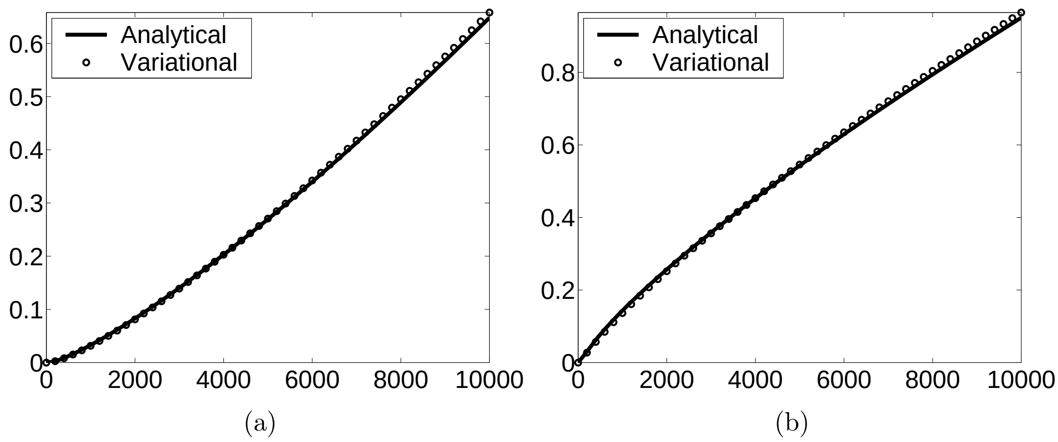


Figure 7: Comparing the WRMS analytical solutions, as given by Equation 117, to the variational solutions for the flow of Herschel-Bulkley fluids in long thin slits with (a) $n = 0.8$, $C = 0.05$ Pa.s^n, $\tau_o = 0.5$ Pa, $B = 0.01$ m, $W = 1.0$ m and $L = 1.2$ m; and (b) $n = 1.25$, $C = 0.025$ Pa.s^n, $\tau_o = 1.0$ Pa, $B = 0.03$ m, $W = 1.0$ m and $L = 1.3$ m. The axes are as in Figure 2, while the differences are about 1.49% and 1.74% respectively.

sufficient to establish the variational method and the optimization principle upon which the method rests. For the Newtonian and power law fluids, full analytical solutions are obtained from the variational method, while for the other fluids mixed analytical-numerical procedures were established and used to obtain the solutions.

Although some of the derived expressions and solutions are not of interest of their own as they can be easily obtained from other non-variational methods, the theoretical aspect of our investigation should be of great interest as it reveals a tendency of the flow system to minimize the total stress which the variational method is based upon; hence giving an insight into the underlying physical principles that control the flow of fluids.

The value of our investigation is not limited to the above mentioned theoretical aspect but it has a practical aspect as well since the variational method can be used as an alternative to other methods for other geometries and other rheological fluid models where mathematical difficulties may be overcome in one formulation based on one of these methods but not the others. The variational method is also more general and hence enjoys a wider applicability than some of the other methods which are based on more special or restrictive physical or mathematical principles.

6 Nomenclature

γ	rate of shear strain (s^{-1})
δ	$\mu_0 - \mu_i$ (Pa.s)
λ	characteristic time constant (s)
μ	fluid shear viscosity (Pa.s)
μ_0	zero-shear viscosity (Pa.s)
μ_i	infinite-shear viscosity (Pa.s)
μ_o	Newtonian viscosity (Pa.s)
μ_r	low-shear viscosity in Ree-Eyring model (Pa.s)
$\sigma_\parallel$	area of slit wall tangential to the flow direction (m^2)
τ	shear stress (Pa)
$\tau_{\pm B}$	shear stress at slit walls corresponding to $z = \pm B$ (Pa)
τ_c	characteristic shear stress in Ree-Eyring model (Pa)
τ_o	yield stress in Casson, Bingham and Herschel-Bulkley models (Pa)
τ_t	total shear stress (Pa)
B	slit half thickness (m)
C	viscosity consistency coefficient in Herschel-Bulkley model (Pa.s^n)
C'	viscosity consistency coefficient in Bingham model (Pa.s)
f	$\lambda^m \gamma_B^m$
${}_2F_1$	hypergeometric function
$F_\perp$	force normal to the slit cross section (N)
g	$1 + f$
I_{Ca}	definite integral expression for Carreau model ($Pa^2.s^{-1}$)
I_{Cr}	definite integral expression for Cross model ($Pa^2.s^{-1}$)
k	viscosity consistency coefficient in power law model (Pa.s^n)
K	viscosity consistency coefficient in Casson model (Pa.s)
L	slit length (m)
m	indicial parameter in Cross model
n	flow behavior index in power law, Carreau and Herschel-Bulkley models
Δp	pressure drop across the slit length (Pa)
P_0	pressure at the slit entrance (Pa)
P_L	pressure at the slit exit (Pa)

Q	volumetric flow rate ($m^3.s^{-1}$)
v	fluid speed in the flow direction ($m.s^{-1}$)
W	slit width (m)
z	coordinate of slit smallest dimension (m)

References

[1] A.H.P. Skelland. *Non-Newtonian Flow and Heat Transfer*. John Wiley and Sons Inc., 1967. 2, 6, 8, 18, 19, 25

[2] R.B. Bird; R.C. Armstrong; O. Hassager. *Dynamics of Polymeric Liquids*, volume 1. John Wiley & Sons, second edition, 1987. 2, 8, 10, 12, 15, 18, 19

[3] T. Sochi. Using the Euler-Lagrange variational principle to obtain flow relations for generalized Newtonian fluids. *Rheologica Acta*, 53(1):15–22, 2014. 2

[4] T. Sochi. Variational approach for the flow of Ree-Eyring and Casson fluids in pipes. *Submitted*, 2014. arXiv:1412.6209. 2

[5] T. Sochi. Slip at Fluid-Solid Interface. *Polymer Reviews*, 51(4):309–340, 2011. 7

[6] P.J. Carreau; D. De Kee; R.P. Chhabra. *Rheology of Polymeric Systems*. Hanser Publishers, 1997. 8, 12, 14, 15, 18, 19

[7] K.S. Sorbie. *Polymer-Improved Oil Recovery*. Blackie and Son Ltd, 1991. 12

[8] R.I. Tanner. *Engineering Rheology*. Oxford University Press, 2nd edition, 2000. 12

[9] R.G. Owens; T.N. Phillips. *Computational Rheology*. Imperial College Press, 2002. 14

7 Appendix

Here we derive a general formula for the volumetric flow rate of generalized Newtonian fluids in rigid plane long thin uniform slits following a method similar to the one ascribed to Weissenberg, Rabinowitsch, Mooney, and Schofield [1], and hence we label the method with WRMS. We then apply it to obtain analytical relations correlating the volumetric flow rate to the pressure drop for the flow of Newtonian and seven non-Newtonian fluids through the above described slit geometry.

The differential flow rate through a differential strip along the slit width is given by

$$dQ = vWdz \tag{67}$$

where Q is the volumetric flow rate and $v \equiv v(z)$ is the fluid speed at z in the x direction according to the coordinates system demonstrated in Figure 1. Hence

$$\frac{Q}{W} = \int_{-B}^{+B} v dz \tag{68}$$

On integrating by parts we get

$$\frac{Q}{W} = [vz]_{-B}^{+B} - \int_{v_{-B}}^{v_{+B}} z dv \tag{69}$$

The first term on the right hand side is zero due to the no-slip boundary condition, and hence we have

$$\frac{Q}{W} = -\int_{v_{-B}}^{v_{+B}} z dv \tag{70}$$

Now, from Equation 14, we have

$$\tau_{\pm B} = \frac{B\Delta p}{L} \tag{71}$$

where $\tau_{\pm B}$ is the shear stress at the slit walls. Similarly we have

$$\tau_z = \frac{z\Delta p}{L} \tag{72}$$

where τ_z is the shear stress at z. Hence

$$\tau_z = \frac{z}{B}\tau_{\pm B} \qquad \Rightarrow \qquad z = \frac{B\tau_z}{\tau_{\pm B}} \qquad \Rightarrow \qquad dz = \frac{Bd\tau_z}{\tau_{\pm B}} \tag{73}$$

We also have

$$\gamma_z = -\frac{dv}{dz} \qquad \Rightarrow \qquad dv = -\gamma_z dz = -\gamma_z \frac{Bd\tau_z}{\tau_{\pm B}} \tag{74}$$

Now due to the symmetry with respect to the plane $z = 0$ we have

$$\tau_B \equiv \tau_{+B} = \tau_{-B} \tag{75}$$

On substituting from Equations 73 and 74 into Equation 70, considering the flow symmetry across the center plane $z = 0$, and changing the limits of integration we obtain

$$\frac{Q}{W} = \int_{\tau_{-B}}^{\tau_{+B}} \frac{B\tau_z}{\tau_{\pm B}} \gamma_z \frac{Bd\tau_z}{\tau_{\pm B}} = 2\left(\frac{B}{\tau_B}\right)^2 \int_0^{\tau_B} \tau_z \gamma_z d\tau_z \tag{76}$$

that is

$$\boxed{Q = 2W\left(\frac{B}{\tau_B}\right)^2 \int_0^{\tau_B} \gamma\tau d\tau} \tag{77}$$

where it is understood that $\gamma = \gamma_z \equiv \gamma(z)$ and $\tau = \tau_z \equiv \tau(z)$.

For **Newtonian** fluids with viscosity μ_o we have

$$\tau = \mu_o \gamma \qquad \Rightarrow \qquad \gamma = \frac{\tau}{\mu_o} \tag{78}$$

Hence

$$Q = 2W\left(\frac{B}{\tau_B}\right)^2 \int_0^{\tau_B} \gamma\tau d\tau = \frac{2W}{\mu_o}\left(\frac{B}{\tau_B}\right)^2 \int_0^{\tau_B} \tau^2 \, d\tau = \frac{2WB^2}{3\mu_o}\tau_B \tag{79}$$

that is

$$\boxed{Q = \frac{2WB^3 \Delta p}{3\mu_o L}} \tag{80}$$

For **power law** fluids we have

$$\tau = k\gamma^n \qquad \Rightarrow \qquad \gamma = \left(\frac{\tau}{k}\right)^{1/n} \tag{81}$$

Hence

$$Q = 2W\left(\frac{B}{\tau_B}\right)^2 \int_0^{\tau_B} \left(\frac{\tau}{k}\right)^{1/n} \tau d\tau = \frac{2W}{k^{1/n}}\left(\frac{B}{\tau_B}\right)^2 \int_0^{\tau_B} \tau^{1+1/n} \, d\tau \tag{82}$$

$$Q = \frac{2W}{k^{1/n}\left(2+1/n\right)}\left(\frac{B}{\tau_B}\right)^2 \tau_B^{2+1/n} = \frac{2WB^2}{k^{1/n}\left(2+1/n\right)}\tau_B^{1/n} \tag{83}$$

that is

$$\boxed{Q = \frac{2WB^2 n}{2n+1}\sqrt[n]{\frac{B\Delta p}{kL}}} \tag{84}$$

When $n = 1$, with $k \equiv \mu_o$, the formula reduces to the Newtonian, Equation 80, as it should be.

For **Ree-Eyring** fluids we have

$$\tau = \tau_c \operatorname{asinh}\left(\frac{\mu_r \gamma}{\tau_c}\right) \qquad \Rightarrow \qquad \gamma = \frac{\tau_c}{\mu_r}\sinh\left(\frac{\tau}{\tau_c}\right) \tag{85}$$

Hence

$$Q = \frac{2W\tau_c}{\mu_r}\left(\frac{B}{\tau_B}\right)^2 \int_0^{\tau_B} \tau \sinh\left(\frac{\tau}{\tau_c}\right) d\tau \tag{86}$$

$$Q = \frac{2W\tau_c^2}{\mu_r}\left(\frac{B}{\tau_B}\right)^2 \left[\tau \cosh\left(\frac{\tau}{\tau_c}\right) - \tau_c \sinh\left(\frac{\tau}{\tau_c}\right)\right]_0^{\tau_B} \tag{87}$$

that is

$$\boxed{Q = \frac{2W\tau_c^2}{\mu_r}\left(\frac{B}{\tau_B}\right)^2 \left[\tau_B \cosh\left(\frac{\tau_B}{\tau_c}\right) - \tau_c \sinh\left(\frac{\tau_B}{\tau_c}\right)\right]} \tag{88}$$

For **Carreau** fluids, the viscosity is given by

$$\mu = \frac{\tau}{\gamma} = \mu_i + \delta\left(1+\lambda^2\gamma^2\right)^{n'/2} \tag{89}$$

where $\delta = (\mu_0 - \mu_i)$ and $n' = (n-1)$. Therefore

$$\tau = \gamma\left[\mu_i + \delta\left(1+\lambda^2\gamma^2\right)^{n'/2}\right] \tag{90}$$

and hence

$$d\tau = \left[\mu_i + \delta\left(1+\lambda^2\gamma^2\right)^{n'/2} + n'\delta\lambda^2\gamma^2\left(1+\lambda^2\gamma^2\right)^{(n'-2)/2}\right] d\gamma \tag{91}$$

Now, from the WRMS method we have

$$\frac{Q\,\tau_B^2}{2WB^2} = \int_0^{\tau_B} \gamma\tau d\tau \tag{92}$$

If we label the integral on the right hand side of Equation 92 with I_{Ca} and substitute for τ from Equation 90, substitute for $d\tau$ from Equation 91, and change the integration limits we obtain

$$I_{\text{Ca}} = \int_0^{\gamma_B} \gamma^2 \left[\mu_i + \delta\left(1+\lambda^2\gamma^2\right)^{n'/2}\right] \left[\mu_i + \delta\left(1+\lambda^2\gamma^2\right)^{n'/2} + n'\delta\lambda^2\gamma^2\left(1+\lambda^2\gamma^2\right)^{(n'-2)/2}\right] d\gamma \tag{93}$$

On solving this integral equation analytically and evaluating it at its two limits we obtain

$$\begin{aligned} I_{\text{Ca}} &= \frac{n'\delta^2\gamma_B\left[{}_2F_1\left(\frac{1}{2},1-n';\frac{3}{2};-\lambda^2\gamma_B^2\right) - {}_2F_1\left(\frac{1}{2},-n';\frac{3}{2};-\lambda^2\gamma_B^2\right)\right]}{\lambda^2} \\ &+ \frac{\left(1+n'\right)\delta^2\gamma_B^3\,{}_2F_1\left(\frac{3}{2},-n';\frac{5}{2};-\lambda^2\gamma_B^2\right)}{3} \\ &+ \frac{n'\delta\mu_i\gamma_B\left[{}_2F_1\left(\frac{1}{2},1-\frac{n'}{2};\frac{3}{2};-\lambda^2\gamma_B^2\right) - {}_2F_1\left(\frac{1}{2},-\frac{n'}{2};\frac{3}{2};-\lambda^2\gamma_B^2\right)\right]}{\lambda^2} \\ &+ \frac{\left(2+n'\right)\delta\mu_i\gamma_B^3\,{}_2F_1\left(\frac{3}{2},-\frac{n'}{2};\frac{5}{2};-\lambda^2\gamma_B^2\right) + \mu_i^2\gamma_B^3}{3} \end{aligned} \tag{94}$$

where ${}_2F_1$ is the hypergeometric function of the given arguments with its real part being used in this evaluation. Now, from applying the rheological equation at the slit wall we have

$$\left[\mu_i + \delta\left(1+\lambda^2\gamma_B^2\right)^{n'/2}\right]\gamma_B = \frac{B\,\Delta p}{L} \tag{95}$$

From the last equation, γ_B can be obtained numerically by a simple numerical solver, such as bisection, and hence I_{Ca} is computed. The volumetric flow rate is then obtained from

$$\boxed{Q = \frac{2WB^2 I_{\text{Ca}}}{\tau_B^2}} \tag{96}$$

For **Cross** fluids, the viscosity is given by

$$\mu = \frac{\tau}{\gamma} = \mu_i + \frac{\delta}{1+\lambda^m\gamma^m} \tag{97}$$

where $\delta = (\mu_0 - \mu_i)$. Therefore

$$\tau = \gamma \left(\mu_i + \frac{\delta}{1 + \lambda^m \gamma^m} \right) \tag{98}$$

and hence

$$d\tau = \left(\mu_i + \frac{\delta}{1 + \lambda^m \gamma^m} - \frac{m\delta\lambda^m\gamma^m}{(1 + \lambda^m \gamma^m)^2} \right) d\gamma \tag{99}$$

If we follow a similar procedure to that of Carreau by applying the WRMS method and labeling the right hand side integral with I_{Cr}, substituting for τ and $d\tau$ from Equations 98 and 99 respectively, and changing the integration limits of I_{Cr} we get

$$I_{\text{Cr}} = \int_0^{\gamma_B} \gamma^2 \left(\mu_i + \frac{\delta}{1 + \lambda^m \gamma^m} \right) \left(\mu_i + \frac{\delta}{1 + \lambda^m \gamma^m} - \frac{m\delta\lambda^m\gamma^m}{(1 + \lambda^m \gamma^m)^2} \right) d\gamma \tag{100}$$

On solving this integral equation analytically and evaluating it at its two limits we obtain

$$I_{\text{Cr}} = \frac{\left[3\delta^2 (m - g) - \{\delta^2 (m - 3) + 2m\delta\mu_i\} g^2 \, {}_2F_1 \left(1, \frac{3}{m}; 1 + \frac{3}{m}; -f\right) + 6m\delta\mu_i g + 2m\mu_i^2 g^2\right] \gamma_B^3}{6mg^2} \tag{101}$$

where

$$f = \lambda^m \gamma_B^m \qquad , \qquad g = 1 + f \tag{102}$$

and ${}_2F_1$ is the hypergeometric function of the given argument with its real part being used in this evaluation. As before, from applying the rheological equation at the wall we have

$$\left(\mu_i + \frac{\delta}{1 + \lambda^m \gamma_B^m} \right) \gamma_B = \frac{B \, \Delta p}{L} \tag{103}$$

From this equation, γ_B can be obtained numerically and hence I_{Cr} is computed. Finally, the volumetric flow rate is obtained from

$$\boxed{Q = \frac{2WB^2 I_{\text{Cr}}}{\tau_B^2}} \tag{104}$$

For **Casson** fluids we have

$$\tau^{1/2} = (K\gamma)^{1/2} + \tau_o^{1/2} \qquad (\tau \geq \tau_o) \tag{105}$$

Hence

$$\gamma = \frac{\left(\tau^{1/2} - \tau_o^{1/2}\right)^2}{K} \tag{106}$$

On applying the WRMS equation we get

$$Q = \frac{2W}{K}\left(\frac{B}{\tau_B}\right)^2 \int_{\tau_o}^{\tau_B} \tau \left(\tau^{1/2} - \tau_o^{1/2}\right)^2 d\tau \tag{107}$$

$$Q = \frac{2W}{K}\left(\frac{B}{\tau_B}\right)^2 \int_{\tau_o}^{\tau_B} \left(\tau^2 - 2\sqrt{\tau_o}\tau^{3/2} + \tau_o\tau\right) d\tau \tag{108}$$

$$Q = \frac{2W}{K}\left(\frac{B}{\tau_B}\right)^2 \left[\frac{\tau^3}{3} - \frac{4\sqrt{\tau_o}\tau^{5/2}}{5} + \frac{\tau_o\tau^2}{2}\right]_{\tau_o}^{\tau_B} \tag{109}$$

that is

$$\boxed{Q = \frac{2W}{K}\left(\frac{B}{\tau_B}\right)^2 \left[\frac{\tau_B^3}{3} - \frac{4\sqrt{\tau_o}\tau_B^{5/2}}{5} + \frac{\tau_o\tau_B^2}{2} - \frac{\tau_o^3}{30}\right]} \tag{110}$$

For **Bingham** fluids we have

$$\tau = \tau_o + C'\gamma \qquad \Rightarrow \qquad \gamma = \frac{\tau - \tau_o}{C'} \qquad (\tau \geq \tau_o) \tag{111}$$

Hence

$$Q = \frac{2W}{C'}\left(\frac{B}{\tau_B}\right)^2 \int_{\tau_o}^{\tau_B} \tau\left(\tau - \tau_o\right) d\tau = \frac{2W}{C'}\left(\frac{B}{\tau_B}\right)^2 \left[\frac{\tau^3}{3} - \frac{\tau_o\tau^2}{2}\right]_{\tau_o}^{\tau_B} \tag{112}$$

that is

$$\boxed{Q = \frac{2W}{C'}\left(\frac{B}{\tau_B}\right)^2 \left[\frac{\tau_B^3}{3} - \frac{\tau_o\tau_B^2}{2} + \frac{\tau_o^3}{6}\right]} \tag{113}$$

When $\tau_o = 0$, with $C' \equiv \mu_o$, the formula reduces to the Newtonian, Equation 80, as it should be.

For **Herschel-Bulkley** fluids we have

$$\tau = \tau_o + C\gamma^n \qquad \Rightarrow \qquad \gamma = \frac{1}{C^{1/n}} (\tau - \tau_o)^{1/n} \qquad (\tau \geq \tau_o) \tag{114}$$

Hence

$$Q = \frac{2W}{C^{1/n}} \left(\frac{B}{\tau_B}\right)^2 \int_{\tau_o}^{\tau_B} \tau (\tau - \tau_o)^{1/n} \, d\tau \tag{115}$$

$$Q = \frac{2W}{C^{1/n}} \left(\frac{B}{\tau_B}\right)^2 \left[\frac{n (n\tau_o + n\tau + \tau) (\tau - \tau_o)^{1+1/n}}{(2n^2 + 3n + 1)}\right]_{\tau_o}^{\tau_B} \tag{116}$$

that is

$$\boxed{Q = \frac{2W}{C^{1/n}} \left(\frac{B}{\tau_B}\right)^2 \left[\frac{n (n\tau_o + n\tau_B + \tau_B) (\tau_B - \tau_o)^{1+1/n}}{(2n^2 + 3n + 1)}\right]} \tag{117}$$

When $n = 1$, with $C \equiv C'$, the formula reduces to the Bingham, Equation 113; when $\tau_o = 0$, with $C \equiv k$, the formula reduces to the power law, Equation 84; and when $n = 1$ and $\tau_o = 0$, with $C \equiv \mu_o$, the formula reduces to the Newtonian, Equation 80, as it should be.