Algebraic approach to the derivative and continuity

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Abstract

Continuity is relevant for the real numbers and functions, namely to understand singularities and jumps. The standard approach first defines the notion of a limit and then defines continuity using limits. Surprisingly, Vredenduin (1969), Van der Blij (1970) and Van Dormolen (1970), in main Dutch texts about didactics of mathematics (journal Euclides and Wansink (1970, volume III)), work reversely for highschool students: they assume continuity and define the limit in terms of the notion of continuity. Vredenduin (1969) also prefers to set the value at the limit point ($x = a$) instead of getting close to it ($x \to a$). Their approach fits the algebraic approach to the derivative, presented since 2007. Conclusions are: (1) The didactic discussions by Vredenduin (1969), Van der Blij (1970) and Van Dormolen (1970) provide support for the algebraic approach to the derivative. (2) For education, it is best and feasible to start with continuity, first for the reals, and then show how this transfers to functions. (3) The notion of a limit can be defined using continuity. The main reason to mention the notion of a limit at all is to link up with the discussion about limits elsewhere (say on the internet). Later, students may see the standard approach. (4) Education has not much use for limits since one will look at continuity. The relevant use of limits is for infinity.

1. Introduction

1.1. An algebraic approach to the derivative

In set theory, functions are sets of ordered pairs {*x*, *f*[*x*]}, with *x* in the domain and *f*[*x*] in the range.

In algebra, a function has a *prescription* or *algebraic expression* that provides an algorithm to calculate the output from the input.

> NB. The term "prescription" fits the Dutch "voorschrift" and the German "Zuordnungsvorschrift". [1] In English the word "description" is common, [2] but the notion of a *recipe* is more useful.

The prescription itself provides information. This information can be used to construct the derivative without use of limits. This becomes the algebraic approach to the derivative.

This algebraic approach to the derivative has been defined in "A Logic of Exceptions" (ALOE), Colignatus (2007, 2011). A proof of concept was "Conquest of the Plane" (COTP), Colignatus (2011). An overview has been given in Colignatus (2016a), including what the (positive and negative) reactions have been. [3]

The following assumes that one has read Colignatus (2016a) so that one knows what the algebraic approach to the derivative is. (It is a definition, and it is clear, so please don't do as if it isn't a definition.)

1.2. A difference to be aware about

Traditionally: If a function has a derivative then it is continuous. If a function is continuous then it doesn't need to have a derivative. An example is Abs[*x*] or |*x*| at *x* = 0.

Algebraically: Using the new scheme, Abs[*x*] or |*x*| would have a derivative, namely {-1, 0, 1} for values *x* < 0, *x* = 0, and *x* > 0.

1.3. Why I haven't discussed continuity before

When the algebraic approach to the derivative was defined in ALOE in 2007, it was clear at the time that the issue of continuity should not present a problem. [4]

However, I was aware that an explicit statement on continuity would be useful. Potentially this might be useful for discussing the different outcome in section 1.2 for example. It wasn't a priority but over the years since 2007 I had some heightened awareness on the notion of continuity. Thus the present paper is important since it provides that explicit statement.

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[1] https://de.wikipedia.org/wiki/Funktion_(Mathematik)

[2] "Different formulas or algorithms may describe the same function. (...) Furthermore, a function need not be described by a formula, expression, or algorithm, nor need it deal with numbers at all: the domain and codomain of a function may be arbitrary sets. " https://en.wikipedia.org/wiki/Function_(mathematics)

[3] This overview Colignatus (2016a) itself has been maltreated by the editors of NAW, see Colignatus (2016b). The latter discussion contains aspects about the approach that are relevant in themselves too.

[4] In COTP in 2011 it was only a challenge to show that the approach worked for trigonometry and the exponential function too, but this was quickly solved, since the approach helped to focus on what matters. When the reader likes a challenge, start with the definition in ALOE and try to show that it works for polynomials, trigonometry and the exponential function. I am wondering whether this compares to suggesting to people to reinvent the bicycle.

Two events clinched the matter.

- First is, that I finalised my ideas on the real numbers, in "Foundations of Mathematics. A Neoclassical Approach to Infinity" (FMNAI), Colignatus (2015). Obviously, the real numbers are a foundation for continuity, and it is useful to have sound foundations.
- Secondly, this November, I got a copy of Wansink (1970). Actually, volumes I, II, III. This is a discussion of didactics of mathematics, for the training of Dutch teachers at that time.

1.4. A surprise in Wansink (1970)

My original source on analysis remains Scheelbeek & Verdenius (1973) that I used as a student. First the notion of a limit is defined, and then this is used to define continuity. The same Weierstrasz epsilon and delta approach can be found in Apostol (1957, 1965), with limit p61, and continuity p67. A function $f[x]$ is continuous in $x = a$ when $\lim[f[x], x \to a] = f[a]$. [5]

Wansink (1970), a main Dutch text for didactics of mathematics, surprised me by presenting two authors Van der Blij (1970) and Van Dormolen (1970) who reason in opposite manner, at least for the didactics in highschool, and I discovered that Vredenduin (1969) has this too:

> If continuity has been given, then one can define limits in terms of this continuity.
>
> *Definition*: It is said that f has a limit value b in point a, or $\lim[f[x], x \to a] = b$, when there is a continuous function g, such that $f[x] = g[x]$ for all $x \neq a$ and $g[a] = b$.

Appendix A contains Van der Blij (1970) p125-126. His objective is to provide a definition of the notion of the limit (p125: definition of "limietbegrip"). We see the same approach by Van Dormolen (1970:204) in the same volume, see **Appendix B**. (Perhaps not surprising, as Van Dormolen wrote his 1982 thesis with supervisor Van der Blij.) For Dutch readers, Vredenduin (1969) also provides clarification for the intuition in topology. His text is online yet a relevant section is in **Appendix C,** in which he follows the same method as Van der Blij. I prefer to focus on Van der Blij since he is most explicit about algebraic simplification.

In the algebraic approach to the derivative, algebra and the prescription (including the statement on domain and range) determine whether there is continuity or discontinuity. With this as a given, we are close to Vredenduin, Van der Blij en Van Dormolen. Thus the idea of 2007 that continuity should not be a problem is resolved in even simpler manner than I was consciously aware of (but don't underestimate intuition via subconsciousness).

1.5. Continuity: not only for functions

Writing this memo, I was a bit amazed that a google on continuity commonly generated texts on continuity of *functions*, [6] but much less about the more basic notion of continuity of the *real numbers*. When you are aware of this though then you can find the proper pages. Still, these pages tend to be about topology for first year students of mathematics, and not what would be needed for junior highschool.

Thus, Colignatus (2016c) presents a logical development for a presentation in junior highschool. This is not a development of didactics yet, but only the logical framework.

Also, Colignatus (2016d) discusses that students in junior highschool must choose between continuing with math *A* (alpha) or math *B* (beta) in senior highschool. A discussion of continuity should not create confusion about a career that uses mathematics.

PM. Some readers might think that a definition of continuity for functions suffices, since when you would use the identity function, then this would generate a notion of continuity for the real

[5] The Weierstrasz definition might have to be adapted for domain and range.
[6] Basically all the same, one wonders why people bother to restate this.

line: lim[*x*, *x* → *a*] = *a*. However, if you are using epsilon and delta, then these are reals themselves, and you could be assuming the notion of continuity that you must prove.

1.6. A structure for continuity

Thus a structure for continuity becomes:

(1) A proper definition for the reals provides us with the real line as the defining notion of continuity: dense and without holes. (Mathematics education improves with a better grasp of what the reals are.) See Colignatus (2016c) and FMNAI (2015). Starting with the reals requires the limit for infinity (namely the notion of infinite decimals).
(2) Functions and algebraic prescriptions may preserve continuity. See section 5 below. (One must also develop what "algebraic prescription" is, see Colignatus (2016a) and its references.)
(3) From this notion of continuity we can define a notion of a limit. Why would we actually do so ? It might be helpful for a discussion like under (2). Also, students will look on the internet and see discussion about limits, and then should be aware that limits in their textbook are defined via use of continuity.
(4) However, we don't need limits for the derivative. [7]

1.7. Structure of this memo

Given the above, we now only have to look at Van der Blij (1970), Van Dormolen (1970) and Vredenduin (1969), with the relevant texts in **Appendices A, B and C**. I start with Van der Blij since he is most explicit for our purposes.

2. Discussion of Van der Blij (1970:125-126)

2.1. General setting

Page 125 provides the general setting.

- Van der Blij clearly shows the tension between analysis and topology. Highschool mathematics still tends to follow analysis (with Cauchy rather than Weierstrasz) while research mathematics has continued in topology. Teachers might develop didactics for approaches that research mathematicians no longer deem most efficient.
- Van der Blij clearly mentions that highschool education "works by examples" rather than develop the general theory, in this case about functions. (For example, after the demise of the New Math, it wouldn't do to state the set-theoretic definition of a function. [8])

In this case, however, I am not convinced that the approaches in both analysis and topology are so relevant.

In the algebraic approach to the derivative, we have no need for topology and its open and closed sets, or Weierstrasz with epsilon and delta.

As said, Van der Blij's objective in p125-126 is to provide a definition of what a limit would be (p125: definition of "limietbegrip").

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[7] The algebraic approach uses the notion of simplification. In some respects it is like replacing the notion of a limit by a notion of simplification, like lim[*f*[*x*], *x* → *a*] = Simplify[*f*][*a*]. The discussion of trigonometry and exponential function however indicates that this is not quite so simple, unless one wants to define "simplification" to be so.
[8] I am in favour however for a thorough re-evaluation of what would still be useful to (re-) introduce. https://en.wikipedia.org/wiki/New_Math

2.2. What is an algebraic expression ?

For us, Van der Blij's discussion becomes relevant at the bottom of page 125, when he states:

> "Let *f* be an explicitly algebraically defined prescription for a function, i.e. an expression that is composed out of a finite number of rational and root functions." (my translation) [9]

I don't think that this is a proper definition of "algebraic expression". However, the earlier discussion about the algebraic approach has shown that there isn't yet a broadly accepted definition, neither for highschool. Vredenduin (1968) is an effort, but limited to elementary operations, and it didn't reach the classrooms. Again, students in highschool learn about algebraic manipulation, but *they learn by example,* and the formal prescription is lacking.

My solution for this problem has been to refer to a computer algebra package *Mathematica,* and in particular to the function Simplify[*expr*]. This is not intended as a mathematical definition, but it provides an empirical framework, that allows researchers to check what algebraic manipulations can be used in practical discussions. Obviously, teachers should arrive at a definition of what an algebraic expression is. But this should not be a bottleneck for the introduction of the algebraic approach to the derivative.

2.3. General *f* versus limited scope

It may be remarked that Van der Blij discusses a general (algebraic) *f.* In ALOE, COTP and Colignatus (2016a), the approach has been restriced to *f* = difference quotient. (This uses a distinction between static and dynamic division, with their application to the derivative.)

2.4. Distinction between "theory" and "practice"

Van der Blij starts with a function *f,* that would be undefined in $x = a$.

Then he explains what is done "in practice". This is a key term, for it creates a distinction between practical use of the limit and the formal use of the limit.

In practice: One finds an associated algebraic expression for function *g*, such that $f[x] = g[x]$ for $x \neq a$. Then he defines: $\lim[f[x], x \to a] = g[a]$. Students are supposed to see without proof that *g* itself is continuous at *a.*

At the top of p126, he gives an example of a quotient that is undefined for $x = 1$. With algebraic simplification, the term $x - 1$ drops out.

(1) Under "Voorbeeld" he (for once sloppily) states that $f[x] = g[x]$ without stating that $x \neq 1$.

(2) The same procedure ("practice") (with proper exclusion of the limit point) is exactly adopted by the algebraic approach to the derivative.

> (2a) One cannot say that Van der Blij is a precursor to the algebraic approach. When I developed the algebraic approach, my inspiration was this practice as well, and Van der Blij (1970) is only an example of an author who restates this practice.
> (2b) However, Van der Blij deserves a compliment for restating this explicitly, for other authors only state the result, and refer to teachers to show this practice on the blackboard. Idem dito for Van Dormolen (1970).
> (2c) The algebraic approach doesn't explicitly mention this other function *g.* It works directly with the algebraic expression. My impression is that Van der Blij was aware that students needed an anchor to understand the practice. He looked for this anchor in this other function *g.* Indeed it is possible to define such a function *g*. However, this

[9] Van der Blij's chapter got published, even though he was so vague. Compare this with the unreasonale demands by the editors of NAW, see Colignatus (2016b).

> also distracts for our purposes. The algebraic approach finds that anchor in the manipulation of the domain. This makes more sense, since when *f* is not defined for *x* = *a,* then is advisable to be explicit on the manipulation of the domain, so that *a* becomes included.

(3) Van der Blij's objective here is to provide a definition of what a limit would be (p125: definition of "limietbegrip"). Now that he has defined what a limit is, he can say that *g* is continuous at *a* since lim[*g*[*x*], *x* → *a*] = *g*[*a*]. (Which would require another *h* again.) [10]

If lim[*f*[*x*], *x* → *a*] has another theoretical definition (e.g. Weierstrasz) then one relies on an unstated proof, that this practice indeed covers the true meaning of lim[*f*[*x*], *x* → *a*] = *g*[*a*].

NB. This differs from our purposes w.r.t. the derivative though. This additional theory (Weierstrasz) is entirely superfluous for the derivative, when the algebraic approach to the derivative can be defined independently by itself. The "practice" becomes the proper theory. The "practice" works because the algebraic expression contains all information that is relevant for understanding the function and the manipulation of the domain. (Thus, if one introduces the Weierstrasz conditions, then one must explain that these fit the practice.)

In fact, the expression "lim[*f*[*x*], *x* → *a*]" is superfluous for the derivative (with *f* now the difference quotient) (provided that one follows the algorithm in the algebraic approach).

2.5. Van der Blij: "Determination of the limit is a purely algebraic issue"

Van der Blij p126 correctly concludes that isses of topology do not surface here.

A key recognition is: *"Determination of the limit is a purely algebraic issue".*

Obviously, this conclusion depends upon the chosen algebraic *f.* Still it is important that Van der Blij recognises algebraic simplification for what it is.

2.6. Remainder of p126: trigonometry and exponential function

Subsequently Van der Blij considers trigonometry, that would require ordering, and the exponential function, that would require topology.

For this, I best refer to Colignatus (2011), "Conquest of the Plane" (COTP).

- Indeed, trigonometry uses this ordering, but this remains an issue of algebra and logic, and there is no "squeezing" in terms of limits. It is logic that 1 ≤ *b* ≤ 1 implies *b* = 1.
- The exponential function appears not to need topology.

3. Discussion of Van Dormolen (1970:204)

Van Dormolen (1970:204) provides a rationale for highschool:

> Students are still reasoning much from intuition, and thus it is reasonable to start with the intuitions on continuity and define the notion of a limit from there.

The alternative function *g* now is *f*.*

Again, the example is a quotient. This *f* and *f** overlap because of algebraic simplification. Students are assumed to see without further proof that *f** is continuous at the point. From this, the limit for *f* can be defined.

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[10] Potentially, this notion of a limit is restricted to continuity only. Indeed, a limit to infinity could be treated differently.

Van Dormolen states that the method might be longer than other definitions, but he doesn't consider this objectionable given the importance of the notion of continuity.

Up to here his presentation is equivalent to the one by Van der Blij.

In his subsequent discussion, Van Dormolen provides more formal definitions of continuity, and states that it is somewhat convoluted to first assume it intuitively, then define the limit by using it, and then develop continuity formally from the limit. He leaves it at this.

PM. Instead, it would have been more interesting to see a discussion how the use of epsilon and delta actually presume a notion of continuity, while such a limit is used to define continuity (i.e. one is asssuming what must be shown).

4. Discussion of Vredenduin (1969:15-16)

Vredenduin (1969:15-16) follows the method by Van der Blij (1970) (or conversely). At first one wonders whether he really does for $x = 2$, but when he starts on his second example for $x = 3$, then one clearly sees that he applies this method.

Vredenduin (1969:15) states: "Het nemen van de limiet is geen dynamisch maar een statisch proces. Vandaar dat ik onder het limietteken schrijf "$x = 2$" en niet "$x \to 2$." "

I would think that "static process" is a contradiction in terms. But the intuition that we should look at $x = 2$ rather than the neighbourhood fits the algebraic approach.

The wonder of this article is that Vredenduin spends so much space on the continuity of functions, while this can be resolved by proper definition of the reals and the approach in the next section.

5. A new topic in the textbook

Thus, we can follow Vredenduin, Van der Blij and Van Dormolen, and assume the notion of continuity, and work from there.

This however leaves us unsatisfied. There is a gap (not to say *discontinuity*) between accepting continuity as a basic notion, and presuming that this is immediately obvious for more intricate functions.

Thus it makes sense to introduce a new chapter in the textbook, that reduces this gap. This chapter would discuss the following principles, given that students have learned about the continuity of the real number line. [11]

- The constant function c and the identity function I: $\mathbb{R} \to \mathbb{R}$ are continuous.
- If a function f: $\mathbb{R} \to \mathbb{R}$ is continuous, then $r\, f$: $\mathbb{R} \to \mathbb{R}$ is continuous too, for real number r.
- If functions f and g : $\mathbb{R} \to \mathbb{R}$ are continuous, then $f + g$ and f x g: $\mathbb{R} \to \mathbb{R}$ are continuous too. Also f / g : $\mathbb{R} \to \mathbb{R}$ is continuous except discontinuous where $g[x] = 0$.
- If functions f and g : $\mathbb{R} \to \mathbb{R}$ are continuous, then f o g and g o f : $\mathbb{R} \to \mathbb{R}$ are continuous (for the composition like $f[g[x]]$).

This should cover polynomials and rational functions. Trigonometry and the exponential function can be inserted later on.

For the above, one might have exercises for different domains and ranges.

[11] See also https://en.wikipedia.org/wiki/Continuous_function#Examples

There is the tricky issue that 1 / *x* might exclude 0 from its domain, and then be said to be continuous on its domain. Such a statement would be accurate. Students should learn that domain and range are key elements in the definition of a function.

The above only covers real valued functions. The algebraic approach to the derivative still uses algebraic prescriptions for functions, in which symbols have their own properties different from pure numbers. Thus the above is helpful for the notion of continuity but not sufficient for the algebraic approach to the derivative. See Colignatus (2016a) and its references for discussion of this.

Students would also want to see applications why continuity is useful.

- The theorems on the intermediate and extreme values would be interesting to discuss. [12]
- Potentially the classification of discontinuities is relevant too, albeit that students would like to see examples from real life where such occur. [13]

6. Conclusions

The didactic discussions by Vredenduin (1969), Van der Blij (1970) and Van Dormolen (1970) provide support for the algebraic approach to the derivative. [14]

For education, it is best and feasible to start with continuity, first for the reals, and then show how this transfers to functions.

The notion of a limit can be defined using continuity. The main reason to mention the notion of a limit at all is to link up with the discussion about limits elsewhere (say on the internet). Later, students may see the reversed approach (though now standard), that first the notion of a limit is defined, and then used to define continuity. [15]

The relevant use of limits is for infinity. This is e.g. used for the creation of the reals (infinite decimals).

Obviously, these conclusions are only logical, and must be seen as tentative for actual application in practice. Empirical testing is required, since it are students who determine what works for them.

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[12] https://en.wikipedia.org/wiki/Continuous_function#Properties

[13] https://en.wikipedia.org/wiki/Classification_of_discontinuities and http://math.mit.edu/~jspeck/18.01_Fall%202014/Supplementary%20notes/01c.pdf

[14] The algebraic approach is both an improvement in didactics and an essential reformulation. It was presented in ALOE first, and ALOE has a textbook format but also re-engineers logic. Readers interested in this research method might look at: http://thomascool.eu/Papers/AardigeGetallen/2016-11-03-Presentation-Math-Ed-Redesign.pdf

[15] With the question whether the definition of the limit doesn't presume continuity in epsilon and delta.

Appendix A. Van der Blij (1970:125-126)

We noemen de verzamelingen van $\mathcal{O}$ de open verzamelingen. De regel 1 zegt dat de lege verzameling en de verzameling V open zijn, 2 zegt dat de doorsnede van twee open verzamelingen open is, 3 zegt dat de vereniging van willekeurig veel open verzamelingen open is.

Met behulp van het begrip open verzameling introduceert men continue afbeeldingen van topologische ruimten in elkaar. Een afbeelding F heet continu wanneer iedere open verzameling in de beeldruimte een open origineel heeft. We gaan hier niet op details in. Het zal duidelijk zijn, dat men topologische groepen kan bestuderen. Dit zijn verzamelingen die zowel een groepstructuur als een topologische structuur dragen. Er is natuurlijk een eis te stellen over de samenhang van deze twee structuren. De eenvoudigste formulering voor zo'n eis is dat de afbeeldingen $i: x \to \bar{x}$, $l_a: x \to x * a$, $r_b: x \to b * x$ alle continu zijn.

Men kan dan nader onderzoeken, of de aanwezigheid van de topologische structuur beperkingen aan de groepsstructuur oplegt en omgekeerd.

Natuurlijk kan men ook topologische lichamen introduceren. Dan zal men eisen dat de verschillende lichaamsbewerkingen continue afbeeldingen zijn.

En nu zijn we ineens weer terug bij de situatie van het onderwijs. Het algebraonderwijs werkt in de eerste leerjaren duidelijk òf alleen met de lichaamsstructuur van de getallen òf met een ordeningsstructuur. De introductie van $\sqrt{2}$ als oneindig voortlopende decimale breuk is vaak een proces, waarbij de ordeningsstructuur gebruikt wordt, maar soms vermengd met iets topologische structuur. Het limietbegrip kan via continuïteit òf direct met behulp van open verzamelingen slechts in een topologische ruimte gedefinieerd worden. (En niet eens in iedere topologische ruimte; als men uniciteit van de limiet eist, moet men restricties aan de topologie opleggen; het moet een hausdorffruimte zijn, dat wil zeggen bij $a \in V$ en $b \in V$ moeten $A \in \mathcal{O}$ en $B \in \mathcal{O}$ bestaan met $a \in A, b \in B$ en $A \cap B = \emptyset$)

Maar daarvan afgezien komt in de bovenbouw bij het limietbewijs de topologische structuur in het geding. In sommige situaties kan men nog zuiver algebraïsch te werk gaan. En zo doet men het vaak op school. Want als regel werkt men terecht met voorbeelden en niet met een algemeen functiebegrip.

Laat f een expliciet algebraïsch gedefinieerd functievoorschrift zijn, d.w.z. een functievoorschrift dat samengesteld is uit een eindig aantal rationale en wortelfuncties.

Laat f in het getal a niet gedefinieerd zijn, maar wel gedefinieerd in een interval A om a, waaruit a weggelaten is.

In de praktijk bepaalt men dan $\lim_{x \to a} f(x)$ door een expliciet algebraïsch gedefinieerd functievoorschrift g te vinden zodat $f(x) = g(x)$ voor $x \in A$, $x \neq a$; terwijl g in a wel gedefinieerd is, en dan te stellen $\lim_{x \to a} f(x) = g(a)$.

Voorbeeld:

$$f(x) = \frac{\sqrt{x}-1}{\sqrt[3]{x}-1} = \frac{(x-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{(x-1)(\sqrt{x}+1)} = \frac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{\sqrt{x}+1}.$$

$$g(x) = \frac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{\sqrt{x}+1}.$$

En dus $\lim_{x \to 1} f(x) = g(1) = \frac{3}{2}$.

Open verzamelingen, omgevingen, enzovoorts, komen hier in de methodiek in het geheel niet naar voren. De bepaling van de limiet is een zuiver algebraïsche zaak, de topologische structuur wordt beslist niet duidelijk gemaakt, en als deze al expliciet er bij besproken wordt, domineert bij de leerlingen toch veelal de algebraïsche structuur als boven geschetst.

Gebruiken we bij limieten dan nooit echt de topologische structuur? Iets gecompliceerder is het bij

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Bij het bewijs gebruiken we, uit aanschouwelijke meetkunde afgeleid,

$$\cos x < \frac{\sin x}{x} < 1.$$

En hieruit deduceren we

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Merkwaardig, we gebruiken de ordeningsstructuur.

Zo op het eerste gezicht functioneert de topologische structuur van de reële getallen nauwelijks bij de limietbepalingen. De situatie bij het bepalen van de afgeleide van a^x voert ons tot

$$\lim_{h \to 0} \frac{a^h - 1}{h}$$

en deze vraagt echt iets meer inzicht in de topologische structuur. Samenvattende zou men concluderen, dat bij de schoolstof in de bovenbouw wel de topologische structuur naar voren komt, maar de behandeling van problemen meestal alleen algebraïsche of ordeningsstructuren benut.

Appendix B. Van Dormolen (1970:204)

$\lim_{x \to 2} (x+2)$. De leerlingen moeten leren inzien, dat deze opgave zich ook leent tot toepassing van de gegeven definitie, ook al is in dit geval het gebruikmaken van gereduceerde omgevingen overbodig. In beide gevallen zal de leerling zich intuïtief wensen te beroepen op continuïteitsoverwegingen. Het is daarom redelijk te pleiten voor een limietdefinitie die uitgaat van het continuïteitsbegrip.

7.2.1.3 Definitie met behulp van continuïteit

Als de functie f continu is in a, dan verstaat men onder de limiet van f voor x naar a het getal $f(a)$.

Als f niet continu is in a, terwijl er een functie f^* bestaat die wel continu is in a en buiten a geheel met f samenvalt, dan verstaat men onder de limiet van f voor x naar a het getal $f^*(a)$.

Voorbeeld.

$$f(x) = \frac{x^2-4}{x-2} \text{ voor elke } x \in \mathbb{R} \setminus \{2\};$$

$f(2)$ is niet gedefinieerd;

$$f^*(x) = x+2.$$

Hier is f^* continu in 2 en valt buiten 2 geheel met f samen.

Dus:

$$\lim_{x \to 2} f(x) = f^*(2).$$

Opmerking

Nu is ook te begrijpen waarom geldt:

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} (x+2).$$

f^* is immers continu in 2, dus per definitie is niet alleen $\lim_{x \to 2} f(x) = f^*(2)$, maar ook $\lim_{x \to 2} f^*(x) = f^*(2)$.

Een bezwaar van deze definitie is, dat zij veel langer is dan de vorige definities. Bovendien eist ze een grondige voorbereiding ten aanzien van het begrip continuiteit, maar aangezien de continuïteit zo'n grote rol speelt in de analyse, kan dit nauwelijks als bezwaar worden aangemerkt.

7.2.2 *Definities van continuïteit*

In het verleden werd er aan deze definities doorgaans weinig of geen aandacht besteed. Intuïtief kan men de leerlingen gemakkelijk duidelijk maken wat er met continuïteit wordt bedoeld. Het is echter bijzonder

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Appendix C. Vredenduin (1969:15-16)

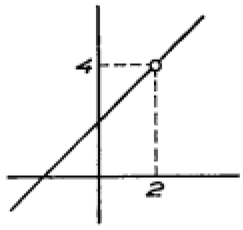
Nu de limieten. Als voorbeeld kiezen we de functie

$$f : x \to \frac{x^2 - 4}{x - 2}.$$

We merken op, dat voor $x \neq 2$ geldt

$$f(x) = x + 2$$

en dat voor $x = 2$ de functie niet gedefinieerd is. De grafiek is dus een rechte lijn minus een punt. De grafiek is getekend in figuur 17.



FIGUUR 17

Het ligt voor de hand het gat in de grafiek op te vullen door de functie voor $x = 2$ de waarde 4 toe te kennen. Beter gezegd: we ontwerpen een nieuwe functie, f^*, gedefinieerd door

$$f^* = f \text{ voor } x \neq 2$$
$$f^*(2) = 4.$$

De functie f^* is gedefinieerd voor $x = 2$ en de functiewaarde voor $x = 2$ is zodanig gekozen, dat f^* continu is.
We noemen nu het getal 4 de continumakende waarde voor $x = 2$ (voor f). Ook zeggen we wel, dat 4 de limiet van f is voor x nadert tot 2. We schrijven:

$$\lim_{x=2} f(x) = 4.$$

Hier stoten we op enkele moeilijkheden. Het vinden van de continumakende waarde en dus van de limiet heeft niets te maken met het laten veranderen van het origineel x. We laten x niet waarden doorlopen, die tot 2 naderen en kijken, wat er dan met de functiewaarde gebeurt. Het nemen van de limiet is geen dynamisch, maar een statisch proces. Vandaar, dat ik onder het limietteken schrijf '$x = 2$' en niet '$x \to 2$'. Te voren heb ik gezegd: 'Ook zeggen we wel, dat 4 de limiet van f is voor x nadert tot 2.' Ik heb dit opzettelijk volgens de traditie zo

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geformuleerd. Herleest u het, dan ziet u hoe onbruikbaar de traditie voor ons is. Wij moeten zeggen: '. . . dat 4 de limiet van f is voor $x = 2$.'
Er is nog iets. We spreken over de limiet van de functie. We schrijven '$\lim\limits_{x=2} f(x)$' en hier staat: de limiet van de functiewaarde. Ik geloof, dat we verstandig doen dit zo te blijven doen. Correcter zou zijn: $\lim\limits_{x=2} x \to f(x)$. Maar we kunnen het deel '$x \to$ ' zonder bezwaar weglaten, omdat het impliciet weergegeven wordt door het '$x =$' onder het limietteken.
We keren terug naar de functie

$$f : x \to \frac{x^2-4}{x-2}\,.$$

We vragen nu naar $\lim\limits_{x=3} f(x)$. We moeten dus een functie f^* ontwerpen, waarvoor geldt

$$f^* = f \text{ voor } x \neq 3$$

en die voor $x = 3$ een zodanige waarde heeft, dat hij continu is. Nu is de functie f reeds continu en gedefinieerd voor $x = 3$. Omdat $f(x) = 5$, kunnen we dus volstaan met te kiezen: $f^*(x) = 5$.
Dus: als een functie f continu is en reeds gedefinieerd voor $x = a$, dan is de limiet voor $x = a$ (de continumakende waarde) gelijk aan $f(a)$. Dit resultaat komt overeen met de klassieke definitie van een continue functie.
De leerling zal het nut van een limietdefinitie beter gaan inzien, zodra hij minder triviale gevallen onder ogen krijgt. Een voorbeeld hiervan is

$$\lim_{x=0} \frac{\sin x}{x}\,.$$

Het grote voordeel van de gegeven behandeling van continuïteit en van limiet is, dat voor de leerling deze moeilijke begrippen gemakkelijk en ook technisch gemakkelijk hanteerbaar gemaakt worden.
Blijven nog over de 'oneigenlijke' limieten, waarin x nadert tot oneindig of tot min-oneindig, of de functiewaarde nadert tot oneindig of min-oneindig. De officiële wiskunde kent fraaie methoden om het voorgaande betoog te generaliseren en zo tot een meer algemene limietdefinitie te komen, waaronder zowel de eigenlijke als de oneigenlijke limieten begrepen worden. Helaas zal het niet mogelijk zijn deze in wiskunde I te behandelen; de tijd zal ontbreken. Degene, die in wiskunde II als keuze-onderwerp topologie kiest, zal een meer algemene limietdefinitie met vrucht kunnen geven. Maar in wiskunde I zullen we ons tevreden moeten stellen met een intuïtieve behandeling van de oneigenlijke limieten.

References

Apostol, F.M. (1957, 1965), "Mathematical analysis", Addison-Wesley

Blij, F. van der (1970), "Structuren", in Wansink (1970), p117-129

Colignatus, Th. (1981 unpublished, 2007, 2011), "A logic of exceptions", http://thomascool.eu/Papers/ALOE/Index.html

Colignatus, Th. (2011), "Conquest of the Plane", http://thomascool.eu/Papers/COTP/Index.html (Check the reading notes.)

Colignatus, Th. (2011-2016), "Reading notes for COTP", http://thomascool.eu/Papers/COTP/2011-08-23-ReadingNotesForCOTP.pdf

Colignatus, Th. (2015), " Foundations of Mathematics. A Neoclassical Approach to Infinity", http://thomascool.eu/Papers/FMNAI/Index.html

Colignatus, Th. (2016a), "An algebraic approach to the derivative", http://thomascool.eu/Papers/Math/2016-08-14-An-algebraic-approach-to-the-derivative.pdf

Colignatus, Th. (2016b), "Breach of integrity of science by the editors of Nieuw Archief voor Wiskunde (NAW) w.r.t. the 2016 article about the algebraic approach to the derivative", http://thomascool.eu/Papers/Math/Derivative/2016-11-21-Breach-integrity-of-science-by-NAW-on-derivative.pdf

Colignatus, Th. (2016c), "Definition of continuity (written out)", https://boycottholland.wordpress.com/2016/11/28/definition-of-continuity-written-out

Colignatus, Th. (2016d), "Lawyers of space and number", https://boycottholland.wordpress.com/2016/11/29/lawyers-of-space-and-number

Dormolen, J. van (1970), "Problemen inzake het onderwijs in de analyse", in Wansink (1970).

Scheelbeek, P.A.J. & W. Verdenius (1973), "Dictaat Analyse", Mathematisch Instituut, RUG

Vredenduin, P.G.J. (1968), "Formele eigenschappen", Euclides, jaargang 43, no 10, p313-319, https://archief.vakbladeuclides.nl/bestanden/43_1967-68_10.pdf

Vredenduin, P.G.J. (1969), "Continuïteit", Euclides 45, no 1, p6-16, https://archief.vakbladeuclides.nl/bestanden/45_1969-70_01.pdf

Wansink, J.H. (1970), "Didactische oriëntatie voor wiskundeleraren III", Wolters-Noordhoff