

Yield and Solidification of Yield-Stress Materials in Rigid Networks and Porous Structures

Taha Sochi

University College London, Department of Physics & Astronomy, Gower Street, London, WC1E 6BT

Email: t.sochi@ucl.ac.uk.

Abstract

In this paper, we address the issue of threshold yield pressure of yield-stress materials in rigid networks of interconnected conduits and porous structures subject to a pressure gradient. We compare the results as obtained dynamically from solving the pressure field to those obtained statically from tracing the path of the minimum sum of threshold yield pressures of the individual conduits by using the threshold path algorithms. We refute criticisms directed recently to our previous findings that the pressure field solution generally produces a higher threshold yield pressure than the one obtained by the threshold path algorithms. Issues related to the solidification of yield stress materials in their transition from fluid phase to solid state have also been investigated and assessed as part of the investigation of the yield point.

Keywords: Keywords: fluid mechanics; yield-stress; threshold yield pressure; threshold solidification pressure; network of conduits; porous media; threshold path algorithms.

1 Introduction

Yield-stress materials are commonplace in nature and industry. They include very common biological fluids like blood [1] as well as many polymeric solutions used in reservoir engineering and pharmaceutical manufacturing. These materials are characterized by behaving like solids below a certain threshold stress and like fluids above. There are many controversies about the nature of these materials, their rheological definition, and even their bare existence. They are problematic both experimentally, as they behave strangely and sometimes unpredictably, and theoretically as they are difficult to model and simulate computationally. There are several rheological models that have been proposed for modeling these materials;

some of the most common ones are Bingham, Herschel-Bulkley, and Casson. However, almost all the available rheological models that characterize the yield-stress behavior are empirical in essence and phenomenological in nature [2-6].

The above-mentioned problems that associate the bulk rheology of yield-stress materials are aggravated by more complications and difficulties when their rheology in porous structures and networks of interconnected conduits is investigated experimentally or theoretically. Several fluid-structure interaction factors emerge in such situations to play intricate defining roles in the overall conduct of such systems. For instance, the effect of tortuosity and shape irregularities of the conduits inside such structures makes the local yield point highly dependent on several geometric and topological factors that are difficult to predict and model [7, 8].

In the mobilization of yield-stress materials through networks of interconnected conduits and porous structures, there is an important issue about how to predict the threshold yield pressure of such materials saturating such media. In this regard, there are two main approaches to predict the yield point: (a) determining the threshold yield pressure dynamically by finding the pressure field which is normally obtained through solving the balance equations of the flow system that are based on the conservation principles and constitutive fluid models, and (b) determining the threshold yield pressure statically through using the threshold path algorithms such as the invasion percolation with memory [9] and the path of minimum pressure [7, 10] which trace the route that minimizes the sum of the threshold yield pressures of the route conduits inside these structures. These algorithms are based on the inert geometry of the individual conduits and the rheology of the yield-stress materials without the involvement of dynamic factors.

Sochi [7] (henceforth called S10) has investigated this problem and concluded that the threshold yield pressure obtained from solving the flow system is generally higher than the one obtained from the threshold path algorithms. This was justified by several factors; the main ones are (a) the rejection of the underlying assumption of the threshold path algorithms that the threshold yield pressure of serially connected conduits is equal to the sum of their individual threshold yield pressures, (b) dynamic factors related to obtaining a stable and consistent pressure field configuration, (c) the effect of the tortuosity on the pressure field and its direct impact on the nodal pressure of the intermediate nodes and hence the yield point of the threshold path, and (d) the communication of these intermediate nodes with the global pressure field through conduits connected to these nodes but are not part of the threshold path.

Recently, Balhoff *et al* [11] (henceforth called BRKMP) conducted a study in which they investigated this issue, among other issues, in detail and challenged the previous findings of S10. They argued that the threshold yield pressure obtained from solving the balance equations must be the same as the one obtained from the threshold path algorithms. They supported their theoretical reasoning by flow simulations in which they used a robust solving scheme based on the Newton-Raphson method in conjunction with the mass conservation and characteristic flow models. They even produced a mathematical proof using a graph theory framework to back their findings.

In this context, we should distinguish between two transition points for yield stress materials between the solid state and the fluid phase. The first one, which we call the yield point, is the transition from the solid-like state to the fluid state; and the second one, which we call the solidification or blockage point, is the transition from the fluid state to the solid state. These two points are in general different due to the effect of the initial flow conditions and hysteresis and hence the experimental and computational searching techniques for these two points should be different as well. However, the two problems are closely linked although there seems to be little interest in the solidification problem due, apparently, to a common belief that the two points are the same. We will discuss the solidification point as part of our investigation to the yield point but we will not go deep into this investigation due to the specific objectives of the current study.

In our view, the yield point should be identified by a gradual and continuous increase in the pressure drop, whether across the bulk material or single conduit or interconnected structure of multiple conduits, on starting from a confirmed solid state point such as zero pressure drop, while the solidification point should be identified by a gradual and continuous decrease in the pressure drop on starting from a well established fluid state point. There is also the possibility of a sudden and non-continuous change in the pressure drop on a yield-stress material in its solid or fluid state which may or may not result in a transition in the state of material. The latter possibility is relevant to identifying the yield point if the material was initially a solid with a sudden pressure increase and to identifying the solidification point if the material was initially a fluid with a sudden pressure decrease. Other possibilities can also be considered but they are of little relevance to the current investigation and hence they will be ignored.

Which method, gradual or sudden, should be used to determine the yield and solidification points is a matter of convention as long as the conditions are stated

unambiguously. However, it is very possible that the yield and solidification points obtained from a sudden change in the pressure drop are not the same as the ones obtained from a gradual change. One potential reason for this is transitional instabilities although other reasons are also possible. In this case, more than one point for yield and solidification, which depend on the pressure application method, should be accepted if it is supported by experimental evidence. More discussion about these issues can be found in [7] and section 4.

In the present paper, we discuss the issue of yield point in detail and challenge the findings of BRKMP. Our main objection to the BRKMP criticism and findings is that what they claim to be the yield point is in fact more appropriate to be the solidification point of the yield-stress materials on a gradual decrease of the pressure from above the sum of thresholds where BRKMP assumed the material has already yielded and hence it is a fluid, to the point of blockage where the system converges to the solid state because it has reached the sum of the threshold yield pressures on the threshold path. We also present two mathematical proofs for our proposal that the threshold yield pressure of an ensemble of serially-connected conduits is in general greater than the sum of their individual threshold yield pressures. One of these proofs is based on the assumption that yield-stress materials prior to reaching their yield point are fluids with very high viscosity, and the other proof is based on the assumption that yield-stress materials are solids prior to yield. A mathematical argument has also been presented to show that finding a mass-conserving consistent pressure field in a fluid-filled ensemble of interconnected conduits is always possible for any type of fluid above its minimum mobilization pressure in the given ensemble, where the minimum mobilization pressure is obtained from the sum of the minimum mobilization pressures of the individual conduits in the ensemble. This argument is key to identifying the circularity in the BRKMP argument.

The non-Newtonian fluids may be classified into two main categories, history-dependent which include viscoelastic and thixotropic/rheopectic, and history independent which are the purely viscous non-Newtonian fluids that also include the Newtonian as a special case. The second category may be equated with the generalized Newtonian fluids if yield-stress materials are accepted in this category. Yield-stress can associate both history-dependent and history-independent fluids. For the purpose of the present paper, these attributes are almost irrelevant as we are mainly interested in the threshold yield point. Although history-dependent and history-independent attributes have very strong impact on the flow, this is generally valid only above the very low-shear-rate regimes, i.e. following yield and mobilization.

The reason is that prior to yield any potential deformation is minimal and hence any non-Newtonian effects, other than yield-stress, are negligible since the fluid is still at its low-shear Newtonian plateau which characterizes almost all non-Newtonian fluids. The low-shear Newtonian plateau can also be justified theoretically by the fact that all the non-Newtonian rheological properties are strongly dependent on the rate of deformation, whether shear or extension. We therefore do not differentiate in this paper between the history-dependent and history-independent fluids as long as they are yield-stress materials, although we will indicate the consequences of these properties on the yield-stress behavior briefly when necessary and where relevant. However, history-dependent and history-independent attributes should have more significant impact on the solidification point but in this study we investigate the solidification point marginally as part of our investigation to the yield point.

We should also remark that in the present study we are only concerned with rigid networks and porous structures; any tangible deformability, such as being elastic or viscoelastic, in these structures requires further modeling considerations and hence complicates the modeling strategies of the yield and solidification processes substantially.

2 Modeling Yield Stress in Porous Structures

In this section we outline our proposals for modeling the mobilization of yield-stress materials in networks of interconnected conduits and porous structures. We would like to insist that in the present paper we consider these aspects from the viewpoint of threshold yield point only with minimal consideration for the subsequent dynamic effects that automatically take place following mobilization which will inevitably change the dynamics of the system, and hence the modeling strategy, fundamentally. These effects must be taken into account thoroughly for a complete and reliable yield-stress flow model. Also, there are still many detailed issues that should be dealt with at the practical levels for implementing such models, such as convergence difficulties and convergence enhancement techniques, which are very serious issues for such studies; but we do not consider these issues here since they are out of the scope of the present paper. Some of these issues have already been discussed in some of our previous papers (e.g. [12, 13]).

Although we presented a limited amount of computational work in this paper, due to its nature, extensive computational work has been done in the background

as part of this investigation to test and verify various possibilities and aspects. We therefore feel obliged to provide a general clarification about the computational framework which the current study relies upon. We have already fully explained this framework in some of our previous publications and hence for the purpose of saving space and avoiding repetition we refer the reader to the following papers: [13–15] where our computational framework is fully explained. More relevant details may also be found in [12, 16] although these are mainly related to a one-dimensional finite element Newtonian flow model. We would also like to clarify that this computational framework is different to the one used in our previous studies (e.g. [7, 10, 17]) and hence there is no ground for potential criticisms based on the computational approach adopted in the previous studies. However, we have no reason to believe that the previous results are incorrect or compromised because of the previous computational framework and modeling strategies which, to the best of our knowledge, are still valid in general.

The minimum pressure drop required to initiate the mobilization of a yield-stress material in its solid state is called the threshold yield pressure. The essential issue that determines the threshold yield pressure of a yield-stress material occupying a network of interconnected conduits or a porous structure is the pressure field configuration inside such structures. Let us assume we have a solid porous structure filled with a yield-stress material and we started from a zero pressure drop and kept varying the pressure drop across the structure either gradually and continuously or through sudden changes. The crucial question then is what is the pressure field configuration inside the porous structure as a function of the applied pressure drop across the structure. If we can *a priori* determine the pressure field spatially inside the structure as a function of the applied pressure drop for any given pressure value then we can easily determine the threshold yield point by simply identifying the minimum pressure drop across the structure that creates a path on which the pressure drop across each one of its conduits exceeds the threshold yield pressure for that conduit. For a perfectly circular cylindrical rigid tube with a constant cross sectional area along its axial length, the threshold yield pressure condition is given by [7, 10]

$$\tau_w = \tau_o \quad \implies \quad \Delta P_t = \frac{2L\tau_o}{R} \quad (1)$$

where τ_w is the shear stress at the tube wall, τ_o is the yield-stress of the fluid, ΔP_t is the pressure drop across the tube at the yield threshold, and L and R are the tube length and radius respectively. It should be remarked that the condition

given by Equation 1 is a necessary but not sufficient condition for yield, as will be discussed later.

It is noteworthy that our definition for the threshold yield pressure of networks and porous structures and how it is determined is based on some implicit assumptions about how the pressure field configuration inside such structures changes in response to the applied pressure drop across these structures. Although these assumptions are not self-evident, they seem to be generally accepted and hence we see no necessity for discussing them in the present paper. However, there is one important assumption that requires some clarification that is the assumption of continuity of yield above the threshold yield pressure which, although it seems intuitive, still requires a physical or mathematical justification. While we will not discuss this issue here, we think the argument presented in Appendix A, whose essence is the possibility of finding a mass-conserving consistent pressure field for any type of fluid above its minimum mobilization pressure, can be easily adjusted to provide such a proof. Due to the fact that the flow rate is a strictly increasing and continuous function of pressure drop, the yield condition of an ensemble, as soon as it is satisfied, will remain so on increasing the pressure drop across the ensemble.

However, no one can completely rule out the possibility of a blockage subsequent to yield at a pressure drop above the threshold yield pressure due to continuous or discrete transformations in the system dynamics, especially with the involvement of complex non-Newtonian rheological factors other than yield-stress, that change the pressure field configuration in a way that affects the yield condition. This possibility may not be realistic in a simple one-dimensional network of serially-connected conduits but it should be realistic for more complex two-dimensional and three-dimensional networks and porous structures. Such a possibility should be seriously considered for a complete yield-stress model which is out of the scope of the current paper as it is mainly focused on the yield point succeeding a total blockage. Anyway, to avoid any possible disputes we could assume that the previous statements in the former paragraphs related, explicitly or implicitly, to the dependency of pressure field inside a structure on the pressure drop across it and subsequent developments are just definitions or assumptions and hence they are part of our modeling strategy and not considered as physical facts.

There are two main approaches for modeling yield-stress materials prior to reaching their yield point whether in the bulk flow, single conduit flow or flow through networks of interconnected conduits and porous structures; these two ap-

proaches are explained in the following subsections.

2.1 Highly-Viscous Fluid Approach

According to this approach, the yield-stress materials prior to reaching their yield point are fluids with very high viscosity. Therefore, they are distinguished by having a viscosity function whose dependency on the shear stress is discontinuous at the yield point. Our modeling choice for the highly-viscous fluid approach is to identify the pressure field prior to yield from solving the balance equations assuming the fluid is Newtonian with a constant viscosity. As indicated early, the Newtonian assumption prior to yield is very realistic one even for the highly non-Newtonian fluids, because at these stages of negligible deformation, all non-Newtonian rheological effects, except yield-stress, are absent as the fluid is still on its low-shear Newtonian plateau.

Therefore, to find the threshold yield pressure we step up on the pressure ladder by starting from a confirmed non-yield point and solve the pressure field at each pressure step using the Poiseuille flow model. The pressure field is then tested at each step to identify a possible inlet-to-outlet spanning path whose all conduits have passed their threshold yield point, as given by Equation 1 for cylindrical tubes. The minimum pressure drop that satisfies this condition is taken to be the threshold yield pressure that defines the yield point. At and above this pressure, the flow model for the mobilized parts will be subject to the adopted yield-stress rheological fluid model such as Herschel-Bulkley. Although the fluid prior to yield, according to this approach, is theoretically assumed to be of very high-viscosity, computationally the value of the Newtonian viscosity is irrelevant to the pressure field solution since the viscosity in the Poiseuille model is a conductance scale factor for the flow rate with no effect on the configuration of the pressure field and hence any value for the viscosity will produce the same pressure field.

Now we test the consequences of this modeling approach and compare the yield point obtained dynamically from solving the pressure field to the one obtained statically from the threshold path algorithms. In this regard, it is easy to verify that the dynamic threshold yield pressure of the structure according to this modeling strategy generally exceeds the sum of the threshold yield pressures of the individual tubes as given by the threshold path algorithms. In Appendix B we presented a mathematical proof for this assertion for the case of a one-dimensional network consisting of an ensemble of serially-connected tubes. We also demonstrate this by a simple example of such a network, shown in Figure 1 with data given in

Table 1, where we can simply verify that the sum of the threshold yield pressures is 450 Pa for a yield-stress value of 5 Pa, while the threshold yield pressure for this value of yield-stress as obtained dynamically from solving the pressure field is about 1664 Pa. Although a general proof for such a statement for two-dimensional and three-dimensional networks are not available currently, we feel that the same principles should apply. Anyway, the special case of one-dimensional networks is sufficient to discredit the BRKMP claim that these two thresholds are equal in general, as will be discussed later in detail. Furthermore, all our simulations using the old and the new computational frameworks with a diversity of two-dimensional and three dimensional networks produced dynamic yield points that are generally larger than the static yield points.



Figure 1: One-dimensional network of serially-connected cylindrical tubes.

Table 1: Data related to the network of serially-connected tubes shown in Figure 1 where ΔP_t stands for the threshold yield pressure of the respective tube as given by Equation 1 and the indices are related to the tubes in Figure 1 from left to right. The threshold yield pressures given in the fourth column are based on a yield-stress value of $\tau_o = 5$ Pa.

Tube Index	R	L	ΔP_t
1	0.020	0.160	80
2	0.015	0.165	110
3	0.010	0.090	90
4	0.017	0.119	70
5	0.022	0.220	100

2.2 Solid-Like Approach

According to this approach, the yield-stress materials prior to reaching their yield point are solid-like substances. A reasonable modeling strategy for this approach is to determine the pressure field from intuitive physical considerations as part of this approach. The most reasonable option for modeling the pressure field in this case is to assume a linear pressure drop across the structure and hence a constant pressure gradient. Any other model for the pressure field requires additional justification.

The propagation of pressure through solid materials may seem strange but it is physically sound since pressure can propagate through solids as in the case of sound wave transmission and reflection for instance. Although solids are normally assumed to be rigid, they are not absolutely rigid since all solids, due to their atomic structure, have a certain degree of elasticity; the yield-stress materials are not an exception as they obviously have such a property.

According to this modeling strategy and the associated assumption about the spatial definition of the pressure field, the pressure field is determined as a function of the pressure drop across the structure with no need for solving the balance equations as it can be obtained from pure geometric considerations such as the proportionality of pressure to the distance from the inlet and outlet boundaries. If a linear pressure drop is adopted to define the pressure field prior to yield then no backtracking will occur. The threshold yield pressure is found by increasing the pressure drop across the structure gradually starting from a point known to be below the threshold yield point of the structure. A test is then carried out at each pressure step to identify a possible connected route that spans the structure from the inlet to the outlet with all its conduits being above their threshold yield pressure. The minimum pressure that satisfies such a condition will be deemed as the threshold yield pressure. At and above this point, the flow in the mobilized part of the system should be determined by solving the balance equations according to the presumed yield-stress theoretical fluid, such as Bingham, that is used to model the flow following mobilization.

The state of the flow system, according to this solid-like scenario, is expected to change radically on reaching the yield point and hence a very different pressure field may replace the pre-mobilization pressure field. These two pressure fields could even be qualitatively different. Furthermore, the system at the transition point may be unstable especially if complex non-Newtonian rheological factors, such as history-dependent effects, are becoming involved in the post yield processes. The occurrence of such instabilities is entirely realistic from the physical viewpoint as such transitional instabilities are commonplace in physical systems, including fluid dynamics. Mathematical models may also be characterized by such instabilities. Anyway, as indicated earlier we are not concerned with these issues in the present paper which is limited in scope to the identification of the threshold yield pressure with a minor interest in other related issues. Any subsequent changes in the dynamics of the system will not change the yield point which took place earlier as this is part of the system history.

With regard to the consequences of this solid-like approach and the adopted modeling strategy, it is easy to verify that the threshold yield pressure of the flow system generally exceeds the sum of the threshold yield pressures of the individual tubes on the threshold path as given by the threshold path algorithms. In Appendix C we presented a mathematical proof for this assertion for the case of a one-dimensional network consisting of serially-connected tubes. We also demonstrate this by the simple example of Figure 1 and Table 1, where we can easily verify that the sum of the threshold yield pressures for a yield-stress value of 5 Pa is 450 Pa, while the threshold yield pressure for this value of yield-stress as obtained dynamically from inspecting the pressure field is about 754 Pa assuming the ensemble is straightly aligned. The required threshold yield pressure gradient will be greater for a tortuous network since the pressure gradient across the entire network will be multiplied by a sinusoidal factor to obtain the component of the pressure gradient in the tube axial direction.

3 Pressure Regimes

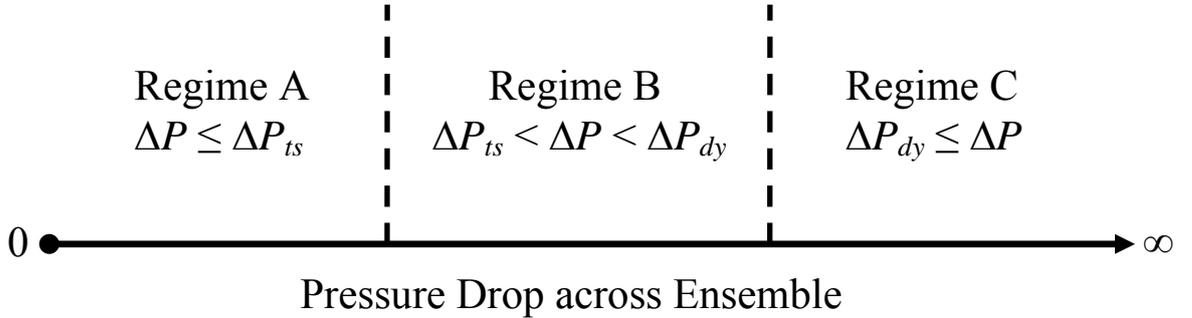


Figure 2: The three pressure regimes for an ensemble of interconnected conduits or a porous structure, where ΔP_{ts} is the pressure drop of threshold sum and ΔP_{dy} is the pressure drop of dynamic yield.

To clarify the situation for identifying the threshold yield and solidification points, we refer to Figure 2 where we identified three mutually-exclusive pressure regimes related to the magnitude of the applied pressure drop across an ensemble of interconnected conduits or a porous structure. For simplicity, we assume the ensemble is a one-dimensional straightly-aligned network like the one depicted in Figure 1 although the classification and associated arguments are valid in general for other types of networks and porous structures. These pressure regimes are

- Regime A where the pressure drop is less than or equal to the sum of the threshold yield pressures of the tubes in the ensemble.
- Regime B where the pressure drop is larger than the threshold sum but less than the threshold yield pressure as identified by the dynamic argument based on solving or inspecting the pressure field.
- Regime C where the pressure drop is greater than or equal to the ensemble threshold yield pressure according to the dynamic argument.

We all agree that in regime A the ensemble is blocked because there is no way to split the pressure drop to ensure simultaneous yield of all the tubes in the ensemble. This is correct whether we applied the pressure drop gradually and continuously from above or from below or we applied the pressure suddenly, as long as we start from a solid state point.

We also agree that in regime C the ensemble is open to the flow because whether we used the rheological model of the yield-stress, like Bingham, or the pre-yield model, like Poiseuille or solid state, the pressure will split in both possibilities such that every tube in the ensemble will reach its yield point. There is also no difference with regard to the sudden or gradual application of such a pressure drop if we ignore, in the case of a sudden application, a possible brief transitional stage during which the pressure adjusts itself to satisfy the requirement of one of the rheological models and hence the system may still be blocked.

As for regime B, there are different scenarios that generally depend on the initial conditions and the method of applying the pressure drop as outlined below

- We should agree that if we start from a pressure drop in stage C where the system is flowing and keep decreasing the pressure drop gradually and continuously then we should have a flow in stage B as well, because the initial condition for the system requires the application of the yield-stress rheological flow model and hence all is needed is the satisfaction of the mass conservation principle which is possible even in regime B according to the mathematical argument of Appendix A. This may be stated in a different way by saying that the smooth variation of the pressure field inside the structure in response to a similar variation in the pressure drop across it requires the continuity of the initial configuration of the pressure field which, qualitatively, is that of a yield-stress rheology. The assumption of a sudden blockage on entering regime B implies a sudden and non-continuous change in the pressure field

configuration which is difficult to imagine and justify physically. Now whether the system will be blocked or not on further decrease beyond the lower limit of regime B is dependent on possible hysteresis, as discussed early.

- If we start from a pressure drop in regime A where the system is blocked and keep increasing the pressure drop continuously then on exceeding the upper limit of this regime the system should be still blocked because at the very edge of regime A we agree that the system is blocked since it is subject to the pre-yield model and according to this model the pressure field is very different to that required for a simultaneous yield of the tubes. It is difficult to imagine that an infinitesimal increase in the pressure drop on passing the upper limit of regime A will change the pressure field configuration suddenly and radically to the configuration required for a simultaneous yield as a consequence of the supposed validity of the adopted yield-stress rheological model on such a trivial transition. This is demonstrated in Figure 3 where we compared the pressure field of a Poiseuille flow with a pressure drop of 450 Pa across the ensemble, which is equal to the threshold sum, with the pressure field of a Bingham flow with a pressure drop of 450.1 Pa across the ensemble, which is just above the threshold sum. As can be seen, these two pressure fields are very different. Such a strong dissimilarity will also be obtained for a solid-like pre-yield approach. Therefore we think the most logical scenario is that the pressure field will keep adjusting itself continuously and smoothly according to the rules of the pre-yield model, whether fluid or solid, on a continuous increase of the pressure drop across the ensemble all the way through regime B and hence the system will yield only when it enters regime C.
- Now if we apply a sudden pressure drop whose value belongs to regime B then the outcome in our view is dependent on two factors: transitional instabilities and the previous pressure regime to which the system was subject prior to the sudden change. Briefly, if the previous pressure is in regime A then the most likely outcome is blockage, but transitional instabilities may lead to a pressure distribution that opens all the tubes simultaneously and hence the system will continue flowing because as soon as the system starts flowing, whether it is in regime B or C it should be subject to the yield-stress rheological fluid model which can sustain a stable mass-conserving flow according to the mathematical argument of Appendix A. On the other hand if the previous pressure drop belongs to regime C then the most likely scenario is mobilization

although instabilities may lead to blockage. Other static and dynamic factors, like hysteresis, should also play a role in these scenarios. Other rheological aspects, especially history-dependent attributes, could also be important in determining the transitional stage and the final outcome.

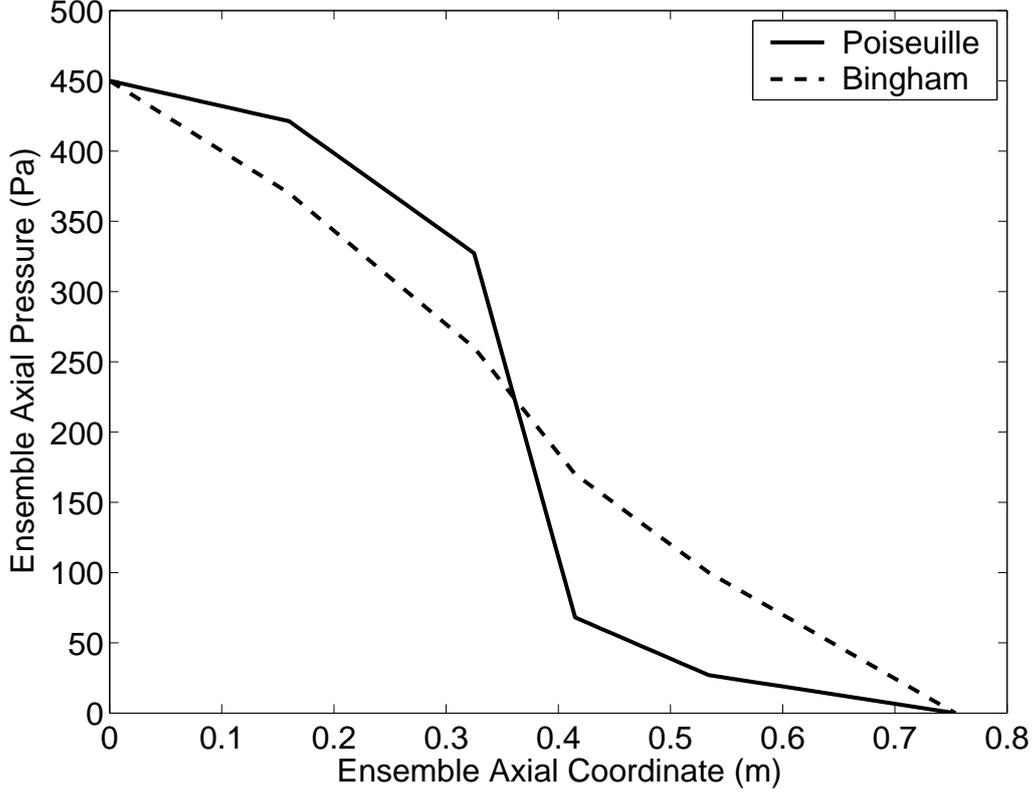


Figure 3: Axial pressure as a function of axial coordinate of the ensemble of Figure 1 for a Poiseuille flow with a pressure drop of 450 Pa and a Bingham flow with a pressure drop of 450.1 Pa. The yield-stress of the Bingham fluid is assumed to be 5 Pa. The Bingham pressure field is obtained by gradual decrease of the pressure drop starting from a high value belonging to regime C where the system is known to have already reached its dynamic yield point. The flow rate for Poiseuille is $Q_P \simeq 2.26 \times 10^{-4} \text{ m}^3 \cdot \text{s}^{-1}$ assuming $\mu = 0.05 \text{ Pa} \cdot \text{s}$, and for Bingham is $Q_B \simeq 2.72 \times 10^{-11} \text{ m}^3 \cdot \text{s}^{-1}$ assuming $C = 0.05 \text{ Pa} \cdot \text{s}$.

4 Discussion

Now, the key question is why a yield-stress fluid model at a pressure just above the threshold sum produces a mass-conserving consistent pressure field configuration with an open path while a solid-like or Newtonian fluid models do not produce

such an open path at such a pressure. In Appendix A we presented a mathematical argument to demonstrate why a mass-conserving consistent pressure field can always be found for a pressure drop above the threshold sum of a yield-stress fluid-structure system.

In reality finding a mass-conserving consistent pressure field for a yield-stress fluid above the threshold sum of an ensemble is not different to finding such a field for a Poiseuille flow except that while for the Poiseuille flow the threshold pressure for mobilization is zero, for the yield-stress fluids the mobilization threshold is the sum of the threshold yield pressures since this sum is the absolute minimum for any possible mobilization assuming that it is split correctly to overcome the yield point for each tube in the ensemble. As we always can find a mass-conserving consistent pressure field for Poiseuille flow above the zero pressure reference level, which seems self-evident although we believe it requires a mathematical proof as outlined in Appendix A, we can find such a pressure field for the yield-stress fluids above the sum of thresholds for the same reason. The latter gives the illusion that this is because the yield point is at the threshold sum whereas the reality is that justifying the search for a consistent pressure field above the sum and below the dynamic yield point is only justified if we assume that the system in regime B is in a yield state which can be justified in the case of solidification process but not in the case of yield process.

The mathematical argument in Appendix A reveals our main objection to the BRKMP reasoning because what BRKMP do in their model is only to find a mass-conserving consistent pressure field above the threshold sum point which can be trivially found. Finding such a pressure field gives the impression that the actual yield point is at the threshold sum whereas in reality searching for such a mass-conserving pressure field in the neighborhood of the threshold sum can only be justified if we assume that the system is already in a yield state above the threshold sum point.

This reveals that all the derived results of BRKMP are in fact based on the very statement that they are supposed to prove and hence they in fact use a circular argument. The matter of fact is that they start from the assumption that the threshold yield pressure of the ensemble is determined by the sum of the threshold yield pressures of the individual tubes; then all they need to take care of is mass conservation above this limit.

Therefore, the BRKMP method, which is supposed to be for identifying the yield point, may be more appropriate to use for identifying the blockage or solid-

ification point because when they start from a pressure point above the threshold sum point assuming the system is already in a fluid state to which a rheological yield-stress fluid model, like Bingham, applies and keep lowering the pressure gradually and continuously as can be concluded from their algorithm (refer e.g. to their equation $\Delta P^* = \Delta P_m(1 + \varepsilon)$), they will inevitably converge to the blockage point at the threshold sum point.

However, in reality, due to hysteresis, the actual solidification point may be below the threshold sum point as indicated previously. Such a hysteresis lagging is commonplace in polymeric and other yield-stress systems, and hence it can delay the solidification to a pressure point below the value of threshold sum when approaching the point from above. So, even if we assume that the yield point from below is the one obtained by the threshold path algorithms it is not necessarily that the solidification point from above is the same as the yield point. This of course implies that the system during this pressure decreasing process will be subject to a different yield-stress rheological model from the one that applies during the yield process or at least to the same rheological model but with different parametric values. Briefly, the points of transition between the solid state and fluid phase for yield-stress materials do not necessarily agree even in the bulk rheology regardless of the extra reasons for this in the *in situ* rheology, which we are concerned with in this paper, and whether the actual yield point for a network or a porous structure is at the threshold sum or not. Experimental evidence has already shown that the two points usually do not agree. The reason for hysteresis in general is the disturbance of the micro-molecular structure during the deformation process in the fluid phase.

In fact even detecting the solidification point from above, ignoring the hysteresis issue, is only legitimate if we start from a confirmed fluid state point as obtained from the dynamic yield condition, i.e. from a point in regime C, and keep lowering the pressure all the way through regime B until we reach the threshold sum point, because as explained early there is no ground in general for assuming a yielded fluid state if we start in regime A or B. In our simulation experiments we experienced exceptional convergence difficulties when we tried to start from pressure points in regime B. We observed that the convergence was easier if we start from above the dynamic yield point in regime C, which is unusual because convergence difficulties usually increase with higher pressure boundary conditions. This is in complete agreement with the BRKMP observations about convergence difficulties in these regimes (refer for example to their statement “The traditional Newton’s

method converges easily if the imposed pressure gradient is significantly higher than the threshold pressure gradient...” although BRKMP may offer a different explanation. The reason for these difficulties in our view is the difficulty of finding a consistent pressure field of a yielded system on starting from the given initial conditions based on the state in regime B.

In brief, we can challenge the underlying assumptions of BRKMP model that lead to such conclusions. The key question that BRKMP should consider is why a pressure drop that is infinitesimally above the sum of the threshold yield pressures applied across a serially-connected ensemble should necessarily split, according to their yield scenario, such that the size of the pressure drop across each tube is infinitesimally above its threshold yield pressure, while at an infinitesimally lower pressure drop across the ensemble (i.e. when the pressure drop was equal to the threshold sum) the pressure field was very different as it was subject to a different rheological model. If we accept this non-evident and controversial scenario, which BRKMP explicitly or implicitly present as a fact and not just as an assumption or a possibility, then all is needed is to satisfy the mass conservation principle which is a trivial thing to do as we demonstrated in Appendix A. Therefore, it is not surprising when BRKMP find that the threshold yield pressure as found from solving the pressure field is identical to the value obtained from the threshold path algorithms because during all the stages of stepping down on the pressure ladder they are using a yield-stress fluid model since they assume, at least implicitly, that the yield-stress fluid has already yielded in regime B and hence a physical flow that satisfies the mass-balance equation will be found inevitably. As soon as they approach their ‘yield’ point from above based on a fluid state assumption, they should converge to a zero flow at the threshold sum and hence the two pressure values will necessarily agree.

5 Criticism

We now address the main criticisms raised by BRKMP to our yield-stress modeling approach and the conclusions that have been reached in S10. We also present some of our criticisms to the BRKMP model as described in their paper.

One of the major criticisms directed to our model is that delaying the yield point beyond reaching the threshold sum of the threshold path algorithms necessitates that some conduits have already reached their yield points (refer to the BRKMP statement “This condition requires that.....and not yield flow.”) and hence cannot

be blocked as implied by our model which requires the yield point to occur at a higher pressure belonging to regime C in a blatant violation to the conduit yield condition as given by Equation 1.

First, according to our model there is no ground for the application of the yield condition of Equation 1 prior to reaching the yield point of the ensemble, because we are not looking for the yield condition of a single tube but we are searching for the yield point of an ensemble or porous structure. Before reaching the yield condition of the ensemble the material is not considered a yield-stress fluid that is subject to the condition of Equation 1, but it is either a highly-viscous Newtonian fluid or a solid state material. Therefore before reaching the dynamic yield point in regime C the flow system is assumed to be Newtonian or solid state and hence no yield-stress does exist. Yield-stress model will take effect only on opening a spanning path that sustains a tangible quantity of flow by reaching the ensemble dynamic yield point.

Second, the possibility that some conduits reach their yield point before the system reaches its yield point occurs not only in the controversial B regime but even in regime A where we all agree that the system in this regime cannot yield; whether we adopted a highly-viscous fluid approach or a solid-like approach prior to yield. Even BRKMP who use, according to our understanding of their model as indicated for example by the second part of their equation (1), a Poiseuille model prior to reaching the threshold sum should accept that some conduits will reach their yield condition as given by Equation 1 in regime A. For example, the sum of threshold yield pressures for the ensemble of Figure 1 and Table 1 is 450 Pa for $\tau_o = 5$ Pa. However, if we apply a pressure drop across the ensemble well below this sum then we will find that some of the tubes have already reached their yield point assuming a Poiseuille flow of a highly-viscous fluid. For instance if we apply a pressure drop of 200 Pa then the pressure drop across the third tube with a consistent pressure field of a Poiseuille flow will be about 115 Pa which is well above its threshold yield pressure of 100 Pa assuming a yield-stress of 5 Pa. The solid-like approach also implies the occurrence of such situations. So even according to the BRKMP modeling strategy such a ‘violation’ to the yield condition in some conduits is inevitable. The setting of the flow in these conduits to zero, as BRKMP seem to suggest, is arbitrary and hence requires justification; moreover this setting is a clear violation of the adopted Poiseuille flow in these conduits in this regime which has obvious consequences on the mass conservation balance. More discussion about this issue will be presented later.

Third, and possibly the most important factor, is that pressure drop is a necessary but not a sufficient condition for fluid flow. Two obvious examples are yield-stress fluids where no flow occurs even with the presence of a pressure drop, and the second is a tube immersed vertically in a body of water. The presumed solidity or fluidity with high viscosity in the first example cannot change the argument which is based on the expectation of a tangible flow of a fluid phase prior to yield as if it was a normal Poiseuille fluid. For the second example the flow upwards will not happen even with the presence of a pressure drop in the upward direction because it is balanced by another force which is the force of gravity in this case. A conduit confined within a non mobilized structure will not flow even if it reached its threshold yield pressure due to a similar balancing force, that is the yield stress force of the surrounding structure which is essentially the same force that prevents flow in a stand-alone tube filled with a yield-stress fluid and subjected to a pressure drop below its yield point.

Another criticism to our model is the violation of local mass balance (refer to “It is unknown why Sochi (2010) obtains...found from search algorithms.” in BRKMP). According to our model, the system before reaching the dynamic yield point as obtained by solving the pressure field, assuming a highly-viscous fluid approach, is subject to the Poiseuille model and hence the mass is conserved locally and globally. The illusion of a violation to the local mass balance arises from imagining that the isolated throats will be automatically subject to the rheological model of the yielded yield-stress fluid as soon as they reach their threshold yield condition of Equation 1. Mass balance violation can only occur if it is not accommodated in the flow model correctly, and hence if the model dictates that the mobilization in the individual throats does not take place automatically as soon as they reach their threshold yield point, but should also associate the flow conductivity condition by being part of a yielded inlet-to-outlet spanning path, then no local or global mass balance violation will occur.

In fact the mere distinction between local and global mass balance, as if they are two separate conditions, is incorrect, because these two conditions are the same in essence due to the fact that the global mass balance is based on the local mass balance of the individual interior pores. This can be proved simply by stepping through the network from the inlet boundary to the outlet boundary to verify that the total outflow must be equal to the total inflow if mass balance is respected on each interior pore [12]. In brief, local and global mass balance should be satisfied if the process is modeled correctly using a consistent yield-stress flow model as

described early.

With regard to our criticism to the BRKMP investigation, in addition to the points that we already made, we should first express our reservation about the graph theory proof. We have a strong suspicion about the capability of the graph theory in principle to determine the outcome of a physical process in such dynamic systems. All the graph theory, and any similar mathematical apparatus, can do is to reproduce the pre-stated assumptions in a technical form with drawing some logical conclusions from the given conditions. In fact the content of the given proof of graph theory may not even be controversial as long as it is related to finding the threshold path from static considerations. The important thing that really needs a proof is the underlying assumptions and conditions which lead to these logical conclusions. The expected outcome of such processes in such dynamic systems is therefore more logical to obtain from dynamic considerations based on the physics of fluid mechanics.

We also observe that, unlike us, BRKMP do not have a model for the solid-like approach. The assumptions of solid-like and highly-viscous fluid approaches are not just mathematical ideals but they correspond to a physical reality, that is the yield-stress fluids should behave in one of these ways or the other and hence for a complete modeling approach both possibilities should be considered. In fact it is physically viable that even some yield-stress materials prior to yield could behave as solid-like while others behave as highly-viscous fluids or a single yield-stress material behaves differently under different physical conditions.

We also notice that what BRKMP describe as “Close inspection” in their statement “Upon convergence, some throats may appear open...(total flow into the network model equals flow out of the model) is found.” may not be sufficient to make such generalizations and hence if this is a possible defect in the model it should be approached in a more formal, systematic and rigorous way than a close inspection. Another point is that “some throats may appear open” just confirms what we stated already about the inevitability of this situation even according to the BRKMP modeling approach; the use of “may appear” to reduce the impact as if we are witnessing a real physical process and not just a model that we created by our own hands does not make any good. We also do not understand the supposed problem in mass conservation as if it is a matter of choice that we need to take care of personally: simply if we set our model correctly and ensured that our code does not contain serious bugs then mass conservation will be taken care of automatically by the model and the code without need to worry about it and try to fix it through

close inspection or arbitrary blocking of some throats or any other means. The Newton-Raphson method as described by BRKMP is sufficiently robust to conserve mass. Yes what should be worrying is a possible inconsistency in the model itself where it is theoretically assumed that no throat can reach its yield condition unless it is part of a connected path whereas the physics of the model requires such a situation to occur, as discussed early. We also do not understand the role or the value of this arbitrary discarding of the isolated throats apart from the possibility of adjusting the model to make it look more consistent.

There are also some other issues which are relatively minor in the BRKMP assessment to S10. For example, there are some misinterpretations of S10, e.g. the meaning of the dynamic effects which are wrongly interpreted as of a viscous nature, like the meaning of this term in Chen *et al* [18], whereas we clearly stated that it is related to the pressure field, and hence some of the BRKMP arguments may not stand as they are. Also, the path of minimum pressure algorithm is not an approximate method but it is rigorous within its validity domain. The algorithm is mainly based on a linear pressure drop assumption prior to mobilization which is mostly relevant to the solid-like approach for un-yielded yield-stress materials. Backtracking in such situations will not be allowed because it does not occur for obvious physical reasons, as indicated early in this paper. We also notice that there is a mention and even discussion of convergence problems with some suggestions about how to overcome these problems and improve the rate of convergence by Sochi in his thesis [10]. Another minor remark is that although a single open path at the threshold pressure gradient is the most common possibility, multiple open paths are also possible.

There are possibly other limitations in the BRKMP yield-stress model which we suspect from reading the method description in their paper, like possible inconsistency in the use of Poiseuille and yield-stress models in the pre and post yield regimes. However reaching a definite conclusion about these issues requires further technical and coding details and more clarifications from Balhoff and coworkers, which are not available to us. There are other controversial issues in BRKMP that can be challenged but they are not related to yield-stress and hence are entirely out of the scope of the present paper.

6 Final Thoughts

Finally, by what means we can verify which model is the ‘correct’ one? Experimental evidence should have the final word about most, but not all, of the previous issues which determine the validity and applicability of any model. There are many limitations in the experimental procedures, their results, interpretations and conclusions. Although we think that experimental evidence can in many circumstances rule in or rule out some of the above mentioned models and scenarios, such as the yield point of an ensemble and if it is at the threshold sum or at the dynamic yield point or may even be at a different point, many other possibilities related to other phenomena, which are more involved and less obvious, may not be possible to assess and reach a conclusion about unequivocally. Some flow systems, like a highly complex non-Newtonian yield-stress fluid in a topologically and geometrically complex porous medium, may be too complex to reach a definite conclusion about their rheological behavior including their yield point due to the involvement of many intricate factors. The quantitative difference between the two methods for determining the yield point, for instance, may be absorbed in the overall error margin of the yield process. The difference between the static yield point as determined by the threshold path algorithms and the dynamic yield point as determined from the pressure field is obviously system dependent and hence the difference between the two methods may not be sufficiently big in some cases for an unambiguous conclusion.

Another limitation of the experimental evidence is that in some circumstances although it can endorse certain possibilities it cannot entirely rule out other possibilities. For example, there is a possibility that there are different types of yield-stress fluids where each type has a distinctive and different yield and rheological behavior. ‘Yield-stress’ is a generic label that can encapsulate many other physical attributes that characterize different yield-stress materials and hence affect the overall behavior of the flow system including its yield and solidification points. Although this may be difficult to imagine with regard to the yield and solidification points, it could have an impact on other rheological attributes that, directly or indirectly, affect these points.

We also should not rule out the possibility of yield-stress models, other than the ones that have already been proposed in the literature including our own model, that could lead to a different and possibly better prediction of the yield and solidification points. Regardless of any model, there is also the possibility of a yield point different to the static and dynamic ones, most likely to be in between, due to the

involvement of other rheological and dynamic factors. The proposed yield and solidification scenarios in the literature including the present paper are mostly based on a pure logical reasoning with an implicit assumption of an ideal yield-stress material, and hence many real-world physical factors are not fully incorporated in these models.

Regardless of all these controversial and uncontroversial issues, even if our criticism to the BRKMP model is rejected, our yield-stress model as proposed in S10 and elaborated in the present paper is at least as valid as the BRKMP model from a pure modeling viewpoint based on the sensibility and consistency criteria, as long as there is no independent and conclusive evidence, experimental or otherwise, with or against one of these models or the other. In this paper we provided sufficient clarifications and justifications to endorse the yield-stress modeling approach of S10 regardless of the validity or invalidity of any other model. We therefore believe that the BRKMP attempt to disqualify the modeling approach of S10 is void.

7 Conclusions

The main conclusions reached in this study is the confirmation of the previous findings by Sochi [7] with regard to the threshold yield pressure of yield-stress materials residing in rigid networks of interconnected conduits or rigid porous structures subject to a pressure field defined by two pressure boundary conditions. The essence of the previous findings is that the dynamic yield point as obtained from solving or inspecting the pressure field is generally higher than the static yield point found by the threshold path algorithms. This is in a complete disagreement with Balhoff *et al* [11] who claimed to have proved that the threshold yield pressure obtained dynamically is identical to the one found by the threshold path algorithms. We demonstrated that what Balhoff *et al* identified is more appropriate to be the solidification point on a gradual and continuous lowering of the pressure drop starting from an established fluid state rather than the yield point of a solid state material. However, even this could be challenged on the basis of the inertial nature of complex fluids that may shift the solidification point to a lower pressure point than the threshold sum.

Nomenclature

μ	fluid dynamic viscosity (Pa.s)
τ_o	yield-stress (Pa)
τ_w	stress at tube wall (Pa)
C	consistency coefficient in Bingham model (Pa.s)
L	tube length (m)
P	pressure (Pa)
ΔP	pressure drop (Pa)
ΔP_{dy}	pressure drop of dynamic yield (Pa)
ΔP_t	threshold pressure drop (Pa)
ΔP_{ts}	pressure drop of threshold sum (Pa)
Q	volumetric flow rate ($\text{m}^3 \cdot \text{s}^{-1}$)
Q_B	flow rate of Bingham model ($\text{m}^3 \cdot \text{s}^{-1}$)
Q_P	flow rate of Poiseuille model ($\text{m}^3 \cdot \text{s}^{-1}$)
R	tube radius (m)

References

- [1] T. Sochi. Non-Newtonian Rheology in Blood Circulation. *Submitted*, 2013. arXiv:1306.2067. [1](#)
- [2] A.H.P. Skelland. *Non-Newtonian Flow and Heat Transfer*. John Wiley and Sons Inc., 1967. [2](#)
- [3] R.B. Bird; R.C. Armstrong; O. Hassager. *Dynamics of Polymeric Liquids*, volume 1. John Wily & Sons, second edition, 1987. [2](#)
- [4] H.A. Barnes; J.E Hutton; K. Walters. *An introduction to rheology*. Elsevier, 1993. [2](#)
- [5] P.J. Carreau; D. De Kee; R.P. Chhabra. *Rheology of Polymeric Systems*. Hanser Publishers, 1997. [2](#)
- [6] H.A. Barnes. The yield stress—a review or ‘ $\pi\alpha\nu\tau\alpha\rho\epsilon\iota$ ’—everything flows? *Journal of Non-Newtonian Fluid Mechanics*, 81(1):133–178, 1999. [2](#)
- [7] T. Sochi. Modelling the Flow of Yield-Stress Fluids in Porous Media. *Transport in Porous Media*, 85(2):489–503, 2010. [2](#), [4](#), [6](#), [23](#)
- [8] T. Sochi. Non-Newtonian Flow in Porous Media. *Polymer*, 51(22):5007–5023, 2010. [2](#)
- [9] H. Kharabaf; Y.C. Yortsos. Invasion percolation with memory. *Physical Review*, 55(6):7177–7191, 1997. [2](#)
- [10] T. Sochi. *Pore-Scale Modeling of Non-Newtonian Flow in Porous Media*. PhD thesis, Imperial College London, 2007. [2](#), [6](#), [21](#)
- [11] M. Balhoff; D. Sanchez-Rivera; A. Kwok; Y. Mehmani; M. Prodanović. Numerical Algorithms for Network Modeling of Yield Stress and other Non-Newtonian Fluids in Porous Media. *Transport in Porous Media*, 93(3):363–379, 2012. [3](#), [23](#)
- [12] T. Sochi. One-Dimensional Navier-Stokes Finite Element Flow Model. *Technical Report*, 2013. arXiv:1304.2320. [5](#), [6](#), [19](#)
- [13] T. Sochi. Pore-Scale Modeling of Navier-Stokes Flow in Distensible Networks and Porous Media. *Computer Modeling in Engineering & Sciences (Accepted)*, 2014. [5](#), [6](#)

- [14] T. Sochi. Flow of Navier-Stokes Fluids in Converging-Diverging Distensible Tubes. *Submitted*, 2013. arXiv:1310.4221. [6](#)
- [15] T. Sochi. Flow of non-Newtonian Fluids in Converging-Diverging Rigid Tubes. *Submitted*, 2013. arXiv:1310.7655. [6](#)
- [16] T. Sochi. Comparing Poiseuille with 1D Navier-Stokes Flow in Rigid and Distensible Tubes and Networks. *Submitted*, 2013. arXiv:1305.2546. [6](#)
- [17] T. Sochi. Pore-scale modeling of viscoelastic flow in porous media using a Bautista-Manero fluid. *International Journal of Heat and Fluid Flow*, 30(6):1202–1217, 2009. [6](#)
- [18] M. Chen; W.R. Rossen; Y.C. Yortsos. The flow and displacement in porous media of fluids with yield stress. *Chemical Engineering Science*, 60(15):4183–4202, 2005. [21](#)

A Mass-Conserving Pressure Field

Assume we have a network consisting of n serially-connected cylindrical tubes which generally have different radii and lengths containing a yield-stress material. The sum of threshold yield pressures of the individual tubes in such an ensemble, ΔP_{st} , is given by

$$\Delta P_{st} = \sum_{i=1}^n \Delta P_{it} \quad (2)$$

where ΔP_{it} is the threshold yield pressure of tube i . Now if we apply a total pressure drop of ΔP_{st} across the ensemble and assume that this total pressure drop is divided such that for each tube a pressure drop equal to its threshold yield pressure ΔP_{it} occurs across its length, then at this total pressure drop ΔP_{st} the flow in the system is zero because all tubes are at their threshold yield pressure. Now let us assume that we added an infinitesimal increase in the pressure drop, $\epsilon > 0$, across the ensemble such that

$$\Delta P_{st} + \epsilon = \sum_{i=1}^n \Delta P_{it} + \sum_{i=1}^n \epsilon_i \quad (3)$$

then there should be in principle a finite minute flow in each tube in the ensemble. Since the flow rate is a continuous function of the pressure drop for each tube, then it is possible to adjust the arbitrary and infinitesimal ϵ_i such that the flow rate in all tubes is the same within a given error tolerance. For this same reason (i.e. the flow rate is a continuous function of pressure drop for each tube) if we now increase ϵ infinitesimally, it should be possible to divide this increase on the pressure drops of the individual tubes such that the flow rate in all tubes is still the same within the given error tolerance. By doing this process of adding an infinitesimal increase to the threshold sum ΔP_{st} repeatedly and dividing the increase on the individual pressure drops appropriately as before, we can reach any pressure drop above the threshold sum ΔP_{st} such that the flow rate in all tubes of the ensemble is the same within the given error tolerance. This in essence is the same as finding a consistent pressure field sustaining a total flow rate in the ensemble that conserves mass.

This mathematical argument can be applied to Poiseuille flow as well to prove that it is always possible to find a consistent pressure field that sustains a mass-conserving flow for any pressure drop greater than zero across such ensembles. The argument can also be generalized to any other characteristic flow above the threshold mobilization pressure of the ensemble for that particular fluid.

B Yield Condition for Fluid Approach

For a network consisting of serially-connected cylindrical tubes containing a yield-stress material assumed to be a highly-viscous fluid prior to yield, the flow of the material can occur *iff* two conditions are simultaneously satisfied: (a) the mass is conserved throughout the ensemble and (b) all tubes pass their threshold yield point simultaneously. In the following we show that for such a network these two conditions require a threshold yield pressure that in general is greater than the sum of the threshold yield pressures of the individual tubes. The threshold yield pressure for a cylindrical tube is given by

$$\Delta P_t = \frac{2\tau_o L}{R} \quad (4)$$

while the Poiseuille flow, which is assumed to model the flow prior to mobilization, in such a tube is given by

$$Q = \frac{\pi R^4 \Delta P}{8\mu L} \quad (5)$$

where ΔP_t is the threshold yield pressure of the tube, τ_o is the yield stress, L and R are respectively the tube length and radius, Q is the volumetric flow rate, μ is the fluid dynamic viscosity, and ΔP is the pressure drop across the tube.

Now let us take the tube with the largest radius in this serially-connected network. Since the flow through the ensemble will not occur unless this tube reaches its threshold yield point, then a necessary condition for the flow to occur is that this tube reaches its yield point. We will see later that this is also a sufficient condition for the flow to occur in the network assuming mass conservation is satisfied. For this tube the flow rate at its threshold pressure is

$$Q_b = \frac{\pi R_b^4 \Delta P_{bt}}{8\mu L_b} = \frac{\pi R_b^4}{8\mu L_b} \frac{2\tau_o L_b}{R_b} = \frac{2\tau_o \pi R_b^3}{8\mu} \quad (6)$$

where b is an index marking this tube, and ΔP_{bt} is the threshold yield pressure of this tube. Due to the mass conservation, this flow rate is the same for all the tubes in the network, that is for any tube other than the one with the largest radius we have

$$Q_i = \frac{\pi R_i^4 \Delta P_i}{8\mu L_i} = \frac{2\tau_o \pi R_b^3}{8\mu} \quad (7)$$

where i is an index marking the other tube. On rearranging and simplifying we

obtain

$$\Delta P_i = \frac{2\tau_o L_i R_b^3}{R_i^4} = \frac{2\tau_o L_i}{R_i} \frac{R_b^3}{R_i^3} = \Delta P_{it} \frac{R_b^3}{R_i^3} \quad (8)$$

where ΔP_{it} is the threshold yield pressure of tube i . Now

$$R_b \geq R_i \quad \Rightarrow \quad \Delta P_i \geq \Delta P_{it} \quad (9)$$

and hence the sum of the actual pressures across the individual tubes for such a flow assuming mass conservation is greater than or equal to the sum of threshold yield pressures of the individual tubes. The equality holds only when all the tubes in the network have the same radii. The condition in Equation 9 also explains why reaching the threshold yield pressure for the tube with the maximum radius is not only a necessary condition but is also a sufficient condition for the flow to occur assuming mass conservation, as indicated earlier.

C Yield Condition for Solid Approach

For a network consisting of serially-connected and straightly-aligned cylindrical tubes containing a yield-stress material which is assumed to be solid-like prior to its mobilization, the flow of the material can occur *iff* all tubes pass their threshold yield point simultaneously. In the following we show that for such a network this condition requires a threshold yield pressure that in general is greater than the sum of the threshold yield pressures of the individual tubes assuming a linear pressure drop which is equivalent to a constant pressure gradient.

We take the tube with the smallest radius in the network. Since the flow through the ensemble will not occur unless this tube reaches its threshold yield point, then a necessary condition for the flow to occur is that this tube reaches its yield point. We will see later that this is also a sufficient condition for the flow to occur in the network. For this tube, indexed by b , the threshold yield pressure is given by

$$\Delta P_{bt} = \frac{2\tau_o L_b}{R_b} \quad (10)$$

and hence the constant pressure gradient across the entire network when tube b is at its threshold yield pressure will be

$$\nabla P = \frac{\Delta P_{bt}}{L_b} = \frac{2\tau_o}{R_b} \quad (11)$$

Now since the pressure drop is assumed linear with respect to the network total length, the pressure drop across any other tube in the network, indexed by i , will be

$$\Delta P_i = \nabla P L_i = \frac{2\tau_o L_i}{R_b} \quad (12)$$

Now since $R_i \geq R_b$ we have

$$\Delta P_{it} = \frac{2\tau_o L_i}{R_i} \Rightarrow \Delta P_i \geq \Delta P_{it} \quad (13)$$

Hence, at the yield point of the ensemble the sum of the actual pressures across the individual tubes in such a network based on the solid-like assumption with a constant pressure gradient across the network is greater than or equal to the sum of threshold yield pressures of the individual tubes. The equality holds only when all the tubes in the network have the same radii. The condition in Equation 13 also explains why passing the threshold yield pressure for the tube with the minimum

radius is not only a necessary condition but is also a sufficient condition for the flow in the network to occur.