An identity for generating special Pythagorean quadruples

Author: Brian Ekanyu

Tel. +256776199444

e-mail: bekanyu@gmail.com

Abstract

This paper proves an identity for generating a special kind of Pythagorean Quadruples by conjecturing that the shortest edge of the Pythagorean box is defined by $a = 1,2,3,4, \ldots ..$ and $b = a + 1$, $c = ab$ and $d = c + 1$. It also shows that $a + d = b + c$ and that the surface area to volume ratio of these Pythagorean boxes is given by $\frac{a}{4}$ where a is the length of the shortest edge.

The identity is:

$$\underline{\boldsymbol{a^2 + (a+1)^2 + a^2(a+1)^2 = [a(a+1)+1]^2}}$$

$\boldsymbol{Proof}$: $\boldsymbol{Consider\ a\ Pythagorean\ box\ of\ dimensions\ a, b, and\ c\ and\ space\ diagonal, d.}$

$$\boldsymbol{Let\ a = a, where\ a = 1, 2, 3, 4, 5, .6, 7, 8, 9, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots}$$

$$\boldsymbol{b = (a+m)}$$

$$\boldsymbol{c = ab}$$

$$\boldsymbol{c = a(a+m)}$$

$$\boldsymbol{d = c + m}$$

$$\boldsymbol{d = a(a+m) + m}$$

$$\boldsymbol{For\ a\ Pythagorean\ box, we\ have:}$$

$$\boldsymbol{a^2 + b^2 + c^2 = d^2. Substituting\ for\ a\ , b, c\ and\ d\ respectively, we\ get:}$$

$$\boldsymbol{a^2 + (a+m)^2 + a^2(a+m)^2 = [a(a+m)+m]^2}$$

$$\boldsymbol{a^2 + a^2 + 2am + m^2 + a^2(a^2 + 2am + m^2) = a^2(a+m)^2 + 2am(a+m) + m^2}$$

$$\boldsymbol{2a^2 + 2am + m^2 + a^4 + 2a^3m + a^2m^2 = a^4 + 2a^3m + a^2m^2 + 2a^2m + 2am^2 + m^2}$$

$$\boldsymbol{(a^4 - a^4) + (2a^3m - 2a^3m) + (a^2m^2 - a^2m^2) + (m^2 - m^2) + 2a^2m + 2am^2 - 2a^2 - 2am = 0}$$

$$\boldsymbol{which\ yields: 2a^2m + 2am^2 - 2a^2 - 2am = 0}$$

$$\boldsymbol{on\ simplification\ we\ get: m^2 + am - a - m = 0}$$

$$\boldsymbol{On\ solving\ the\ above\ quadratic\ equation, we\ get\ the\ values\ of\ m\ as\ m = 1\ and\ m = -a.}$$

$$\boldsymbol{The\ only\ value\ of\ m\ that\ satisfies\ the\ Pythagorean\ condition\ is\ m = 1.}$$

$$\boldsymbol{Examples\ of\ identities\ generated\ by\ the\ above\ identity\ are\ shown\ in\ the\ table\ below:}$$

a	*b*	*c*	*d*
1	2	2	3
2	3	6	7
3	4	12	13
4	5	20	21
5	6	30	31
6	7	42	43
7	8	56	57
8	9	72	73
9	10	90	91
10	11	110	111
11	12	132	133
12	13	156	157
13	14	182	183
14	15	210	211
15	16	240	241
16	17	272	273
17	18	306	307
18	19	342	343
19	20	380	381
20	21	420	421
21	22	462	463
22	23	506	507
23	24	552	553
24	25	600	601
25	26	650	651
26	27	702	703
27	28	756	757
28	29	812	813
29	30	870	871
30	31	930	931
31	32	992	993
32	33	1056	1057
33	34	1122	1123
34	35	1190	1191
35	36	1260	1261
36	37	1332	1333
37	38	1406	1407
38	39	1482	1483
39	40	1560	1561
40	41	1640	1641
41	42	1722	1723
42	43	1806	1807
43	44	1892	1893
44	45	1980	1981
45	46	2070	2071
46	47	2162	2163

47	48	2256	2257
48	49	2352	2353
49	50	2450	2451
50	51	2550	2551
51	52	2652	2653
52	53	2756	2757
53	54	2862	2863
54	55	2970	2971
55	56	3080	3081
56	57	3192	3193
57	58	3306	3307
58	59	3422	3423
59	60	3540	3541
60	61	3660	3661
61	62	3782	3783
62	63	3906	3907
63	64	4032	4033
64	65	4160	4161
65	66	4290	4291
66	67	4422	4423
67	68	4556	4557
68	69	4692	4693
69	70	4830	4831
70	71	4970	4971
71	72	5112	5113
72	73	5256	5257
73	74	5402	5403
74	75	5550	5551
75	76	5700	5701
76	77	5852	5853
77	78	6006	6007
78	79	6162	6163
79	80	6320	6321
80	81	6480	6481
81	82	6642	6643
82	83	6806	6807
83	84	6972	6973
84	85	7140	7141
85	86	7310	7311
86	87	7482	7483
87	88	7656	7657
88	89	7832	7833
89	90	8010	8011
90	91	8190	8191
91	92	8372	8373
92	93	8556	8557
93	94	8742	8743
94	95	8930	8931

95	96	9120	9121
96	97	9312	9313
97	98	9506	9507
98	99	9702	9703
99	100	9900	9901
100	101	10100	10101

$$Ad\ infinitum. From\ the\ Becker - Sievert\ identity:$$

$$n^2 + n^2 + \left(\frac{n}{2}\right)^2 = \left(\frac{3n}{2}\right)^2\ where\ n\ is\ an\ even\ number. It\ can\ be\ observed\ that\ the\ Becker - Sievert\ is\ used\ to\ generate\ Pythagorean\ quadruples\ which\ are\ multiples\ of\ the\ premitive\ quadruple$$

$$(1\ 2\ 2\ 3)$$

Theorem 1: There are infinitely many Pythagorean quadruples such that $a + d = b + c$

Proof:

$$Let\ a + b + c = d + k$$

$$a + (a + 1) + a(a + 1) = a(a + 1) + 1 + k$$

$$k = 2a$$

$$\therefore a + b + c = d + 2a$$

$$a + d = b + c$$

$$Therefore\ there\ are\ infinitely\ many\ Pythagorean\ quadruples\ for\ which\ a + d = b + c$$

Theorem 2: There are infinitely many Pythagorean boxes whose surface area to volume ratio is given by $\frac{4}{a}$ where a is the length of the shortest edge.

Proof:

$$Consider\ a\ Pythagorean\ box\ whose\ dimensions\ are:$$

$$a = a$$

$$b = a + 1$$

$$c = a(a + 1)$$

$$volume = abc$$

$$volume = a(a + 1).a(a + 1)$$

$$volume = a^2(a + 1)^2$$

$$surface\ area = 2(ab + ac + bc)$$

$$surface\ area = 2[a(a + 1) + a^2(a + 1) + a(a + 1)]$$

$$surface\ area = 2a(a + 1)(2a + 2)$$

$$surface\ area = 4a(a+1)^2$$

$$\frac{surface\ area}{volume} = \frac{4a(a+1)^2}{a^2(a+1)^2}$$

$$\frac{surface\ area}{volume} = \frac{4}{a}$$

$There\ are\ infinitely\ many\ Pythagorean\ boxes\ whose\ surface\ area\ to\ volume\ ratio\ is\ \frac{4}{a}\ where a\ is\ the\ length\ o$

References:

1. *R.A. Beauregard and E.R. Suryanarayan, Pythagorean boxes, Math. Magazine 74(2001), 222-227.*
2. *R.D. Carmichael, Diophantine Analysis, New York: John Wiley and Sons, 1915.*
3. *R. Spira, The Diophantine equation* $x^2 + y^2 + z^2 = m^2$, *Amer. Math. Monthly Vol.69(1962), No. 5, 360-365.*