Discussing a new way to conciliate large scale and small scale physics

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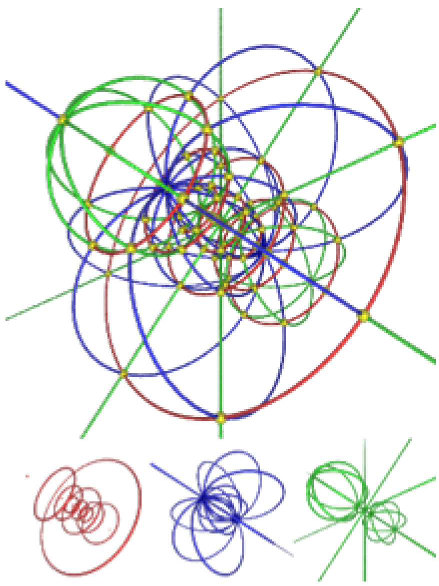
Content

Abstract

Interactions are produced, at small scale, by Lorentz transformations around extra dimensions. As a simple example, we include simultaneously a "Kaluza-Klein fifth dimension" and minimal coupling in Klein-Gordon equation applied to Hydrogen (all equations can be written in dimensionless form). Instead of solving the last separable equation for $\Psi(R)$, we require one more eigeinvalue equation, and require that the eccentricity of the system vanishes, to deduce the energy levels. With 4 spatial dimensions, there are naturally 6 rotations and 2 angular momenta (a classical one with parity+ and a spin with parity-). The SO(4) degeneracy and Schrodinger's energy levels are deduced, but the fine structure requires a modification : we give an example with a linear equation. We observe that the extra degree of freedom naturally disappears at classical scale (objects made of a large number of elementary particles).
We then observe that the quantum principle of minimal coupling (here produced by Lorentz transformations) is analogous to a modification of the metric inside the wave function. We use the corresponding metric (no coordinate singularity, the central one being naturally solved by the Lorentz transformation with extra dimensions) to describe gravitation : the deduced equation of motion reduces, in the low field approximation, to the equation given by general relativity.
More generally, extra dimensions may be usefull in particles physics : conservation of lepton numbers could be understood as conservation of momentum along other dimensions, and unconvenient divergences could be solved...



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Main results – Introduction

W. Pauli to W. Heisenberg (1936) :
"If, instead of Dirac's equation, one assumes as a basis the old scalar Klein-Gordon relativistic equation, it possesses the following properties: the charge density may be either positive or negative and the energy density is always ≥ 0, it can never be negative. This is exactly the opposite situation as in Dirac's theory and exactly what one wants to have. [...] Still am I happy to beat against my old enemy, the Dirac theory of the spinning electron"

We start from

$$E^2 = p^2c^2 + m_0^2c^4 \Leftrightarrow 0 = (iE)^2 + p^2c^2 + m_0^2c^4 \tag{1}$$

$$\Leftrightarrow 0 = (iE)^2 + p_x^2c^2 + p_y^2c^2 + p_z^2c^2 + m_0^2c^4 \tag{2}$$

$$\Leftrightarrow 0 = (iE)^2 + p_x^2c^2 + p_y^2c^2 + p_z^2c^2 + p_w^2c^2 + m_0^2c^4 \quad with \quad p_w^2 = 0 \tag{3}$$

and perform a rotation of angle $i\sigma$ (imaginary for attractive potential, real for repulsive potential)

$$iE \to iE' = iEcos(i\sigma) - p_w sin(i\sigma) = iEcos(i\sigma) = iEcosh(\sigma) \tag{4}$$

$$p_w(=0) \to p'_w = p_w cos(i\sigma) + iEsin(i\sigma) = iEsin(i\sigma) = -Esinh(\sigma) \neq 0 \; (real) \tag{5}$$

and now write, for interacting matter,

$$E^2cosh(\sigma)^2 = p_x^2c^2 + p_y^2c^2 + p_z^2c^2 + p_w'^2c^2 + m_0^2c^4 \quad with \quad p_w'^2 \neq 0 \tag{6}$$

Taking minimal coupling as a first ordrer approximation,

$$cosh(\sigma)^2 = (1 + \frac{k}{ER})^2 \quad , k = \frac{q^2}{4\pi\varepsilon_0}, R^2 = w'^2 + x^2 + y^2 + z^2 \; (w' \neq 0) \tag{7}$$

$$(E + \frac{k}{R})^2 = P^2c^2 + m_0^2c^4 \quad , P^2 = p_x^2 + p_y^2 + p_z^2 + p_w'^2 \tag{8}$$

$$(E + \frac{k}{R})^2\Psi = -\hbar^2c^2(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial w^2}) + m_0^2c^4\Psi \tag{9}$$

or in hyperspherical coordinates,

$$R^2(E + \frac{k}{R})^2\Psi = c^2(R^2\mathbf{P_R}^2 + \mathbf{J}^2)\Psi + R^2m_0^2c^4\Psi \tag{10}$$

where

$$R^2\mathbf{P_R}^2\Psi = -\hbar^2[R^2\frac{\partial^2\Psi}{\partial R^2} + 3R\frac{\partial\Psi}{\partial R}] \tag{11}$$

$$\mathbf{J}^2 = -\hbar^2[\frac{1}{sin^2\gamma}\frac{\partial}{\partial\gamma}(sin^2\gamma\frac{\partial\Psi}{\partial\gamma}) + \frac{1}{sin^2\gamma}((\frac{1}{sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Psi}{\partial\theta}) + \frac{1}{sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2})] \quad (12)$$

The six rotations (included in $\mathbf{J}^2$) in four dimensions are associated to the two following angular momenta

$$\vec{L} = \begin{bmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{bmatrix} \qquad \vec{K} = \begin{bmatrix} xp_w - wp_x \\ yp_w - wp_y \\ zp_w - wp_z \end{bmatrix} = \vec{r}p_w - w\vec{p} \quad (13)$$

$\vec{K}$ will be the spin of the theory identified with the classical Laplace-Runge-Lenz-Pauli vector. If we call $\mathbf{P_{xyz}}$ the parity operator $(x, y, z \rightarrow -x, -y, -z)$.

$$\mathbf{P_{xyz}}\vec{L} = \vec{L} \quad , \quad \mathbf{P_{xyz}}\vec{K} = -\vec{K} \quad (14)$$

Spin and angular momentum have opposite parity. Quaternions are not used in this paper but the previous vectors exhibit quaternion algebra, which is both the algebra of Pauli matrices and the intimate structure of the electromagnetic theory : Maxwell historically wrote his theory with quaternions. Quaternions obey anti commutation relations, like Dirac and Pauli matrices.

It can be shown (analytically and geometrically, two ways exist) that $\mathbf{J}^2\Psi = j(j+2)\hbar^2\Psi$ and since equation (9) contains 4 eigenvalue equations, we require, in hyperspherical coordinates, 4 eigenvalue equations :

$$\mathbf{L_z^2}\Psi = m^2\hbar^2\Psi; \mathbf{L^2}\Psi = l(l+1)\hbar^2\Psi; \mathbf{J^2}\Psi = j(j+2)\hbar^2\Psi; R^2\mathbf{P_R^2}\Psi = n^2\hbar^2\Psi \quad (15)$$

We justify n=1, leaving equation (10) without operators. Solving for R and requiring that the eccentricity of the system (discriminant for R) vanishes immediatly produces :

$$E_j = m_0c^2\sqrt{1-\alpha_j^2} \approx m_0c^2 - \frac{1}{2}m_0c^2\alpha_j^2 \quad (16)$$

which are Schrodinger-Pauli energy levels, $\alpha_j = k/(\hbar c(j+1))$.
This results comes with

$$R_j = (j+1)^2\frac{\sqrt{1-\alpha_j^2}\ \hbar}{\alpha m_0 c} \approx (j+1)^2\frac{\hbar}{\alpha m_0 c} \quad (Bohr\ radii) \quad (17)$$

and

$$P_j = m_0 c \frac{\alpha_j}{\sqrt{1-\alpha_j^2}} \quad , \quad p = m_0 c \frac{\beta}{\sqrt{1-\beta^2}} \quad (\beta = v/c) \tag{18}$$

Momentum of relativistic interacting electron — Momentum of relativistic free electron (for comparison)

Everything is there for a non relativistic Hydrogen, if we replace Klein-Gordon by Schrodinger equation. The fine structure requires to replace $j+1$ by $a + \sqrt{b^2 - \alpha^2}$ with a and b integers. We examine the question and give an example. Multiplyed by $(\hbar c)^{-2}$, equation (10) is dimensionless (natural).

If we look at a block of elementary interacting particles, with angles $\pm i\sigma$ and average their center of mass, the extra dimension disappears since $sinh(\sigma)$ is an odd function : it is then a property of rotations (details in the paper).

To introduce gravitation, we look at the connection between the coupling and the wave function, and observe :

$$E = \gamma m_0 c^2 \Longrightarrow E = \gamma m_0 c^2 + V \Leftrightarrow E - V = \gamma m_0 c^2 \tag{19}$$

$$E\Psi = \hbar \frac{\partial}{\partial t}\Psi \rightarrow (E-V)\Psi = E(1-\frac{V}{E})\Psi = \hbar \frac{\partial}{\partial t'}\Psi = \hbar \frac{\partial t}{\partial t'}\frac{\partial}{\partial t}\Psi \tag{20}$$

we identify $E(1-V/E)$ with $Ecosh(\sigma')$, and then

$$\frac{\partial t}{\partial t'} = (1-\frac{V}{E}) \Leftrightarrow t' = \frac{t}{1-\frac{V}{E}} = t(\frac{E}{E-V}) = t(1+\frac{V}{E-V}) = t(1+\frac{V}{\gamma m_0 c^2}) \tag{21}$$

Fixing, for gravitation (inertial mass $\leftrightarrow$ gravitationnal mass, or energy $\gamma m_0 c^2$ (mass+kinetic) $\leftrightarrow$ gravitationnal charge) :

$$V = -\frac{GM\gamma m_0}{r} = -\frac{GM\gamma m_0 c^2}{c^2 r} \tag{22}$$

produces the metric (without extra dimension for an usual object):

$$c^2 dt^2 (1 - \frac{GM}{c^2 r})^2 = dx^2 + dy^2 + dz^2 + ds^2 \tag{23}$$

with t as main parameter, the motion becomes, with $u = 1/r$,

$$(1-\frac{GMu}{c^2})^2 (\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2 - (1-\frac{GMu}{c^2})^2 ((\frac{m_0 c^2}{L_0})^2 + u^2) \tag{24}$$

Taking the low field approximation : $(1-\frac{GMu}{c^2})^2 \to (1-\frac{2GMu}{c^2})$,

$$(1-\frac{2GMu}{c^2})(\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2 - (1-\frac{2GMu}{c^2})((\frac{m_0c^2}{L_0})^2 + u^2) \quad (25)$$

This equation can be compared with the equation of motion in general relativity

$$(\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2 - (1-\frac{2GMu}{c^2})((\frac{m_0c^2}{L_0})^2 + u^2) \quad (26)$$

where we have the same roots for $du/d\phi = 0$ and then the same aphelion and perihelion. If we examine Mercury :

$G = 6.67384\ 10^{-11}, r = 59.91\ 10^9, M = 1.989\ 10^{30}$ and test with Microsoft Excel :
$\frac{2GM}{c^2r} \approx 4.6\ 10^{-8} << 1$ (Microsoft Excel 2007 gives :"$1 - 4.6\ 10^{-8} = 1$") , then MS Excel can not distinguish the two equations. We show that, with simply

$$E - V = \gamma m_0c^2 \Leftrightarrow E + \frac{GM\gamma m_0c^2}{c^2} = \gamma m_0c^2 \Leftrightarrow E = \gamma m_0c^2(1-\frac{GM}{c^2r}) \quad (27)$$

equation (24) can be deduced without the metric. Compared to Schwarzchild metric, our metric has no coordinate singularity (at $r = 2GM/c^2$) and the central singularity ($r = 0$) is solved at small scale by extra dimensions : the four dimensionnal radius R (and P) can not vanish since interaction means w (and p_w)$\neq 0$. Fixing $r = GM/c^2$ would be meaningless,

$$c^2dt^2(1-\frac{GM}{c^2r})^2 = dx^2 + dy^2 + dz^2 + ds^2 (= 0?) \quad (28)$$

but the theory explicitly uses minimal coupling as a low field (small angle) approximation, so divergences and inconsistences are solved, while quantum mechanic and gravitation are described from the simple principle : rotations in space-time with extra space dimensions. The angles σ and σ' are conventionnal, the theory is adaptable.

Detailed calculations

I – Atoms : Quantum Hydrogen with spin

The present approach can of course be included in Schrodinger equation. We start from Klein-Gordon equation with a "Kaluza-Klein fifth dimension" and minimal coupling :

$$(E + \frac{k}{R})^2\Psi = -\hbar^2 c^2(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial w^2}) + m_0^2 c^4\Psi \tag{29}$$

and now write this equation (multiplied by R^2) with the Laplacian in hyperspherical coordinates

$$R^2(E + \frac{k}{R})^2\Psi = c^2(R^2\mathbf{P_R}^2 + \mathbf{J}^2)\Psi + R^2 m_0^2 c^4\Psi \tag{30}$$

where

$$R^2\mathbf{P_R}^2\Psi = -\hbar^2[R^2\frac{\partial^2\Psi}{\partial R^2} + 3R\frac{\partial\Psi}{\partial R}] \tag{31}$$

$$\mathbf{J}^2 = -\hbar^2[\frac{1}{sin^2\gamma}\frac{\partial}{\partial\gamma}(sin^2\gamma\frac{\partial\Psi}{\partial\gamma}) + \frac{1}{sin^2\gamma}((\frac{1}{sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Psi}{\partial\theta}) + \frac{1}{sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2})] \tag{32}$$

The six rotations (included in $\mathbf{J}^2$) in four dimensions are associated to two following angular momenta :

$$\vec{L} = \begin{bmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{bmatrix} \qquad \vec{K} = \begin{bmatrix} xp_w - wp_x \\ yp_w - wp_y \\ zp_w - wp_z \end{bmatrix} = \vec{r}p_w - w\vec{p} \tag{33}$$

$\vec{K}$ will be the spin of the theory. If we call $\mathbf{P_{xyz}}$ the parity operator $(x, y, z \rightarrow -x, -y, -z)$, then

$$\mathbf{P_{xyz}}\vec{L} = \vec{L} \quad , \quad \mathbf{P_{xyz}}\vec{K} = -\vec{K} \tag{34}$$

Spin and angular momentum have opposite parity. Indeed, the Laplace-Runge-Lenz-Pauli vector K_L (called Laplace vector later), used by Pauli to deduce Schrodinger's energy levels (the same year) has the same algebra. Here we give its usual form and show the equivalence, with $k' = GMm_0$:

$$\vec{K_L} = \frac{1}{\sqrt{-2m_0\epsilon}}(\vec{p} \wedge \vec{L} - m_0 k'\frac{\vec{r}}{r}) \quad , \quad \epsilon \ classical \ energy < 0 \tag{35}$$

since

$$\vec{K_L} = \frac{1}{\sqrt{-2m_0\epsilon}}(m_0^2\vec{v}\wedge\vec{r}\wedge\vec{v} - m_0k'\frac{\vec{r}}{r}) \tag{36}$$

$$\vec{K_L} = \frac{1}{\sqrt{-2m_0\epsilon}}(m_0^2v^2 - m_0\frac{k'}{r})\vec{r} - m_0^2(\vec{v}.\vec{r})\vec{v} \tag{37}$$

$$\vec{K_L} = (\frac{m_0^2v^2 - m_0\frac{k'}{r}}{\sqrt{-2m_0\epsilon}})\vec{r} - (\frac{\vec{p}.\vec{r}}{\sqrt{-2m_0\epsilon}})\vec{p} \tag{38}$$

$$\vec{K_L} = (p_w)\vec{r} - (w)\vec{p} \tag{39}$$

where we can check that, as required,

$$\dot{w} = \frac{\dot{\vec{p}}.\vec{r} + \vec{p}.\dot{\vec{r}}}{\sqrt{-2m_0\epsilon}} = \frac{m_0v^2 - \frac{k'}{r}}{\sqrt{-2m_0\epsilon}} = \frac{p_w}{m_0} \tag{40}$$

We define the total angular momentum $\vec{J} = \vec{K} + \vec{L}$ with the norm

$$J^2 = K^2 + L^2 \tag{41}$$

since $\vec{K}.\vec{L} = 0$, if quaternions are not used (easy to check : by example the first term yp_zxp_w will be canceled with $-xp_zyp_w$). Looking at quantum operators, **K** and **L** are not independant since

$$[\mathbf{L_i}, \mathbf{L_j}] = \hbar\mathbf{L_k}\ ,\ [\mathbf{K_i}, \mathbf{K_j}] = \hbar\mathbf{L_k} \tag{42}$$

Following Pauli, we then define :

$$\mathbf{A} = (\mathbf{L}+\mathbf{K})/2\ ,\ \mathbf{B} = (\mathbf{L}-\mathbf{K})/2 \quad \rightarrow \quad [\mathbf{A_i}, \mathbf{A_j}] = \hbar\mathbf{A_k}\ ,\ [\mathbf{B_i}, \mathbf{B_j}] = \hbar\mathbf{B_k} \tag{43}$$

since $\mathbf{P_{xyz}}\vec{A} = \vec{B}$ and $\mathbf{P_{xyz}}\vec{B} = \vec{A}$, we deduce $|\vec{A}| = |\vec{B}|$.

It is known from the general theory of angular momenta (see D. J. Griffith, 1994 for a demonstration) that

$$\mathbf{A^2}\Psi = a(a+1)\hbar^2\Psi \tag{44}$$

with a integer or half integer, while

$$\mathbf{A^2}\Psi = |\frac{\mathbf{L}+\mathbf{K}}{2}|^2\Psi = \frac{\mathbf{J^2}}{4}\Psi \rightarrow \mathbf{J^2}\Psi = 2a(2a+2)\hbar^2\Psi = j(j+2)\hbar^2\Psi \tag{45}$$

with j integer. $\mathbf{J^2}\Psi = j(j+2)\hbar^2\Psi$ can be deduced by solving the equation analytically (done in annex), the corresponding solution being given by hyperspherical harmonics. The geometrical approach is simpler. Equation (30) then takes the form

$$R^2(E + \frac{k}{R})^2\Psi = (c^2R^2\mathbf{P_R^2} + j(j+2)\hbar^2c^2 + m_0^2c^4)\Psi \tag{46}$$

We will now proceed by analogy with Kepler problems. We recall (classical mechanic with $k' = GMm_0$) :

$$\epsilon = \frac{p^2}{2m_0} - \frac{k'}{r} = \frac{(\vec{p}.\vec{r})^2}{2m_0 r^2} + \frac{(\vec{p} \wedge \vec{r})^2}{2m_0 r^2} - \frac{k}{r} = \frac{p_r^2}{2m_0} + \frac{L^2}{2m_0 r^2} - \frac{k'}{r} \tag{47}$$

Aphelion ($r+$) and perihelion ($r-$) correspond classically to the geometrical configuration $p_r = m_0 \frac{dr}{dt} = 0$ and are given by

$$2m_0|\epsilon|r^2 - 2m_0 kr + L^2 = 0 \tag{48}$$

$$\rightarrow r_{\pm} = \frac{m_0 k'}{2m_0|\epsilon|} \pm \sqrt{(\frac{m_0 k'}{2m_0|\epsilon|})^2 - \frac{L^2}{2m_0|\epsilon|}} = \frac{m_0 k'}{2m_0|\epsilon|} \pm \sqrt{\Delta} \tag{49}$$

(Here, $m_0 k \rightarrow$ classical limit of Ek/c^2 and $2m_0|\epsilon| \rightarrow -(E^2 - m_0^2 c^4)/c^2$)

Circular motion corresponds to aphelion = perihelion ($r+ = r- \Leftrightarrow \Delta = 0$) or

$$-K_L^2 = L^2 + \frac{m_0 k'^2}{2\epsilon} = 0 \quad (\epsilon < 0) \tag{50}$$

It is well known that the norm of the Laplace vector determines the eccentricty of the orbit. To determine the energy levels of Hydrogen, we will require that the 4D eccentricity of the system vanishes. Going back to the wave equation :

$$R^2(E + \frac{k}{R})^2 \Psi = (c^2 R^2 \mathbf{P^2} + R^2 m_0^2 c^4)\Psi \tag{51}$$

$$R^2(E + \frac{k}{R})^2 \Psi = (c^2(R^2 \mathbf{P_R^2} + \mathbf{J^2}) + R^2 m_0^2 c^4)\Psi \tag{52}$$

with the equivalence between operators and vectors :

$$J^2 = R^2 P^2 - (\vec{R}.\vec{P})^2 \;\; ; \;\; \mathbf{J^2} = R^2 \mathbf{P^2} - R^2 \mathbf{P_R^2} \;\; (definition) \tag{53}$$

The right hand side of above is the definition of the Laplacian, we check the left handside :

$$J^2 = L^2 + K^2 \tag{54}$$

$$J^2 = |xp_y - yp_x|^2 + |yp_z - zp_y|^2 + |zp_x - xp_z|^2 + |xp_w - wp_w|^2 + |yp_w - wp_y|^2 + |zp_w - wp_z|^2 \tag{55}$$

$$J^2 = r^2 p^2 - (\vec{r}.\vec{p})^2 \quad + \quad |\vec{r} p_w - w\vec{p}|^2 \tag{56}$$

$$J^2 = r^2 p^2 - (\vec{r}.\vec{p})^2 + r^2 p_w^2 - 2wp_w \vec{r}.\vec{p} \; + w^2 p^2 \tag{57}$$

$$J^2 = r^2 P^2 - (\vec{r}.\vec{p})^2 - 2wp_w \vec{r}.\vec{p} \; + w^2 p^2 \tag{58}$$

$$J^2 = r^2 P^2 - (\vec{r}.\vec{p})^2 - 2wp_w \vec{r}.\vec{p} + w^2(P^2 - p_w^2) \tag{59}$$

$$J^2 = R^2 P^2 - (\vec{r}.\vec{p})^2 - 2wp_w \vec{r}.\vec{p} - w^2 p_w^2 \tag{60}$$

$$J^2 = R^2P^2 - (\vec{r}.\vec{p} + wp_w)^2 = R^2P^2 - (\vec{R}.\vec{P})^2 \tag{61}$$

And we note

$$R^2 = constant \Leftrightarrow R\dot{R} = 0 \Leftrightarrow \gamma m_0 R\dot{R} = \vec{R}.\vec{P} = \vec{r}.\vec{p} + wp_w = 0 \Leftrightarrow \vec{r}.\vec{p} = -wp_w \tag{62}$$

Going back to equation (46), instead of solving $\Psi(R)$ (like Schrodinger or Dirac), we observe that the Laplacian in cartesian coordinates gives 4 eigenvalue equations,

$$-\hbar^2(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial w^2}) \rightarrow \mathbf{p^2_{x_i}}\Psi = \alpha_{x_i}\hbar^2\Psi \ , \ x_i = x, y, z \ or \ w \tag{63}$$

and require, in hyperspherical coordinates, 4 eigenvalue equations :

$$\mathbf{L^2_z}\Psi = m^2\hbar^2\Psi; \mathbf{L^2}\Psi = l(l+1)\hbar^2\Psi; \mathbf{J^2}\Psi = j(j+2)\hbar^2\Psi; R^2\mathbf{P^2_R}\Psi = n^2\hbar^2\Psi \tag{64}$$

where n is to be determined. The analogy with Kepler problems suggest $n = 0$ but this value is forbidden since interaction produced from Lorentz transformation implies

$$R \neq 0, P \neq 0 \ while \ R^2\mathbf{P^2_R} = R^2\mathbf{P^2} \ if \ \mathbf{J^2} = 0 \tag{65}$$

$$R^2\mathbf{P^2_R}\Psi = -\hbar^2(R^2\frac{\partial^2}{\partial R^2} + 3R\frac{\partial}{\partial R})\Psi = n^2\hbar^2\Psi \tag{66}$$

has for solutions

$$C_1R^{\alpha-} \ and \ C_2R^{\alpha+} \ with \ C_1 \ and \ C_2 \ constants \ and \ \alpha\pm = -1 \pm \sqrt{1-n^2} \tag{67}$$

Restricting the theory to rotations we require $\alpha+ = \alpha- \Rightarrow n = 1$. Equation (46) can now be written without operators

$$R^2(E + \frac{k}{R})^2\Psi = c^2(1\hbar^2 + j(j+2)\hbar^2)\Psi + R^2m_0^2c^4\Psi \tag{68}$$

$$[(E^2 - m_o^2c^4)R^2 + 2EkR + k^2 - (j+1)^2\hbar^2c^2]\Psi = 0 \tag{69}$$

with solutions $R\pm$ (where $E < m_0c^2$):

$$R_\pm = -\frac{Ek}{E^2 - m_0^2c^4} \pm \frac{\sqrt{E^2k^2 - (E^2 - m_0^2c^4)(k^2 - (j+1)^2\hbar^2c^2)}}{E^2 - m_0^2c^4} \tag{70}$$

$$R_\pm = -\frac{Ek}{E^2 - m_0^2c^4} \pm \sqrt{\Delta} \tag{71}$$

We can then require that the eccentricity vanishes :

$$E^2k^2 - (E^2 - m_0^2c^4)(k^2 - (j+1)^2\hbar^2c^2) = 0 \tag{72}$$

$$E^2(j+1)^2\hbar^2c^2 = m_0^2c^4((j+1)^2\hbar^2c^2 - k^2) = 0 \tag{73}$$

$$E_j = m_0c^2\sqrt{1-\alpha_j^2}, \quad \alpha_j = k/(\hbar c(j+1)) \tag{74}$$

The 4D radius R then takes the values :

$$R_j = \frac{-E_j k}{E_j^2 - m_0^2 c^4} = -\frac{m_0 c^2 \sqrt{1-\alpha_j^2}\ \alpha \hbar c}{(1-\alpha_j^2 - 1) m_0^2 c^4} \tag{75}$$

$$R_j = (j+1)^2 \frac{\sqrt{1-\alpha_j^2}\ \hbar}{\alpha m_0 c} \approx (j+1)^2 \frac{\hbar}{\alpha m_0 c} \quad (Bohr\ radii) \tag{76}$$

From

$$R^2 \mathbf{P}^2 \Psi = (R^2 \mathbf{P_R}^2 + \mathbf{J}^2)\Psi = (j+1)^2 \hbar^2 \Psi \tag{77}$$

we deduce

$$\mathbf{P_j}\Psi = (j+1)\hbar / R_j \Psi \tag{78}$$

rewritten as

$$P_j = m_0 c \frac{\alpha_j}{\sqrt{1-\alpha_j^2}} \qquad , \qquad p = m_0 c \frac{\beta}{\sqrt{1-\beta^2}} \quad (\beta = v/c) \tag{79}$$

Momentum of relativistic interacting electron — Momentum of relativistic free electron (for comparison)

This approach then naturally produces a spin and Schrodinger's energy levels in the classical limit (but not the fine structure). We now verify that the required degeneracy is produced (since the value of n is constrained)

We show in annexe by an analytical approach that : $j(j+2) > l(l+1) > m^2$ with j, l, m integers, the value $j = -1$ being obsiously forbidden. This is justified geometrically here. The fact that m is an integer can be deduced without knowing Ψ (complex or real) $\rightarrow$ if $\Psi(\phi)$ solution of

$$\mathbf{Lz}^2 \Psi(\phi) = -\hbar^2 \frac{\partial^2}{\partial \phi^2} \Psi(\phi) = \hbar^2 \Psi(\phi) \tag{80}$$

$$\Psi'(\phi) = \Psi(m\phi)\ solution\ of\ \mathbf{Lz}^2 \Psi' = m^2 \hbar^2 \Psi' \tag{81}$$

Our geometry requires $\Psi'(\phi) = \Psi'(\phi + 2\pi) \rightarrow m$ is an integer, and if m integer, l is an integer (general theory of angular momentum). We previously justified that j is an integer.

We take the time to observe that if $x, y, z, w \neq 0$

$$\vec{L} = \begin{bmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{bmatrix} = \begin{bmatrix} yz(p_z/z - p_y/y) \\ zx(p_x/x - p_z/z) \\ xy(p_y/y - p_x/x) \end{bmatrix} \vec{K} = \begin{bmatrix} xp_w - wp_x \\ yp_w - wp_y \\ zp_w - wp_z \end{bmatrix} = \begin{bmatrix} xw(p_w/w - p_x/x) \\ yw(p_w/w - p_y/y) \\ zw(p_w/w - p_z/z) \end{bmatrix} \tag{82}$$

$L_z \neq\ 0 \Rightarrow p_y/y \neq p_x/x \Rightarrow L^2 > L_z^2$ and $\vec{K} \neq \vec{0}$
and then , $J^2 = L^2 + K^2 > L^2\ \ \rightarrow J^2 > L^2 > L_z^2$ (no equalities allowed)

Examining the degeneracy, it is known (and easily verified) that for a given value of $l(l+1)$, there are $2l+1$ possible values for m.
For a given value of $l(l+1)$, there 2 possible values of l (this point justifies that our degeneracy is twice the one obtained by Schrodinger, since solving the wave function $\Psi(R)$ with his approach requires $l > 0$). The degeneracy is then

$$\sum_{l=0}^{j} 2(2l+1) = 2(j+1) + 4\sum_{l=0}^{j} l = 2(j+1) + 2j(j+1) = 2(j+1)^2 \qquad (83)$$

If $j = 0$ there are two states $|j, l, m>$: $|0, 0, 0>$ and $|0, -1, 0>$
With an other equation (or an other interpretation, see later), the degeneracy could be fixed differently, with $n = 0$, $n = j$ or $l(l+1) = j(j+1) \rightarrow K^2$ minimal, by example.

The previous approach did not produces the fine structure, in our opinion, for a geometrical reason. Considering a 3D relativistic Kepler problem, with (x, y) as plane of the motion, $(z = 0 \rightarrow \theta = atan(\sqrt{x^2+y^2}/z) = atan(r/z) = \pi/2)$ we write

$$E = \gamma m_0 c^2 - \frac{k}{r} \Leftrightarrow (E + \frac{k}{r})^2 = p_r^2 c^2 + \frac{L_z^2 c^2}{r^2} + m_o^2 c^4 \qquad (84)$$

This equation is the one used by Sommerfeld to deduce the required energy levels and it can be written in a dimensionless form (the previous equation multiplied by : $\hbar^{-2} c^{-2} r^2$)

$$\hbar^{-2} c^{-2} [r^2(E^2 - m_o^2 c^2) + 2Ekr] = \hbar^{-2}[r^2 p_r^2 + L_z^2 - (k/c)^2] \qquad (85)$$

Obviously, if one form had to be natural, it would be the dimensionless one. It appears from it that $r \neq 0 \Leftrightarrow L_z^2 - \alpha^2\hbar^2 \neq 0$: all structures we observe around us (planetary motion, galaxies...) have angular momentum. The fact that we work on a plane could be linked to a stereographic projection, often used to describe the Hydrogen atom. When we describe the motion of a particle obeying equation (85), we use the polar coordinates (r, ϕ). Indeed, surppressing θ, $\Psi(\theta, \phi, \gamma)$ becomes $\Psi(\phi, \gamma)$ or $\Psi(\phi, r)$ with $r = Rsin\gamma$. If one computes $r(\phi)$ from equation (85) one will find :

$$u = \frac{1}{r} = A + Bcos(\sqrt{1 - (\frac{k}{L_z c})^2}\phi) \qquad (86)$$

Relativity produces a shift of the perihelion, in the physical plane (x, y). The equation is usually solved for $u = 1/r$. This is linked to the Fourier transform : u is analogous to a momentum the same way that $\nu = 1/T$ is linked to energy $\hbar\nu$. Solving for the Hydrogen analytically, it is assumed by Schrodinger and Dirac that (in 3D) Ψ is seperable

$$\Psi(r, \theta, \phi) = \Re(r)\Theta(\theta), \Phi(\phi) \tag{87}$$

which is required if Ψ has a Fourier transform. And indeed, energy and momentum are the variables of the Fourier tranform. Then space could not exist without momentum, time without energy, and vice versa, since they are conjugate variables, through the Fourier transform.

With $k/c = \alpha\hbar$ and $\vec{p_r}.\vec{L}z = 0$ ($\vec{L}$ being orthogonal to the plane of the motion) we can now right :

$$\hbar^{-2}c^{-2}[r^2(E^2 - m_o^2c^4) + 2Ekr] = \hbar^{-2}|r\vec{p_r} + \sqrt{1 - (\frac{\alpha\hbar}{L_z})^2}\vec{L_z}|^2 \tag{88}$$

Taking the square root :

$$\hbar^{-1}c^{-1}\sqrt{r^2(E^2 - m_0^2c^4) + 2Ekr} = \hbar^{-2}|r\vec{p_r} + \sqrt{1 - (\frac{\alpha\hbar}{L_z})^2}\vec{L_z}| \tag{89}$$

It is here temptating, with this linear form, to require

$$[r\mathbf{P_r} - \sqrt{1 - (\frac{\alpha\hbar}{m})^2}\mathbf{L_z}]\Psi(r, \phi) = -ir\frac{\partial\Psi}{\partial r} - i\sqrt{1 - (\frac{\alpha\hbar}{L_z})^2}\frac{\partial\Psi}{\partial\phi} \tag{90}$$

$$r\mathbf{P_r} - \sqrt{1 - (\frac{\alpha\hbar}{m})^2}\mathbf{L_z}\Psi(r, \phi) = (n' + \sqrt{m^2 - \alpha^2})\Psi \rightarrow \Psi = e^{i(nln(r)+m\phi)} \tag{91}$$

and taking the square would then gives

$$(\hbar^{-2}c^{-2}r^2(E^2 - m_0^2c^2) + \hbar^{-1}c^{-1}2E\alpha r)\Psi = (n' + \sqrt{m^2 - \alpha^2})^2\Psi \tag{92}$$

$(j + 1)$ has been replaced by $(n' + \sqrt{m^2 - \alpha^2})$. Solving for r,

$$r\pm = \frac{-\hbar cE\alpha}{E^2 - m_0^2c^4} \pm \Delta \tag{93}$$

$$\rightarrow \Delta = 0 = \frac{\sqrt{E^2\alpha^2 - (E^2 - m_0^2c^4)(n' + \sqrt{m^2 - \alpha^2})^2}}{E^2 - m_0^2c^4} \tag{94}$$

$$E^2(\alpha^2 + n'^2 + 2n'\sqrt{m^2 - \alpha^2} + m^2 + \alpha^2) = m_0^2c^4(n'^2 + 2n'\sqrt{m^2 - \alpha^2} + m^2 + \alpha^2) \tag{95}$$

$$E^2 = m_0^2c^4\frac{(n'^2 + 2n\sqrt{m^2 - \alpha^2} + (m)^2 - \alpha^2)}{(n'^2 + 2n'\sqrt{m^2 - \alpha^2} + m^2)} \tag{96}$$

$$E^2 = \frac{m_0^2c^4}{\frac{(n'^2+2n\sqrt{m^2-\alpha^2}+m^2)}{(n'^2+2n'\sqrt{m^2-\alpha^2}+m^2-\alpha^2}) } \tag{97}$$

$$E = \frac{m_0c^2}{\sqrt{1+\frac{\alpha^2}{(n'+\sqrt{m^2-\alpha^2})^2}}}(\ Dirac - Sommerfeld\ energy\ levels) \tag{98}$$

Here m^2 should be an integer > 0 since the geometrical (natural) equation gives $L_z^2 - \alpha^2 \neq 0$ if $r \neq 0$. A good suggestion for a stereographic projection, is to rewrite the Laplacian :

$$(E + \frac{k}{R})^2\Psi = c^2(\mathbf{P_R}^2 + \frac{\mathbf{J}^2)}{R^2}\Psi + m_0^2c^4\Psi \tag{99}$$

where

$$\mathbf{P_R}^2\Psi = -\hbar^2[\frac{\partial^2\Psi}{\partial R^2} + \frac{3}{R}\frac{\partial\Psi}{\partial R}] \tag{100}$$

$$\frac{\mathbf{J}^2}{R^2} = -\frac{\hbar^2}{R^2}[\frac{1}{sin^2\gamma}\frac{\partial}{\partial\gamma}(sin^2\gamma\frac{\partial\Psi}{\partial\gamma}) + \frac{1}{sin^2\gamma}((\frac{1}{sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Psi}{\partial\theta}) + \frac{1}{sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2})] \tag{101}$$

and to observe that, in spherical coordinates $R^2sin^2\gamma = r^2 = x^2 + y^2 + z^2$. All rotations are included in three dimensions. But $\mathbf{P_R}^2$ is not contained in three dimensions, so $(j+1)$ is a 4D value. From a 3D point of view, $(j+1)$ should be replaced to take this point into account. Wondering why all planets and galaxies are oganized in rotating systems, we believe that a good theory of the Hydrogen atom should give the answer : it had to be the case. We hope we have convinced the reader that Dirac equation is not the only possible approach for Hydrogen. We have in mind that Dirac matrices could be replaced by quaternions, by example.

II – From the quantum world to the human scale

We start from a free system and perform a rotation of angle $\pm i\sigma$ in the energy-momentum space :

$$0 = (iE)^2 + p_x^2c^2 + p_y^2c^2 + p_z^2c^2 + m_0^2c^4 \tag{102}$$

$$\Leftrightarrow 0 = (iE)^2 + p_x^2c^2 + p_y^2c^2 + p_z^2c^2 + p_w^2c^2 + m_0^2c^4 \quad with \quad p_w^2 = 0 \tag{103}$$

$$iE \to iE' = iEcos(\pm i\sigma) + p_wsin(\pm i\sigma) = iEcos(\pm i\sigma) = iEcosh(\sigma) \tag{104}$$

$$p_w(=0) \to p'_w = p_w cos(\pm i\sigma) + iE sin(\pm i\sigma) = iE sin(i \pm \sigma) = \mp E sinh(\pm\sigma) \neq 0 \, (real) \tag{105}$$

$$E^2 cosh(\sigma)^2 = p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + p_w'^2 c^2 + m_0^2 c^4 \quad with \quad p_w'^2 \neq 0 \tag{106}$$

A similar structure happens in the usual space. Then, if, *from our point of view*, we compute the center of gravity of an usual object (made of an enormous amount of elementary particles), the averaging will suppress the extra degree of freedom producing the SO(4) symmetry (because of the ± sign), which remains detectable only at the elementary scale. This is a property of rotations. We justify the use of "from our point of view" : Schrodinger considered matter as being pure configuration, Pauli believed that we can not understand matter without understanding our mind studying it, and Einstein thought space and matter (and time) are intrinsically connected (one could not exist without the other(s)). We believe that all this is connected to Ψ. We now examine how the classical theory manifests the initial symmetry :
with,

$$cosh(\sigma)^2 = (1 + \frac{k}{ER})^2 \quad , k = \frac{q^2}{4\pi\varepsilon_0}, R^2 = w'^2 + x^2 + y^2 + z^2 \tag{107}$$

$$(E + \frac{k}{R})^2 = P^2 c^2 + m_0^2 c^4 \quad , P^2 = p_x^2 + p_y^2 + p_z^2 + p_w'^2 \tag{108}$$

If one then suppress the extra degree of freedom (considering an usual object), the previous equation becomes

$$(E + \frac{k}{r})^2 = p^2 c^2 + m_0^2 c^4 \tag{109}$$

We see that k/r modifies our initial symmetry of equation (102). Taking the classical limit :

$$2m_0\epsilon = p_r^2 + \frac{L^2}{r^2} - 2m_0\frac{k}{r} \tag{110}$$

$$2m_0\epsilon = \frac{(\vec{r}.\vec{p})^2}{r^2} + \frac{L^2}{r^2} - 2m_0\frac{k}{r} \tag{111}$$

$$r^2 2m_0\epsilon - (\vec{r}.\vec{p})` + 2m_0 kr = L^2 \tag{112}$$

$$r^2 - \frac{(\vec{r}.\vec{p})^2}{2m_0\epsilon} + \frac{kr}{\epsilon} = \frac{L^2}{2m_0\epsilon} \tag{113}$$

$$r^2 - \frac{(\vec{r}.\vec{p})^2}{2m_0\epsilon} + \frac{kr}{\epsilon} = \frac{L^2}{2m_0\epsilon} \tag{114}$$

$$x^2 + y^2 + z^2 + w^2 + \frac{kr}{\epsilon} = \frac{L^2}{2m_0\epsilon} < 0 \; (since \; \epsilon < 0) \tag{115}$$

We notice here the connection with our previous considerations : we used an imaginary angle $i\sigma$ to describe an attractive potential, and when the extra degree of freedom disappears for usual objects, we then obtain, in the classical theory, the equation of a sphere of imaginary radius. The radius would be positive if the angle was real (and then the potential would be repulsive).

III – Gravitation :

We first observe that the quantum principle of minimal coupling (without extra-dimensions for usual objects) corresponds to (calling E the total energy of the system and V the interaction potential, assumed to depend on x, y, z but assumed to not depend on t explicitly):

$$E = \gamma m_o c^2 \Longrightarrow E = \gamma m_0 c^2 + V \Leftrightarrow E - V = \gamma m_0 c^2 \tag{116}$$

$$E\Psi = \hbar \frac{\partial}{\partial t}\Psi \rightarrow (E - V)\Psi = E(1 - \frac{V}{E})\Psi = \hbar \frac{\partial}{\partial t'}\Psi = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t}\Psi \tag{117}$$

with

$$\frac{\partial t}{\partial t'} = (1 - \frac{V}{E}) \Leftrightarrow t' = \frac{t}{1 - \frac{V}{E}} = t(\frac{E}{E - V}) = t(1 + \frac{V}{E - V}) = t(1 + \frac{V}{\gamma m_0 c^2}) \tag{118}$$

Fixing, for gravitation (inertial mass $\leftrightarrow$ gravitationnal mass, or energy $\gamma m_0 c^2$ (mass+kinetic) $\leftrightarrow$ gravitationnal charge) :

$$V = -\frac{GM\gamma m_0}{r} = -\frac{GM\gamma m_0 c^2}{c^2 r} \tag{119}$$

produces the metric :

$$c^2 dt^2 (1 - \frac{GM}{c^2 r})^2 = dx^2 + dy^2 + dz^2 + ds^2 \tag{120}$$

With t as differentation parameter, we write, as prelimaries :

$$Ecosh(\sigma') = E - V = \gamma m_0 c^2 \Leftrightarrow E + \frac{GM\gamma m_0 c^2}{c^2} = \gamma m_0 c^2 \tag{121}$$

$$E = \gamma m_0 c^2 (1 - \frac{GM}{c^2 r}) \Leftrightarrow \gamma = \frac{E}{m_o c^2 (1 - \frac{GM}{c^2 r})} \tag{122}$$

$$\frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2} = 1 - (\frac{dr}{dct})^2 - r^2 (\frac{d\phi}{dct})^2 \tag{123}$$

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{p}\wedge\vec{r})}{dt} = \frac{d\vec{p}}{dt}\wedge\vec{r} + \frac{d\vec{r}}{dt}\wedge\vec{p} = \vec{F}\wedge\vec{r} = \gamma m_0\vec{v}\wedge\vec{v} = 0 \tag{124}$$

using (122) and the definition of $(\vec{p}\wedge\vec{r})$

$$L = \gamma m_0 r^2\frac{d\phi}{dt} = \frac{E}{c^2(1-\frac{GM}{c^2r})}r^2\frac{d\phi}{dt} \tag{125}$$

$$d\phi = \frac{Lc^2}{E}(1-\frac{GM}{c^2r})\frac{1}{r^2}dt \Leftrightarrow r^2\frac{d\phi}{dt} = \frac{Lc^2}{E}(1-\frac{GM}{c^2r}) \tag{126}$$

And we now start, in spherical coordinates with $\theta = \pi/2$:

$$ds^2 = (1-\frac{GM}{c^2r})^2c^2dt^2 - dr^2 - r^2d\phi^2 \Leftrightarrow \frac{ds^2}{c^2dt^2} = (1-\frac{GM}{c^2r})^2 - 1 + 1 - (\frac{dr^2}{c^2dt^2}) - r^2(\frac{d\phi^2}{c^2dt^2}) \tag{127}$$

and using (123) and (122),

$$\frac{ds^2}{c^2dt^2} = (1-\frac{GM}{c^2r})^2 - 1 + (\frac{m_oc^2}{E})^2(1-\frac{GM}{c^2r})^2 \tag{128}$$

From (127, left hand side),

$$dr^2 = [(1-\frac{GM}{c^2r})^2 - \frac{ds^2}{c^2dt^2} - r^2\frac{d\phi^2}{c^2dt^2}]c^2dt^2 \tag{129}$$

$$dr^2 = [(1-\frac{GM}{c^2r})^2 - \frac{ds^2}{c^2dt^2} - \frac{1}{r^2c^2}(r^2\frac{d\phi}{dt})^2]c^2dt^2 \tag{130}$$

using now (128) and (126)

$$dr^2 = [(1-\frac{GM}{c^2r})^2 + 1 - (1-\frac{GM}{c^2r})^2 - (\frac{m_oc^2}{E})^2(1-\frac{GM}{c^2r})^2 - \frac{1}{r^2c^2}(\frac{Lc^2}{E})^2(1-\frac{GM}{c^2r})^2]c^2dt^2 \tag{131}$$

$$dr^2 = [1-(\frac{m_oc^2}{E})^2(1-\frac{GM}{c^2r})^2 - \frac{1}{r^2c^2}(\frac{Lc^2}{E})^2(1-\frac{GM}{c^2r})^2]c^2dt^2 \tag{132}$$

using (126),

$$\frac{dr^2}{d\phi^2} = \frac{[1-(\frac{m_oc^2}{E})^2(1-\frac{GM}{c^2r})^2 - \frac{1}{r^2c^2}(\frac{Lc}{E})^2(1-\frac{GM}{c^2r})^2]c^2dt^2}{[(\frac{Lc^2}{E})^2(1-\frac{GM}{c^2r})^2\frac{1}{r^4c^2}]c^2dt^2} \tag{133}$$

$$\frac{dr^2}{d\phi^2} = r^4c^2(\frac{E}{Lc^2})^2\frac{1}{(1-\frac{GM}{c^2r})^2} - r^4c^2(\frac{m_0}{L})^2 - r^2 \tag{134}$$

$$\frac{dr^2}{r^4d\phi^2} = (\frac{E}{Lc})^2\frac{1}{(1-\frac{GM}{c^2r})^2} - c^2(\frac{m_0}{L})^2 - \frac{1}{r^2} \tag{135}$$

$$(1-\frac{GM}{c^2r})^2(\frac{dr}{r^2d\phi})^2 = (\frac{E}{Lc})^2 - (1-\frac{GM}{c^2r})^2((\frac{m_0c}{L})^2 + \frac{1}{r^2}) \tag{136}$$

with

$$u = \frac{1}{r} \quad ; \quad (\frac{du}{d\phi})^2 = (-\frac{\frac{dr}{d\phi}}{r^2})^2 = (\frac{dr}{r^2 d\phi})^2 \tag{137}$$

$$(1 - \frac{GMu}{c^2})^2 (\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2 - (1 - \frac{GMu}{c^2})^2 ((\frac{m_0 c^2}{L})^2 + u^2) \tag{138}$$

Taking the low field approximation : $(1 - \frac{GMu}{c^2})^2 \rightarrow (1 - \frac{2GMu}{c^2})$,

$$(1 - \frac{2GMu}{c^2})(\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2 - (1 - \frac{2GMu}{c^2})((\frac{m_0 c^2}{L})^2 + u^2) \tag{139}$$

This equation can be compared with the equation of motion in general relativity (See Magnan reproducing Wheeler and Weinberg for an example)

$$(\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2 - (1 - \frac{2GMu}{c^2})((\frac{m_0 c^2}{L})^2 + u^2) \tag{140}$$

where we have the three same roots for $du/d\phi = 0$ and then the same aphelion and perihelion. If we examine Mercury :

$G = 6.67384\ 10^{-11}, r = 59.91\ 10^9, M = 1.989\ 10^{30}$ and test with Microsoft Excel :
$\frac{2GM}{c^2 r} \approx 4.6\ 10^{-8} << 1$ (Microsoft Excel 2007 gives :"$1 - 4.6\ 10^{-8} = 1$") , then MS Excel can not distinguish the two equations.

Compared to Schwarzchild metric, our metric has no coordinate singularity (at $r = 2GM/c^2$) and the central singularity ($r = 0$) is solved at small scale by extra dimensions. Our metric has been deduced from quantum mechanic, such that, in this discussion, small scale and large scale physics are highly connected. We could say that "*it is the same thing, at an other scale*", both scales being easily adaptable in this theory (to energy levels or motion) : the angles of interaction σ and σ' are conventionnal (first order approximations).
We now show that our definition of energy was sufficient to deduce the equation of motion (137), without the use of ds^2. From

$$E - V = \gamma m_0 c^2 \Leftrightarrow E + \frac{GM\gamma m_0 c^2}{c^2} = \gamma m_0 c^2 \Leftrightarrow E = \gamma m_0 c^2 (1 - \frac{GM}{c^2 r}) \tag{141}$$

we deduce :

$$(\frac{E}{\gamma m_0 c^2})^2 = (1 - \frac{v^2}{c^2})(\frac{E}{m_0 c^2})^2 = (\frac{E}{m_0 c^2})^2 [1 - (\frac{dr^2}{c^2 dt^2}) - r^2 (\frac{d\phi^2}{c^2 dt^2})] = (1 - \frac{GM}{c^2 r})^2 \tag{142}$$

from which we deduce

$$r^2 (\frac{d\phi^2}{c^2 dt^2}) = 1 - (\frac{m_0 c^2}{E})^2 (1 - \frac{GM}{c^2 r})^2 - (\frac{dr^2}{c^2 dt^2}) \tag{143}$$

Then with (141) and the definition of $|\vec{L}| = |\vec{p} \wedge \vec{r}| = |\gamma m_0 r^2 (\frac{d\phi}{dt})|$

$$r^2 \frac{d\phi}{dt} = \frac{L}{\gamma m_0} = \frac{Lc^2}{E}(1 - \frac{GM}{c^2 r}) \Leftrightarrow r^2(\frac{d\phi^2}{c^2 dt^2}) = \frac{1}{r^2 c^2}(r^4 \frac{d\phi^2}{dt^2}) = (\frac{Lc}{rE})^2 (1 - \frac{GM}{c^2 r})^2 \tag{144}$$

we rewrite (142)

$$[1 - (\frac{Lc}{rE})^2 (1 - \frac{GM}{c^2 r})^2 - (\frac{dr^2}{c^2 dt^2})](\frac{E}{m_0 c^2})^2 = (1 - \frac{GM}{c^2 r})^2 \tag{145}$$

$$[1 - (\frac{Lc}{rE})^2 (1 - \frac{GM}{c^2 r})^2 - (\frac{dr^2}{c^2 dt^2})] = (1 - \frac{GM}{c^2 r})^2 (\frac{m_0 c^2}{E})^2 \tag{146}$$

$$1 - (\frac{Lc}{rE})^2 (1 - \frac{GM}{c^2 r})^2 - (1 - \frac{GM}{c^2 r})(\frac{m_0 c^2}{E})^2 = (\frac{dr^2}{c^2 dt^2}) \tag{147}$$

$$[1 - (1 - \frac{GM}{c^2 r})^2 [(\frac{Lc}{rE})^2 + (\frac{m_0 c^2}{E})^2]] c^2 dt^2 = dr^2 \tag{148}$$

rewriting (143)

$$r^2 d\phi^2 = [1 - (\frac{m_0 c^2}{E})^2 (1 - \frac{GM}{c^2 r})^2] c^2 dt^2 - dr^2 \tag{149}$$

(148) and (149) give

$$(\frac{r d\phi}{dr})^2 = -1 + \frac{[1 - (\frac{m_0 c^2}{E})^2 (1 - \frac{GM}{c^2 r})^2]}{[1 - (1 - \frac{GM}{c^2 r})^2 [(\frac{Lc}{rE})^2 + (\frac{m_0 c^2}{E})^2]]} \tag{150}$$

$$(\frac{r d\phi}{dr})^2 = \frac{[-1 + (1 - \frac{GM}{c^2 r})^2 [(\frac{Lc}{rE})^2 + (\frac{m_0 c^2}{E})^2]] + [1 - (\frac{m_0 c^2}{E})^2 (1 - \frac{GM}{c^2 r})^2]}{[1 - (1 - \frac{GM}{c^2 r})^2 [(\frac{Lc}{rE})^2 + (\frac{m_0 c^2}{E})^2]]} \tag{151}$$

$$(\frac{r d\phi}{dr})^2 = \frac{[(1 - \frac{GM}{c^2 r})^2 [(\frac{Lc}{rE})^2]}{[1 - (1 - \frac{GM}{c^2 r})^2 [(\frac{Lc}{rE})^2 + (\frac{m_0 c^2}{E})^2]} \tag{152}$$

$$(\frac{dr}{r d\phi})^2 = \frac{[1 - (1 - \frac{GM}{c^2 r})^2 [(\frac{Lc}{rE})^2 + (\frac{m_0 c^2}{E})^2]}{[(1 - \frac{GM}{c^2 r})^2 [(\frac{Lc}{rE})^2]} \tag{153}$$

$$(\frac{dr}{r d\phi})^2 = \frac{[1 - (1 - \frac{GM}{c^2 r})^2 [(\frac{m_0 c^2}{E})^2]}{[(1 - \frac{GM}{c^2 r})^2 [(\frac{Lc}{rE})^2]} - 1 \tag{154}$$

$$(1 - \frac{GM}{c^2 r})^2 (\frac{dr}{r d\phi})^2 = \frac{[1 - (1 - \frac{GM}{c^2 r})^2 [(\frac{m_0 c^2}{E})^2]}{[(\frac{Lc}{rE})^2]} - (1 - \frac{GM}{c^2 r})^2 \tag{155}$$

$$(1 - \frac{GM}{c^2 r})^2 (\frac{dr}{r d\phi})^2 = [(\frac{rE}{Lc})^2 - (1 - \frac{GM}{c^2 r})^2 [(\frac{m_0 c r}{L})^2] - (1 - \frac{GM}{c^2 r})^2 \tag{156}$$

$$(1 - \frac{GM}{c^2 r})^2 (\frac{dr}{r d\phi})^2 = [(\frac{rE}{Lc})^2 - (1 - \frac{GM}{c^2 r})^2 [(\frac{m_0 c r}{L})^2 + 1] \tag{157}$$

We recognize (136). The equation of the motion can be adapted by modifying the definition of the coupling. The key point is that singularities are solved and a deep connection with quantum mechanic is suggested : it is the same thing. The reader may wonder what would happen if he/she goes in a region of space, next to a central mass, such that $r = GM/c^2$, since the metric we used is :

$$c^2dt^2(1-\frac{GM}{c^2r})^2 = dx^2+dy^2+dz^2+ds^2(=0?) \tag{158}$$

We recall that the theory we are discussing is a low field (or more precisely small angle) approximation. The coupling angle should be replaced by

$$E^2cosh(\sigma)^2 = (E+\frac{k}{R})^2\Psi = E^2(1+\frac{k}{ER})\Psi \rightarrow E^2(1+\frac{k}{ER}+\sum_{i=2}a_i(\frac{k}{ER})^i)^2\Psi \tag{159}$$

if one wants to describe strong field (large angle) physics.

If the reader believes this paper is interesting, thanks for sharing.

References

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Chapter 4.3 Angular momentum

Magnan's website or paper on Arxiv :
https://lacosmo.com/PrecessionOfMercury/index.html
https://arxiv.org/pdf/0712.3709v1.pdf

Others :
A good description of stereographic projection and relativistic Laplace-Runge-Lenz-Pauli vector can be found here :

G. Torres Del Castillo, The hydrogen atom via the four-dimensional spherical harmonics http://www.ejournal.unam.mx/rmf/no535/RMF005300512.pdf

Uri Ben-Yaacov, Laplace-Runge-Lenz symmetry in general rotationally symmetric systems
https://arxiv.org/pdf/1005.1817v2.pdf

Annexe1 – Hyperspherical harmonics, analytical method

We start from 4D Klein-Gordon equation with Minimal Coupling, and follow Schroedinger (1926) and Dirac (1928):

$$(E+\frac{k}{R})^2\Psi = -\hbar^2c^2(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}+\frac{\partial^2\Psi}{\partial w^2})+m_0^2c^4\Psi \tag{160}$$

$$\Psi(x,y,z,w)=\Psi(R,\theta,\phi,\gamma)=\Re(R)\Theta(\theta),\Phi(\phi)\Gamma(\gamma), \tag{161}$$

$$\begin{aligned}(E+\frac{k}{R})^2\Psi = -\hbar^2c^2[&\frac{\partial^2\Psi}{\partial R^2}+\frac{3}{R}\frac{\partial\Psi}{\partial R}+\frac{1}{R^2}\frac{1}{sin^2\gamma}\frac{\partial}{\partial\gamma}(sin^2\gamma\frac{\partial\Psi}{\partial\gamma})\\ &+\frac{1}{R^2sin^2\gamma}(\frac{1}{sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Psi}{\partial\theta})\\ &+\frac{1}{R^2sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2})]+m_0^2c^4\Psi\end{aligned}$$

Multiplying by R^2/Ψ gives (with separation of variables)

$$\begin{aligned}R^2[(E+\frac{k}{R})^2-m_0^2c^4]=-\hbar^2c^2[&R^2\frac{\partial^2\Re}{\partial R^2}+\frac{3\ Re}{Re}\frac{\partial\Psi}{\partial R}+\frac{1}{\Gamma}\frac{1}{sin^2\gamma}\frac{\partial}{\partial\gamma}(sin^2\gamma\frac{\partial\Gamma}{\partial\gamma})\\ &+\frac{1}{sin^2\gamma}((\frac{1}{\Theta sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Theta}{\partial\theta})+\frac{1}{\Phi sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2})]\end{aligned}$$

$$\begin{aligned}\hbar^2c^2R^2[(E+\frac{k}{R})^2-m_0^2c^4]+[R^2\frac{\partial^2\Re}{\partial R^2}+\frac{3\ Re}{Re}\frac{\partial\Psi}{\partial R}]&=-\frac{1}{\Gamma}\frac{1}{sin^2\gamma}\frac{\partial}{\partial\gamma}(sin^2\gamma\frac{\partial\Gamma}{\partial\gamma})\\ &-\frac{1}{sin^2\gamma}((\frac{1}{\Theta sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Theta}{\partial\theta})\\ &+\frac{1}{\Phi sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2})]\\ &=A\end{aligned}$$

$$Asin^2\gamma+\frac{1}{\Gamma}\frac{\partial}{\partial\gamma}(sin^2\gamma\frac{\partial\Gamma}{\partial\gamma})=-(\frac{1}{\Theta sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Theta}{\partial\theta})+\frac{1}{\Phi sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2})=B \tag{162}$$

$$\frac{sin\theta}{\Theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Theta}{\partial\theta}) + Bsin^2\theta = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = C \tag{163}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = -C\Phi \rightarrow \Phi \sim cos(\sqrt{C}\phi + \phi_0) \tag{164}$$

with $C = m^2$,m integer since our assumed geometry requires $\Phi(\phi) = \Phi(\phi + 2\pi)$

$$\frac{sin\theta}{\Theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Theta}{\partial\theta}) + Bsin^2\theta - m^2 \rightarrow sin\theta\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Theta}{\partial\theta}) + Bsin^2\theta\Theta - m^2\Theta = 0 \tag{165}$$

$$u = cos(\theta), \frac{\partial}{\partial\theta} = \frac{\partial u}{\partial\theta}\frac{\partial}{\partial u} = -sin\theta\frac{\partial}{\partial u}, sin^2\theta = 1 - u^2 \tag{166}$$

$$(1-u^2)\frac{\partial}{\partial u}((1-u^2)\frac{\partial\Theta(u)}{\partial u}) + B(1-u^2)\Theta(u) - m^2\Theta(u) = 0 \tag{167}$$

$$\frac{\partial}{\partial u}((1-u^2)\frac{\partial\Theta(u)}{\partial u}) + B\Theta(u) - \frac{m^2}{1-u^2}\Theta(u) = 0 \tag{168}$$

setting $\Theta = (1-u^2)^a\Theta'$

$$-2u\frac{\mathrm{d}\Theta}{\mathrm{d}u} = 2u(2au(1-u^2)^{a-1}\Theta' - (1-u^2)^a\frac{\mathrm{d}\Theta'}{\mathrm{d}u}) = (1-u^2)^a[\frac{4au^2}{1-u^2}\Theta' - 2u\frac{\mathrm{d}\Theta'}{\mathrm{d}u}] \tag{169}$$

$$(1-u^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}u^2} = (1-u^2)^a[(\frac{4a(a-1)u^2}{1-u^2} - 2a)\Theta' - 4au\frac{\mathrm{d}\Theta'}{\mathrm{d}u} + (1-u^2)\frac{\mathrm{d}^2\Theta'}{\mathrm{d}^2u}]$$

$$(1-u^2)\frac{\mathrm{d}^2\Theta'}{\mathrm{d}u^2} - (2u+4a)\frac{\mathrm{d}\Theta'}{\mathrm{d}u} + (B-2a)\Theta' + \frac{4a^2u^2 - m^2}{1-u^2}\Theta' = 0 \tag{170}$$

with $a = |m|/2$,

$$(1-u^2)\frac{\mathrm{d}^2\Theta'}{\mathrm{d}u^2} - (2+2|m|)u\frac{\mathrm{d}\Theta'}{\mathrm{d}u} + (B - |m|(|m|+1))\Theta' = 0 \tag{171}$$

$$(1-u^2)\frac{\mathrm{d}^2\Theta'}{\mathrm{d}u^2} - (2+2|m|)u\frac{\mathrm{d}\Theta'}{\mathrm{d}u} + (B - |m|(|m|+1))\Theta' = 0 \tag{172}$$

with $\Theta' = \sum_{i=0} a_i u^i$

$$\sum_{i=0} i(i-1)a_iu^{i-2} + \sum_{i=0}[-i(i-1) - (2+2|m|)i + ((B - |m|(|m|+1)]a_iu^i = 0 \tag{173}$$

$$\sum_{i=0}(i+2)(i+1)a_{i+2}u^i + \sum_{i=0}[B - (|m|+i+1)(|m|+i)]a_iu^i = 0 \tag{174}$$

This serie will be finite if $B = (|m| + i + 1)(|m| + i) = l(l + 1)$

$$Asin^2\gamma + \frac{1}{\Gamma}\frac{\partial}{\partial\gamma}(sin^2\gamma\frac{\partial\Gamma}{\partial\gamma}) = B \to A\Gamma + \frac{\partial^2\Gamma}{\partial\gamma^2} + 2cotan\gamma\frac{\partial\Gamma}{\partial\gamma} - \frac{l(l+1)}{sin^2\gamma})\Gamma = 0 \quad (175)$$

$$with\ \Gamma(\gamma) = \Gamma'(cos\gamma),\ v = cos\gamma, \frac{\partial}{\partial\gamma} = \frac{\partial v}{\partial\gamma}\frac{\partial}{\partial v} = -sin\gamma\frac{\partial}{\partial v} \quad (176)$$

$$sin^2\gamma = 1 - v^2, \frac{\partial^2\Gamma}{\partial\gamma^2} = -sin\gamma\frac{\partial}{\partial v}(-\sqrt{1-v^2})\frac{\partial\Gamma'}{\partial v} = (1-v^2)\frac{\partial^2\Gamma'}{\partial^2 v} - v\frac{\partial\Gamma'}{\partial v} \quad (177)$$

one obtains,

$$(1-v^2)\frac{\partial^2\Gamma'}{\partial v^2} - 3v\frac{\partial\Gamma'}{\partial v} + A\Gamma'(v) - \frac{l(l+1)}{(1-v^2)}\Gamma'(v) = 0 \quad (178)$$

$$(1-v^2)\frac{\partial^2\Gamma'}{\partial v^2} - 3v\frac{\partial\Gamma'}{\partial v} + A\Gamma'(v) - \frac{l(l+1)}{(1-v^2)}\Gamma'(v) = 0, setting\ \Gamma' = (1-v^2)^a\Gamma'' \quad (179)$$

$$-3v\frac{\partial\Gamma'}{\partial v} = 3v(2av(1-v^2)^{a-1}\Gamma'' - (1-v^2)^a\frac{\partial\Gamma''}{\partial v}) = (1-v^2)^a[\frac{6av^2}{1-v^2}\Gamma'' - 3v\frac{\partial\Gamma''}{\partial v}] \quad (180)$$

$$(1-v^2)\frac{\partial^2\Gamma'}{\partial v^2} = (1-v^2)[(4av^2(a-1)(1-v^2)^{a-2} - 2a(1-v^2)^{a-1})\Gamma'' - 4av(1-v^2)^{a-1}\frac{\partial\Gamma''}{\partial v} + (1-v^2)^a\frac{\partial^2\Gamma''}{\partial^2 v}]$$

$$(1-v^2)\frac{\partial^2\Gamma'}{\partial v^2} = (1-v^2)^a[(\frac{4a(a-1)v^2}{1-v^2} - 2a)\Gamma'' - 4av\frac{\partial\Gamma''}{\partial v} + (1-v^2)\frac{\partial^2\Gamma''}{\partial^2 v}] \quad (181)$$

$$(1-v^2)\frac{\partial^2\Gamma''}{\partial v^2} - (3+4a)v\frac{\partial\Gamma''}{\partial v} + (B-2a)\Gamma'' + \frac{2a(2a+1)v^2 - l(l+1)}{1-v^2}\Gamma'' = 0 \quad (182)$$

with $a = |l|/2$

$$(1-v^2)\frac{\partial^2\Gamma''}{\partial v^2} - (3+2|l|)v\frac{\partial\Gamma''}{\partial v} + (A - |l|(|l|+2)\Gamma'' = 0\ and\ setting\ \Gamma'' = \sum_{n=0} a_i v^i \quad (183)$$

$$\sum_{n=0} i(i-1)a_i v^{i-2} - \sum_{n=0}[i(i-1) + (3+2|l|)i + A - |l|(|l|+2)]a_i v^i = 0 \quad (184)$$

$$\sum_{n=0}(i+2)(i+1)a_{i+2}v^i - \sum_{n=0}[i(i-1) + (3+2|l|)i - A + |l|(|l|+2)]a_i v^i = 0 \quad (185)$$

$$\sum_{n=0}(i+2)(i+1)a_{i+2}v^i + \sum_{n=0}[A - (|l|+i)(|l|+i+2)]a_i v^i = 0 \quad (186)$$

This serie is finite if $A = (|l| + i)(|l| + i + 2) = j(j+2)$

Annexe2 – Special relativity from new postulates

We start in a 4D space : x, y, z, s. A particle is at rest, its momentum $m_0 c = m_0 v_s$ being aligned with the direction $\vec{s}$. All particles move at the speed of light in 4D. If the particle is pushed by someone in the direction $\vec{r} = \vec{x} + \vec{y} + \vec{z}$ such that it has now a velocity $\vec{v}$, conservation of momentum in the direction $\vec{s}$ requires :

$$m_0 c = m' v'_s = m' \sqrt{c^2 - v^2} \rightarrow m'(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{187}$$

We define the energy of the particle as :

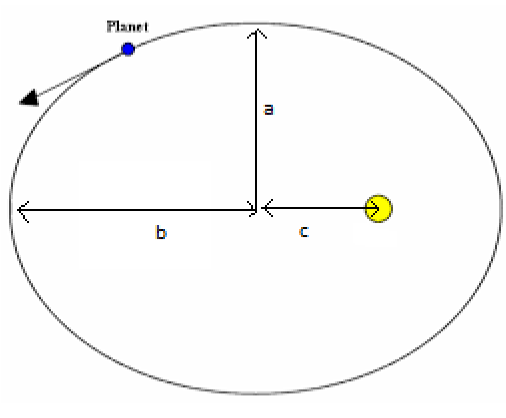
$$E = |P_t c| \tag{188}$$

where P_t is the momentum of particle in 4D. Then,

$$E^2 = P_t^2 c^2 = p^2 c^2 + m_0^2 c^4 \tag{189}$$

$$E = \vec{P}_t . \vec{V}_t = \vec{p} . \vec{v} + p_s v_s = \vec{p} . \vec{v} + m_0 c \sqrt{c^2 - v^2} = \vec{p} . \vec{v} - L_a \tag{190}$$

where L_a is the Lagrangian of a free particle. We keep the special relativity but we can now say that $\vec{s} = \vec{s_1} + .. + \vec{s_n}$ producing what De Broglie called the "internal space" of particles. Maybe other forces (including the dimension w used in the paper) could be integrated here. We don't require invariance of the speed of light, principle of relativity or the Lorentz length contraction, which is interesting : if we consider a planet orbiting far away from its star, such that the motion reduces to Keplerian motion, with $c^2 = b^2 - a^2$:

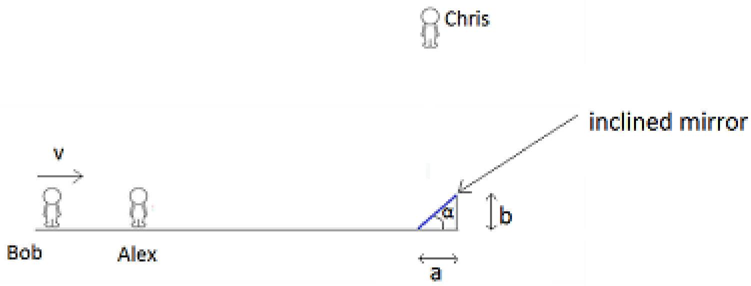


and now imagine that de path of the planet is really materialized (by a structure made of iron, by example), if Bob is moving in the horizontal direction with velocity v such that $b\sqrt{1 - v^2/c^2} = a$, Bob will then see the iron as a circle, according to Lorentz length contraction. But $c \neq 0$ so Bob can deduce he is moving. If a reader could give us an explanation we woud be interested. Other examples are given in the next pages.

Exemple 1

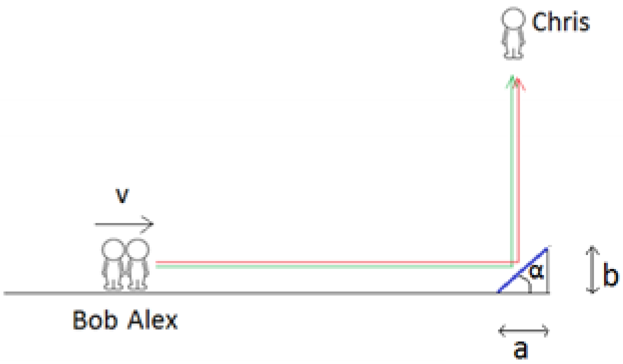
Initial configuration :

Alex, Chris and a mirror (as heavy as we want) are at rest by respect to each other. $a = b$: the angle of inclination of the mirror is then $\alpha = atan(b/a) = \pi/4$ When Bob, moving at speed $v \approx c$ but $v < c$ and Alex, are at the same position, they both emit a photon (the momentum of the photon is as low as we want, in both reference frames).



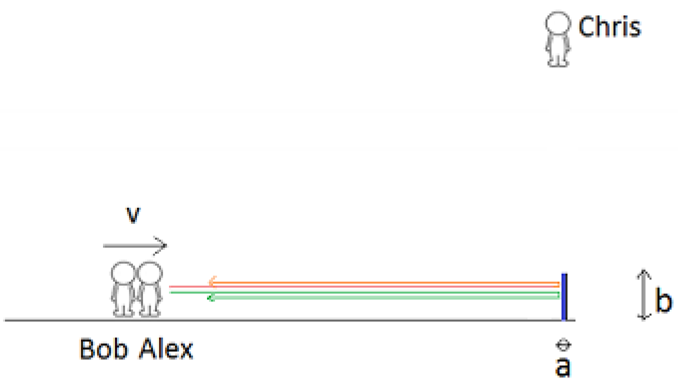
Emission of photons, in Alex's reference frame:

In Alex's reference frame, Chris will receive the two photons.



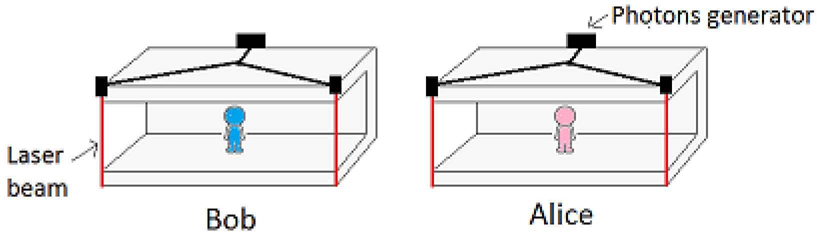
Emission of photons, in Bob's reference frame:

In Bob's reference frame, according to special relativity, the length a is Lorentz contracted: since $v \approx c$ we have $a \approx 0$ and then $\alpha = atan(b/a) \approx \pi/2$ The two photons then come back.

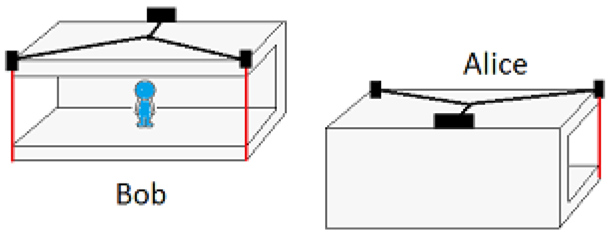


On this example, we don't understand how to conciliate the Lorentz length contraction with the principle of relativity.

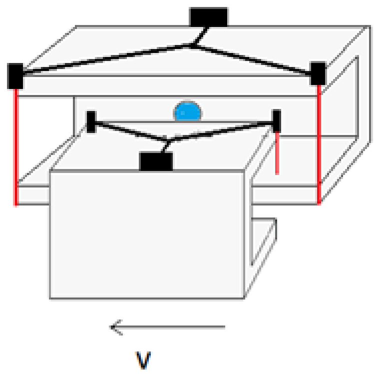
Exemple 2
Bob and Alice in two identical spacecrafts:



Bob and Alice are in two identical spacecraft: at the top of each spacecraft, a photon generator continuously emits pairs of photons, producing two vertical laser beams. A mechanism, under each spacecraft is connected to a bomb. If both laser beams are simultaneously cut, the bomb has to explode. The vertical distance between two photons (vertically moving downward) in laser beams is sufficiently small to consider that the laser beam is (almost) a continuous vertical line.



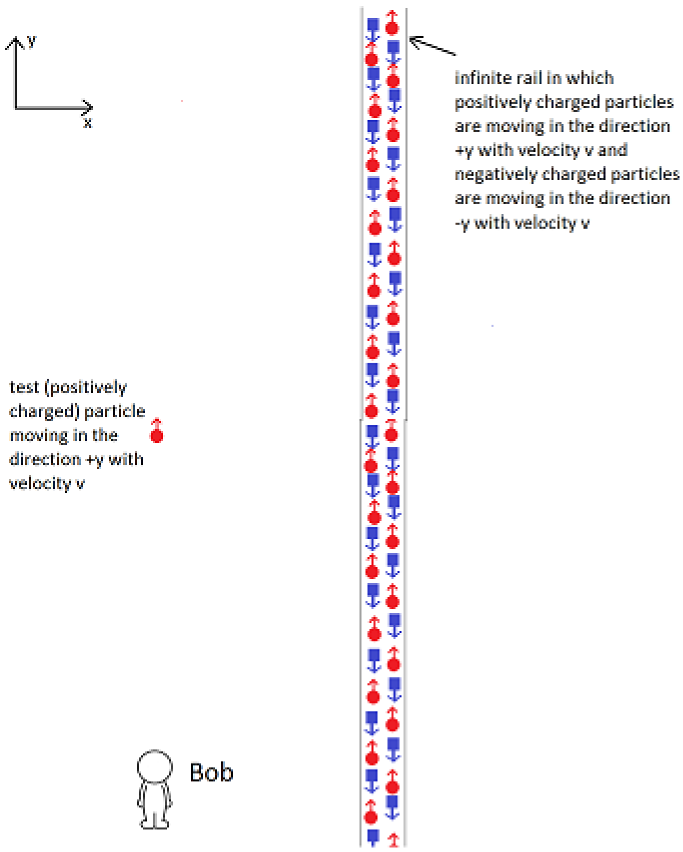
At the beginning of the experiment, Bob and Alice are together, at rest at the same place. Bob goes on the left, and Alice goes on the right. They accelerate symmetrically. Then they stop (symmetrically) their respective spacecraft, and go back to the initial point. To come back, they first accelerate symmetrically, and then shut down their motor, moving then at constant velocity, Bob moving from the left to the right, Alice moving from the right to the left.



Bob is seeing Alice moving at a velocity v. According to special relativity, the length of Alices spacecraft is contracted: when Alice is in front of Bob, both laser beams (of Alice's spacecraft) are cut simultaneously: Alice's bomb has to explode while Bob's bomb can't explode (his lasers beams can't be cut simultaneously). According to special relativity, the situation should be symmetric. In this example, we dont see how to conciliate Lorentz length contraction with the principle of relativity.

Exemple 3

Configuration, in Bob's reference frame :

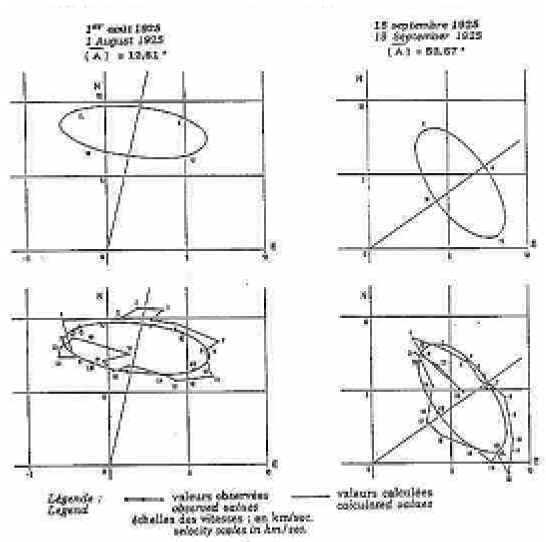


Inside the rail, negatively and positively charged particles are homogeneously distributed such that the electric static potential induced by the rail is V = 0 outside the rail. The system is stationary ($\frac{\partial \vec{A}}{\partial t} = 0$). According to the Lorentz force $\vec{F} = q\vec{E} + \vec{v} \wedge \vec{B}$ (which then reduces to $\vec{F} = \vec{v} \wedge \vec{B}$ in the present case, where $\vec{v}$ is the velocity of the test particle), the test particle will be attracted by the rail (the moving charges in the rails creates an attractive magnetic field in Bob's reference frame).

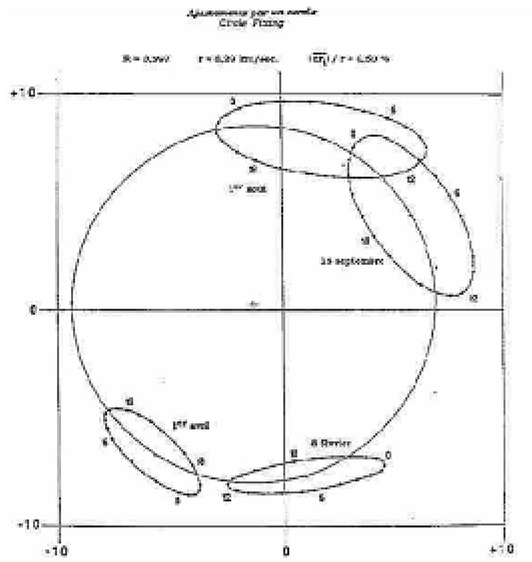
If we examine the situation in the test particle's reference frame, we still have $\vec{E} = \vec{0}$ but now $\vec{v} = 0$ since the test particle is necessarily at rest in its own reference frame, which means $\vec{F} = 0$. In this example, we dont see how to conciliate de Lorentz force with the principle of relativity.

Brief note

The invariance of the speed of light is one of the two assumptions building Einsteins special relativity in 1905. The experimental validation of this assumption has been made initially by Michelson and Morley, in a series of famous experiments performed at the end of the nineteenth century. Naturally, experimental data are associated to uncertainties, but measurements were estimated compatible with the invariance of the speed of light. A few years later, Miller and Morley repeated measurements. It was then considered that the speed of light was not isotropic and showed regularities. Einstein was aware of the new data, according to a letter he wrote to Milikan in 1925: "I believe that I have really found the relationship between gravitation and electricity, assuming that de Millers experiment are based on a fundamental error'. Millers results were reanalyzed by Maurice Alley (Nobel Prize in Economics Sciences) who estimated that the regularities were so troubling that they could not be caused by any undesired effect like temperature. Precisely, in various papers, he illustrated data on hodographs:



Example of hodograph



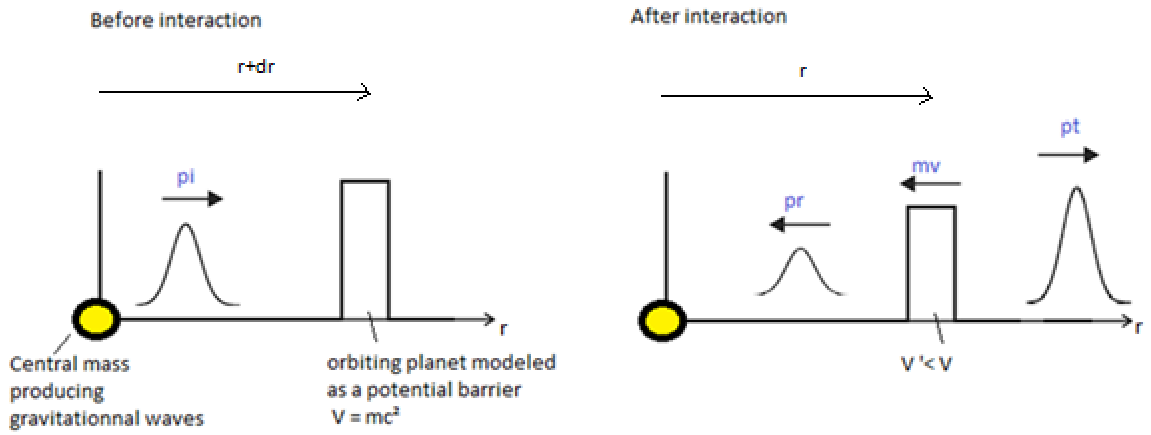
Combining different hodographs, Alley indicates that their respective center belong to the circumference of a circle

According to Alley, the speed of light is then not invariant, and its variations show cosmological regularities. Both Miller and Alley consider that those experiments can only be done outside (not inside a laboratory). In this description given by Alley we have some difficulties to imagine an undesired effect randomly producing such graphs. It is hard to imagine Alley (Nobel Prize in Economic Sciences) voluntarily lying too.

If a reader could give us informations or explanations concerning the examples discussed in this annexe, we would be interested.

Annexe3 – Example for galaxy rotation curves

Here we assume that gravitation is transmitted by waves (or particles, with a quantized phenomenon). The central mass is assumed to emit gravitational waves with momentum p_i. If a planet far away is attracted, conservation of linear momentum requires that the transmitted wave p_t contains more momentum that the incident wave p_i, since the planet is attracted. And then the extra energy must have been taken from the planet, changing its (inertial) mass, the gravitationnal charge.



Here the conservation laws are :
Conservation of linear momentum : $p_i = p_t - p_r - mv \Rightarrow p_t > p_i$
Conservation of energy : $V' < V$
$V - V'$ should increase with V, assuming proportionnality as an example :

$$V(r+dr) - V(r) = (\frac{1}{r_0})V(r)dr \Rightarrow V(r) = V_0\ e^{r/r_0} \tag{191}$$

then

$$E = m(r,\dot{r})c^2(1 - \frac{GM}{c^2 r}) \quad , \quad r = r(t), \dot{r} = \dot{r}(t) \tag{192}$$

$$E = \gamma e^{r/r_0} m_0 c^2 (1 - \frac{GM}{c^2 r}) \quad , \quad \gamma = 1/\sqrt{1 - \dot{r}^2/c^2} \tag{193}$$

$$\begin{aligned} 0 = \frac{\mathrm{d}E}{\mathrm{d}t} &= (\gamma^3 \ddot{r}\dot{r}/c^2) e^{r/r_0} m_0 c^2 (1 - \frac{GM}{c^2 r}) \\ &+ \gamma(\frac{\dot{r}}{r_0} e^{r/r_0}) m_0 c^2 (1 - \frac{GM}{c^2 r}) \\ &+ \gamma e^{r/r_0} m_0 c^2 (\frac{GM\dot{r}}{c^2 r^2}) \end{aligned}$$

$$0 = \gamma e^{r/r_0} m_0 c^2 \dot{r}\ (\gamma^2 \ddot{r}(1 - \frac{GM}{c^2 r}) + \frac{1}{r_0}(1 - \frac{GM}{c^2 r}) + \frac{GM}{c^2 r^2}) \tag{194}$$

$$0 = \gamma^2 \ddot{r}/c^2 (1 - \frac{GM}{c^2 r}) + \frac{1}{r_0}(1 - \frac{GM}{c^2 r}) + \frac{GM}{c^2 r^2} \tag{195}$$

$$\ddot{r} = -\gamma^{-2}(\frac{\frac{GM}{r^2}}{1 - \frac{GM}{c^2 r}} + \frac{c^2}{r_0}) \tag{196}$$

where, for galaxies,

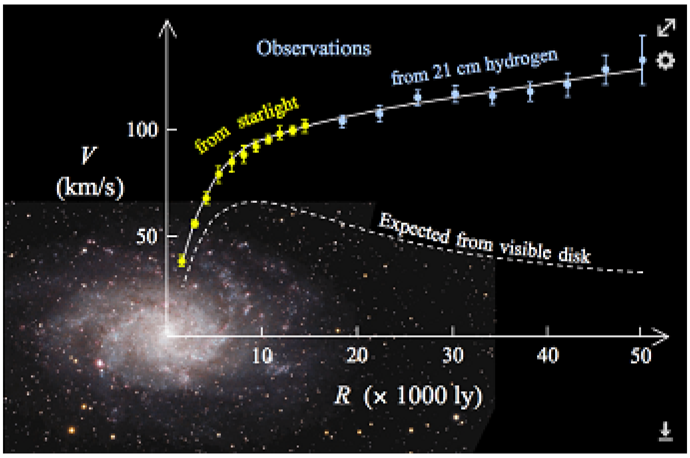
$$\gamma \approx 1\ ;\ 1 - \frac{GM}{c^2 r} \approx 1\ ;\ |\ddot{r}| \approx \frac{v^2}{r} \tag{197}$$

the equation becomes

$$v^2 \approx \frac{GM}{r} + \frac{rc^2}{r_0} \tag{198}$$

$$v^4 \approx (\frac{GM}{r})^2 + 2GM\frac{c^2}{r_0} + (\frac{rc^2}{r_0})^2 \tag{199}$$

where $c^2/r_0 = 0$ classically. This should be compared with :



If we now examine a photon case with $\hbar\nu \leftrightarrow \gamma m_0 c^2$, emitted from a faraway star (radius R_s) and received at a given distance r, conservation of energy implies :

$$E = e^{R_s/r_0}\hbar\nu(1 - \frac{GM}{c^2 R_s}) = e^{r/r_0}\hbar\nu'(1 - \frac{GM}{c^2 r}) \tag{200}$$

where naturally

$$e^{R_s/r_0} \approx 1\ ;\ 1 - \frac{GM}{c^2 R_s} \approx 1\ ;\ 1 - \frac{GM}{c^2 r} \approx 1 \tag{201}$$

we then write

$$\nu' \approx \nu e^{-r/r_0} \approx \nu(1 - \frac{r}{r_0}) \tag{202}$$

such that if we justify galaxy rotation curves by increasing gravitation, we produced a large scale redshift. The large scale redshift and galaxy rotation curves being currently justified by "*dark matter*" and "*dark energy*". We don't believe the example is correct, but we believe that the problems of "*dark matter*" and "*dark energy*" are induced by a misunderstanding of quantum principles.