An introduction to F-notation and the prove of the cartesian product of natural number is countably infinite

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Abstract

This paper will introduce a new notation named as F-notation. This notation will help us to prove the statement, "The cartesian product of natural numbers is countably infinite".

Introduction

Many many years ago, Human beings got the knowledge of counting. Since then, the concept relating to counting is probably the greatest discovery in the initial phase of mathematics. We introduced the number systems like $\mathbb{R}, \mathbb{Q}, \mathbb{N}$ etc. And now, we divided the number systems into two category on basis of counting i.e. Countable sets and Uncountable sets. This concept advances *The Set Theory* which was first proposed by George Cantor and followed by Richard Dedekind [1].
First, we will try to prove the given statement. And at the middle of procedure, we will introduce the F-notation.

Theorem. *The set* $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

Proof. We know, $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. So, the cartesian graph will be;

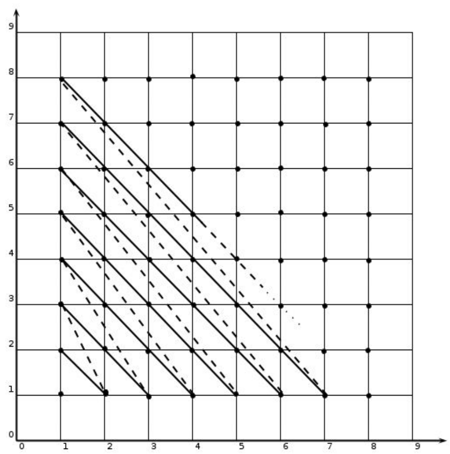

Figure 1: Cartesian product of Natural numbers

Definition 1. *The ordered pair can be written as* $a_x + b_y$ *where a is the first entry which lies on x co-ordinate and b is the second entry which lies on y co-ordinate. And,* $a_x + b_y$ *will give the solution as* $a + b$*.*
i.e. $a_x + b_y = a + b$

First, we will pick out the diagonal elements (i.e. ordered pair) from the above figure. From the definition 1, we can write;
$1_x + 1_y = 2$
$1_x + 2_y = 2_x + 1_y = 3$
$1_x + 3_y = 2_x + 2_y = 3_x + 1_y = 4$
$1_x + 4_y = 2_x + 3_y = 3_x + 2_y = 4_x + 1_y = 5$
$1_x + 5_y = 2_x + 4_y = 3_x + 3_y = 4_x + 2_y = 5_x + 1_y = 6$
So, the pattern would be;
$1_x + n_y = 2_x + (n-1)_y = 3_x + (n-2)_y = 4_x + (n-3)_y = ... = n_x + 1_y = n + 1$

Definition 2. *Let us define a notation named as F-notation and given by*

$$F_{k=1}^{n}$$

It states that, when the different operation has a same solution then, it can be written in one functional notation. i.e. $f(x)$

Applying definition 2, we can write;

$$F_{k=1}^{n} k_x + (n-k+1)_y = n + 1 \tag{1}$$

Since, the images are unique. Hence, we can say equation 1 is a function. So, it becomes;
$f(n) = n + 1$
To test above function, whether it is countably infinite or not. For this, we have to prove whether it bijective or not.
Test 1:
Let,
$f(n_1) \neq f(n_2)$
$\Rightarrow (n_1 + 1) \neq (n_2 + 1)$
$\therefore n_1 \neq n_2$
Hence, the function is one-one.
Test 2:
Let, $m = f(n) = n + 1$
$\Rightarrow m = n + 1$
$\Rightarrow n = m - 1$
$\therefore$ The inverse function is, $f^{-1}(m) = (m - 1)$.
Since, $m \geq 2$ and $n \geq 1 \mid m \in \mathbb{N}$
Thus, $f^{-1}(m) = (m-1) \in \mathbb{N}$ i.e. $f^{-1}(m) \geq 1$.
The inverse of the given function $f^{-1}(m) = (m - 1) \in \mathbb{N}$ where $f^{-1}(m) \geq 1$ that satisfy the domain elements. But as far we know, $f(n) = n + 1 \in \mathbb{N}$ as $f(n) \geq 2$. This means, Range is equal to Co-domain. Hence, the function is onto.
So, The cartesian product of natural numbers is countably infinite.

□

References

[1] Ferreireacute;. The early development of set theory. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Fall 2016 edition, 2016.