The meaning of ‘flat’: deducing general relativity from Newtonian physics.

The motivation and intended audience.

In an earlier paper[1] I showed that Newton's laws generalise to the theory of special relativity. In this paper I show that special relativity in turn implies general relativity.

I have two audiences in mind. I hope it will help those who, like me, left school decades ago but still like to understand today's physics and are dissatisfied with inexact analogies. The second target audience are those academic physicists who do not realise how readily relativity can be derived using only those assumptions that are implicit in Newton's laws: I want to be able to reference this argument in later papers rather than having to remake it.

The organisation of the paper

This paper consists of a fairly short argument, followed by a number of appendices which expand on some of the ideas introduced or referenced in the main text. I suggest you read the main text, and then decide whether and which appendices you are interested in.

To understand general relativity, you need to understand what we mean by 'curved' and by 'flat'. You do not need any mathematics in order to understand curvature: I will explain all you need by reference to the surface of the earth. I show how we can allow for the possibility that space/time is curved without needing to know whether it is, and in that way we can resolve an apparent clash between special relativity and Newton's law of gravitation. I show that Newton's law of gravitation suggests that a suitable definition of mass and stress makes both Einstein's field equation and Newton's laws definitively correct.

What is curvature?

There are two meanings to curvature, and both can be illustrated by thinking about the curvature of the earth's surface. The distance from the North pole to the South pole is 20,000 km if you measure along the surface of the earth, but if you could drill a hole through the middle of the earth, the distance would shorter, about 12,750 km. This means the earth's surface is **extrinsically** curved, because the shortest distance does not stick to the surface.

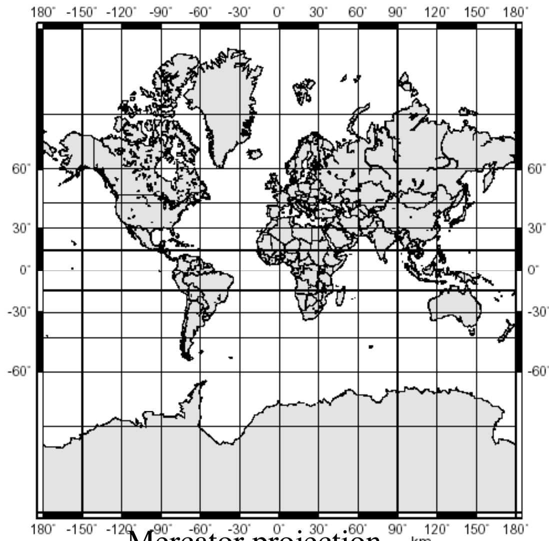
The other sort of curvature becomes obvious when you try to map the earth's surface. A map of an **intrinsically** curved surface must always be distorted. Two maps of the world are shown below. Both are distorted, meaning that the scale varies over the map. The poles are drawn to a bigger scale than the equator. If we are printing a map of the world on a flat piece of paper (or displaying it on a flat screen), there must be some distortion somewhere.

For those who are interested, appendix A describes intrinsic and extrinsic curvature in more detail. From now on in the main text, assume I am talking about intrinsic curvature.

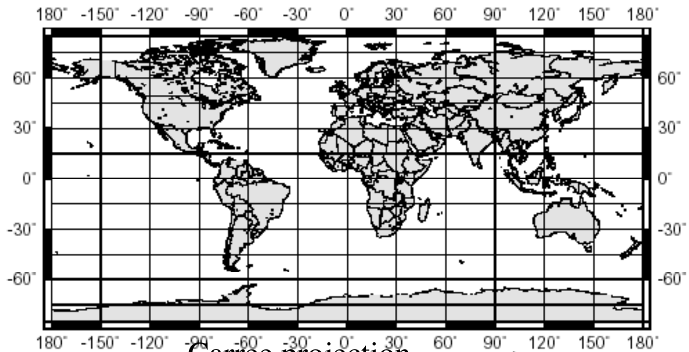
A map can represent a surface but we need a model to represent a three dimensional object or space, and to represent a space/time we need a working model, one that evolves with time. Appendix B discusses briefly the need to think about models rather than think directly about “reality”.

If the surface we are mapping is curved, then we cannot draw a map of the surface to a constant scale. And if the space we are modelling is curved, then we cannot build a model of the space to a constant scale. And if the space time we are modelling is curved, then either the scale will vary across the model, or the model will run at different speeds in different parts or as the model evolves, or model time will be biased (not Newtonian), or some mixture of these.

[1] http://vixra.org/abs/1607.0364: “Deducing Special Relativity from Newtonian Physics”



Mercator projection



Carree projection

(Maps produced by Martin Weinelt's "Online Map Creation" service, at http://www.aquarius.ifm-geomar.de/)

I'm writing about maps drawn on flat paper and models constructed in flat space time. We can draw a map on a curved piece of paper[2], but we must build a model in flat space, because we can't conjure up any curved space in the laboratory in which to build the model. You may not be able to visualise curved space/time, but you should be able to visualise a distorted model: think for example about a Spitting Image puppet.

Although a model of curved space will be distorted in some way, the corollary is not true. For example, a Spitting Image puppet is a distorted model of something in flat space, and a map of the London underground or the New York subway is distorted to improve readability.

Smooth surfaces and local calculations

The surface of the earth is not smooth, but maps of the world assume it is. This process of smoothing is a sort of averaging process that is common in physics. Pressure and density, for example, are smoothed quantities: at an atomic or molecular level a pressure is a scattering of individual blows, and a density will be made up of small concentrations of matter separated by relatively enormous spaces.

If you look at a small enough area of a smooth curved surface, it looks flat, which is why we can draw a map of a city without distortion. If you look at the surface of that city, it won't look flat and it has lots of corners: we smoothed out the curvature in order to draw the map. By contrast, if you look at a small volume of space/time it does seem flat, so much so that for millennia nobody imagined it could be otherwise. Town planners use standard geometry because the town is so small that the curvature of the earth's surface can be ignored, and when considering large volumes of space/time we can use Newton's laws within any small part.

Gravity on a universal scale

We can hit problems when we try to apply Newton's law of gravity to an imagined universe.

For example, consider a simplified universe where the density of matter (smoothed on a very large scale) is roughly the same everywhere. Choose any large sphere within this universe and consider how it is affected by gravity. The mass inside the sphere will pull the edge towards the centre, but at the centre of the sphere gravity should cancel. However, since everywhere is the same everywhere should behave in the same way, and if the centre will not move, nothing should move.

Alternatively, imagine a large sphere with a uniform distribution of stars inside the sphere, but empty space outside the sphere. Newton's law of gravity tells us that the acceleration of the edge

[2] A globe is a map drawn on curved paper.

towards the centre should be proportional to the radius of the sphere[3]. The larger the sphere, the bigger the gravitational pull, and at enormous distances or after enormous times, items at the edge of the sphere will be falling in faster than the speed of light, which would seem to conflict with special relativity.

We can resolve these apparent paradoxes by considering a model of space that changes scale with time.

The earth is 150 million kilometres from the sun, so imagine a sphere of space 300 million kilometres in diameter, filled with pebbles (1cm steel ball bearings) roughly 3 metres apart. I've chosen those figures to make the total mass inside the sphere roughly equal to the mass of the sun.[4]

Now imagine a model of this space. As noted above, Newton's law of gravitation can be re-worded in this special case: things twice as far from the centre will accelerate twice as fast towards the centre, so everywhere in the sphere we can satisfy Newton's law of gravitation by changing the scale of the model.

Now suppose this sphere of space is any sphere chosen from anywhere in a universe which is the same everywhere (with a little smoothing). Everywhere is filled with the pebbles, initially at roughly 3 metre intervals. Whatever model we take for this universe, Newton's law of gravitation is satisfied everywhere if the scale of the model changes at an appropriately accelerating rate. The volume of space represented by the model shrinks, but the pebbles remain the same size. Every model pebble stays at the same place in the model, but grows a little larger in situ reflecting the change of scale of the model.

After roughly 61 days the pebbles will start to touch, and immediately an immense pressure will develop. That is as far Newton's laws can take our model, because Newton's law of gravitation does not take pressure into account.[5] But as long as there was no pressure, we were able to use Newton's law to calculate how the model would develop.

Imagine a laboratory set up in the centre of the model, and the experimenters in the laboratory will watch the pebbles in the laboratory, and around about them. Over the 61 days, the pebbles drift gently towards the centre of the ship, but the one in the middle of the ship doesn't move at all. Towards the end of the 61 days, the pebbles outside the laboratory will start to nudge the laboratory, very gently, then with rapidly increasing pressure, and the laboratory will be violently crushed. In fact, set up your laboratory anywhere in the space, and if it is not moving relative to the surrounding pebbles at the start, the experience within the laboratory will be the same.

We now have enough to enable us to deduce Einstein's field equation, but before doing that you may wish to think about the implications of trying to model the solar system without introducing the distortion necessary to accommodate curvature. This is described in appendix D.

Mass and curvature.

A distorted model allows but does not require that space/time be flat. However, the resultant behaviour of the model tells us the space/time it represents is indeed curved, and tells us what the curvature is. Since the only entity in that universe is mass, it is a reasonable guess that the mass is associated with the curvature. And if mass is associated with curvature, then that association must hold in any reference frame.

Special relativity tells us that if we wish to convert the mass of a body from one reference frame to another we must convert mass and momentum together as a single object. However, a distorted model of space/time introduces a further consideration. If you work on a small enough volume of space/time, you can treat it as flat, but 'small enough' depends on how accurate you need to be. For

[3] The mass inside the sphere of radius r is proportional to r^3, and the acceleration of the edge towards the centre is proportional to mass divided by r^2.
[4] The calculations are given in appendix F
[5] Counterintuitively, pressure increases the gravitational effect.

perfect accuracy you must work on a point and use integration to calculate mass or curvature of anything bigger than a point.[6] That means we define curvature as the property of a point, and relate the density of mass and momentum to the curvature at a point. As a result, if we wish to convert density from one reference frame to another, we must convert the density of mass, momentum and stress together as a single object. This 'single object' is known as the stress energy tensor and its values are set out in the table below.

	Time	Space (x)	Space (y)	Space (z)
Time	**T^{tt}** Mass density	**T^{tx}** Momentum in x direction	**T^{ty}** Momentum in y direction	**T^{tz}** Momentum in z direction
Space (x)	**T^{xt}** (= T^{tx})	**T^{xx}** Pressure in x direction	**T^{xy}** Shear in z direction	**T^{xz}** -shear in y direction
Space (y)	**T^{yt}** (= T^{tx})	**T^{yx}** (= T^{xy})	**T^{yy}** Pressure in y direction	**T^{yz}** Shear in x direction
Space (z)	**T^{zt}** (= T^{tx})	**T^{zx}** (= T^{xz})	**T^{zy}** (= T^{yz})	**T^{zz}** Pressure in z direction

The various effects of inertia at a point, namely the densities of mass momentum and stress, arranged in a table known as the stress energy tensor.

The reason these physical values are thus grouped together and the way in which they are related is described in appendix G. It suffices here to say that the entries in the table are related by four differential equations: one expresses the law of conservation of mass and the other three express the conservation of momentum in each direction.

If curvature maps to the stress energy tensor, then the curvature must transform in the same way when converted to another reference frame, and must obey the same four differential equations. The Einstein tensor is the only curvature quantity that fits this specification, so

$$G^{ab}=kT^{ab}+X^{ab}$$

meaning $G^{tt}=kT^{tt}+X^{tt}$, $G^{tx}=kT^{tx}+X^{tx}$, $G^{ty}=kT^{ty}+X^{ty}$, etc, to $G^{zz}=kT^{zz}+X^{zz}$, where X^{ab} is that part of the Einstein curvature which is not related to the mass, or at least, to mass of which we are aware.

That is not a very challenging assertion, since it leaves the Einstein curvature unconstrained. I will return to this equation, but first I want to clarify what I mean by mass.

What counts as mass?

We constantly make assumptions, and usually those assumptions are obvious. If we are talking about the speed of a spacecraft soon after liftoff, we might talk about its speed relative to the launch pad. A little later we might talk about its speed relative to the earth's centre. Later still, we might talk about its speed relative to the sun. As the spacecraft approaches its destination, we might talk about its speed relative to that destination.

Where space/time is curved, the assumptions become much less obvious. Consider again the simple universe described earlier. If we attempt to model the universe keeping the scale of the model constant, then we choose an arbitrary centre and gravity makes everything fall towards that centre,

[6]In a similar way, when calculating the rate of collapse of the universe, Newton's law of gravitation tells us the second order rate of change of size, and twofold integration takes us to the size at any given moment of model time.

and as they pick up speed they become more massive. But if we allow the scale of the universe to change, then everything remains stationary and their mass is their rest mass.

You might think that all you need to do is specify the model, and the location in the model, but that is not enough. For good reason, the measurements are often adjusted when we talk about a measurement at some distant point. For example, if you take a one kilogram load up to the moon, you give it potential energy. What is the mass of the body on the moon? There are now two contenders. You might say it is the mass as measured on the moon (exactly one kilogram) or the mass it would have if dropped back to earth (one kilogram plus the energy that has been converted from potential energy to kinetic energy).

There are in fact several different ways of defining mass at a distance. I am not going to expound or critique such definitions. I just want to emphasise that I am talking about local mass, where we can assume without introducing significant error that space is Euclidean, time runs the same speed everywhere, and we can use relativistic units for time and space.

Einstein's field equation

Return to the equation $G^{ab}=kT^{ab}+X^{ab}$. If k is a constant in time and across space, then X^{ab} must obey the same differential equation as G^{ab} and T^{ab}. In that case we can assert that X^{ab}/k is the stress energy tensor for some matter that we do not yet fully understand, and the equation becomes:

$$G^{ab}=kT^{ab}$$

and both Einstein's field equation and Newton's laws become true by definition.

We now have two names for the same thing. The stress energy tensor T^{ab} is an object defined by the ways in which it is detected in classical physics, and the Einstein tensor G^{ab} is the same object defined as a curvature of space/time. However, it is convenient to be able to subdivide T^{ab}. For example, T_e^{ab} might be the stress energy tensor of a point within an electron, and T_f^{ab} the stress energy at that same point due to an electric field. In this case momentum and energy are not conserved for the electron and the field independently, but any energy or momentum lost or gained by the electron is gained or lost by the field. I think of G^{ab} as being the complete object, and T^{ab} as a part of the object, and write Einstein's field equation as $G^{ab}=k\Sigma T^{ab}$.

Summary.

To me, general relativity consists of the realisation that space/time is curved, and that there is an identity between mass and curvature which is expressed by Einstein's field equation. In that sense, I have shown that general relativity follows from Newton's laws without any further assumptions.[7] I dropped Newton's assumption that simultaneity was universal (implied by his reference to God's clock), and I dropped the Greeks' assumption that space is flat.

Anyone who is confident that they know what mass and energy are and believes that mass causes curvature may feel that I have thrown away the meaning of Einstein's field equation. I argue that mass and curvature are two aspects of the same basic entity. Mass and matter take numerous forms, and I consider dark energy and dark matter (for example) to be new forms of matter.

[7]Possibly Newton would not have insisted that his gravitational 'constant' was truly constant. Einstein's field equation assumes it to be so.

An overview of the appendices.

A: Extrinsic & intrinsic curvature.

A short history of curvature, and a more detailed description of extrinsic and intrinsic curvature.

B: Models and reality.

This explains why we cannot appeal to 'reality' in order to decide how reality behaves.

C: Reading maps & models.

This describes the distortion that you expect to get when mapping a curved surface.

D: Curvature and the solar system.

This explains the problems we can expect if we attempt to model the solar system without allowing for the curvature of space/time.

E: Other curvature.

Curvature can be divided into Einstein curvature and Weyl curvature. We detect the Einstein curvature as mass, momentum, and stress: we detect the Weyl curvature as gravity. This appendix expands on this.

F: a model of gravity on a universal scale.

This gives the calculations underlying the description of a collapsing universe. It includes a proof that space/time is definitely not flat.

G: The stress energy tensor for the pebbles and variations.

This shows special relativity affects how we treat density, and derives the relationship between the densities of mass, momentum and stress.

H: general relativity & cosmology .

This describes in a bit more detail the model used to consider a universe that is everywhere the same. It shows how the model has evolved, refined by our observations of the universe.

Appendix A: Extrinsic and intrinsic curvature.

More than 2000 years ago the greeks developed geometry, and tried to pin down exactly what they could prove and what they had to define or assume. They defined a plane surface as one that was extrinsically flat, and they found they had to assume something called the parallel postulate, which turns out to be an assumption that a plane is also intrinsically flat. For centuries mathematicians thought it must be possible to prove the parallel postulate, but they could not find a proof. In the 17th century John Wallis showed that the parallel postulate was equivalent to an assumption that one could draw a triangle to any scale without distortion.[8]

In flat space, any surface that is extrinsically flat is also intrinsically flat, so the parallel postulate can be thought of as an assumption that space is flat. Space/time *is* so nearly flat that whenever you think of something curved, you will be thinking of something that is extrinsically curved. But the reverse is not true: something that is **ex**trinsically curved may yet be **in**trinsically flat. Look at four examples, a football, a roll of paper, the surface of a conical dunce's hat, and a curved line: these are all extrinsically curved, but only the football is also intrinsically curved. You can unroll a roll of paper and lay it out flat, and you can cut open a dunce's hat and lay it out flat. Anything that can be straightened out and laid flat is intrinsically flat. In fact a line can never be intrinsically curved: a piece of string (for example) can always be straightened out: so when we talk about a curved line, we always mean an extrinsically curved line.

When we talk about the curvature of space/time, 'curvature' means 'intrinsic curvature'. Except when writing about a line, 'extrinsic curvature' should be written in full.

Flat versus curved.

Something flat has zero curvature, just as something stationary has zero velocity. Once you realise that, you will realise that it is improbable that anything real is truly flat. To assume that space/time is flat is a big assumption, to assume it may be curved is no assumption at all.

I am not strictly consistent in my use of 'curved'. I have just used '*may be curved*' meaning '*may have non-zero curvature*'; in other places I use '*curved*' more assertively to mean '*has non-zero curvature*'. I hope the context makes the meaning is obvious.

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[8] John Wallis definition of flat (and hence of 'not flat') is not quite the same as mine, because it serves a slightly different purpose. John's definition has the advantage that it is a test for flatness that needs nothing more than the surface being tested. I am not aiming to supply a definition: I want to show how we can describe space/time in a way that allows for the possibility that it is curved without the necessity of deciding in advance whether or not it is. That said, it is easy to see that if a surface fails John's test for flatness, it fails mine.

The information about John Wallis is drawn from Eric W. Weisstein et al. "Parallel Postulate." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/ParallelPostulate.html

Appendix B: Models and reality

After the Greeks had investigated geometry, 'reality' was fairly obvious. Space was assumed to be flat (Euclidean), and time ran at the same speed everywhere. It was natural to look upon the earth as stationary, with the universe moving around it. After Copernicus it became more natural to think of the sun as the centre of the universe, and that the earth moves round the sun. But movement is relative, and to say that the earth moves implies there is some reference frame that defines 'not moving'. The model provides that reference frame.

Still today, most people feel that 'in reality' the earth moves round the sun, Newton's law of gravitation seems to confirm this: it is much easier to work out the effects of gravity if we assume the lighter body orbits the heavier.[9] A model where the sun is not moving is ideal if you want to calculate how to send a rocket to the moons of Saturn, but if you want to predict the weather on earth a better model to choose would have the earth rotating but not otherwise moving. And if you want to plan your journey from New York to San Francisco, the model you naturally choose has the earth not even rotating.

When you talk about velocities of objects close at hand the meaning is obvious. If you talk about a car going at 60mph, you mean 60mph relative to the earth, and you assume that the earth stays the same size and is not moving. When you talk about the speed of rocket on its way to Mars or the speed of a galaxy 4000 light years away, the meaning is less clear[10]. In each case we mean something slightly different by 'speed', and in each case there are unspoken underlying assumptions.

The curvature of space/time forces us to use distorted models, killing the idea that there is one 'correct, realistic' model. We can describe how events near a point in space time would appear to a person at that point, but descriptions of distant events are sometimes more poetic than precise.

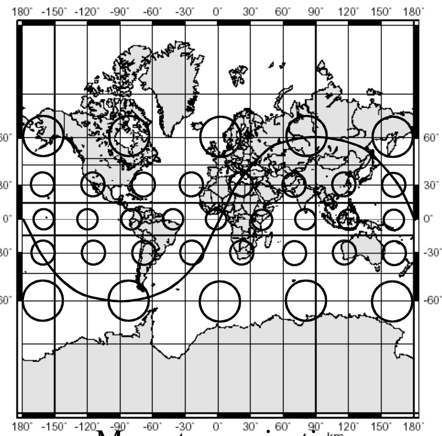
[9] A more sophisticated version of reality will keep the centre of mass of the sun and the earth still, and the sun and the earth will both orbit that centre. This approach stops looking so obvious when you add in all the other planets and their moons, and then consider the solar system as a small part of the milky way.

[10] For example, a quote from http://phoenix.lpl.arizona.edu/faq.php reads: *The Phoenix spacecraft is travelling at approximately 74,000 mph (120,000 km/h). Another number seen in the media is 14,000 mph (22,500 km/h). The 14,000 mph is the speed Phoenix will be travelling with respect to Mars.*

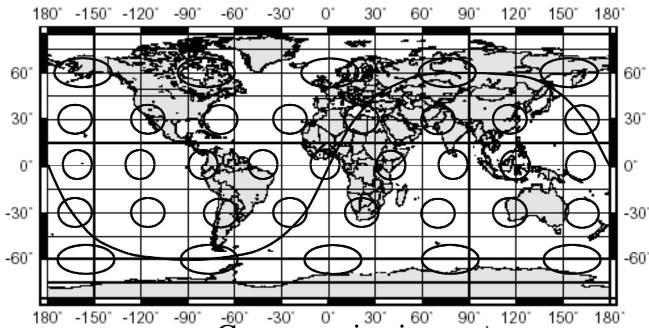
Appendix C: Reading maps and models.

It is helpful to know the sort of distortion that you are likely to find in a map. We imagine circles of (say) 2000km diameter on the ground, and show these on the map. These are called Tissot's Indicatrix, and they show the scale at that point in the map. In the Mercator map below, the scale gets larger as you move away from the equator. In the Carree map, the scale in the EW direction gets larger, but the scale in the NS direction remains constant.

If you draw a straight line on the ground, it forms part of a great circle. A great circle has been drawn on the two maps below, and shows another form of distortion: straight lines may become curved in the map or model. The distortion of the circles depends on the scale of the map in the neighbourhood of the circle: the bending of the line is caused by the fact that the scale changes slightly as you cross the neighbourhood of the line.



Mercator projection



Carree projection

Maps with Tissot's indicatrix and a great circle added to give a feel for the distortion.

A geodesic is a line that is straight on the surface or in the space being mapped, but it may be curved in the model: the great circle shown on the above maps is an example. Conversely, a line that is straight on the map or model may be curved on the surface or in the space we are describing: the lines of constant latitude shown on the above maps are examples.

Some extra complications arise as we move from a surface to a space. The shortest distance between two points on a surface is a straight line, but if the surface itself is extrinsically curved then the line may not be a straight line in space. A great circle is a straight line on the earth's surface, but it is curved in space. A straight line in space may not be a straight line in space/time if the space is extrinsically curved.[11]

The distortion in models of space/time can be hard to envisage unless the model is very simple and idealised, or the distortion is very slight. Appendix D describes the very slight distortion that is inherent in a model of the solar system.

[11] The space is extrinsically curved if time is biased. Local time defined by the sun is obviously biased: if you fly from London to New York, it takes under 3 hours by local time, but the return journey will take just under 12 hours. Even Universal Time is very slightly biased to compensate for the earth's rotation.

Appendix D: Curvature and the solar system.

Imagine trying to map the surface of the earth. We can do so by putting markers on the ground (in England they are called “trig points”), and measuring the distance between the markers. Actually, in practice we measure angles as well, but we could get by just measuring distances. Now suppose we assume that the earth is flat. When we come to draw the map we will find that the measurements are not consistent, and we will assume that reflects errors made in the measurement.

If we were mapping a small enough part of the earth’s surface, the apparent errors would be tiny, possibly too small to detect, and we would ignore them and draw the map. A flat map of a curved surface must be slightly distorted, and the map we draw thinking the earth was flat is no exception. We don’t know where the distortion is, but it is there, caused by our ‘fixing up’ the apparent errors.

The effect of this distortion is that measurements made on the map are slightly misleading. Straight lines drawn on the map may not be quite straight on the ground. Angles drawn on the map may not be quite true. If we knew what and where the distortion was, we could use the measurements made on the map to calculate the correct distances, plot the true straight lines, or calculate the actual angles. But if we use the map without even realising it is distorted, we will just find the map is very slightly unreliable.

Much the same happens when we map space/time. Suppose we want to build a working model of the solar system, and we assume space is flat, and that the scale of space remains constant, and assume that time runs at the same rate everywhere, and we build the model as best we can. Suppose we now stop the model in order to look at a model of space at a single instant of time. A straight line drawn in the model will be almost a straight line in space. An angle measured in the model will be almost the same as the corresponding angle in space. A distance measured in the model will be almost to scale of the corresponding distance in space. There will be distortion there, but it will be very slight, probably far too slight for us to measure even today.

If we now let our model run, for a few seconds everything will move in a straight line unless it hits something.[12] But after a few seconds, the effect of gravity becomes noticeable, and it becomes obvious that things in the model are not going to move in straight lines in our model. If we take our slightly distorted model of space, and model time correctly, we will find that time runs very slightly slower in some parts than in others. This is a consequence of gravitational red shift. Remember the maps of the world showing a great circle as a curved line? [13] The line was curved because the scale of the map changed slightly across the neighbourhood of the line. In our model of the solar system the line which a planet follows is curved because the scale of time changes slightly as you cross the neighbourhood of the planet’s path.

If we ignore the slight variation in the rate of time, but assume there is a ‘force of gravity’ pushing anything from areas where time runs quickly towards areas where time runs slowly, then we can predict the bending of the path of the planets and other bodies almost exactly. The error that results from assuming space/time is flat and gravity is a force are tiny if we are mapping a volume that is as small and as peaceful as the solar system, but it can make it hard to accept that the planets travel in a straight line in space time.[14]

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[12] I’m only looking at big things, like a planet. If you drop a stone it hits the ground in much less than a second, telling you that the stone and the ground can’t both be going in a straight line.
[13] See appendix C.
[14] Remember: a body travels in a straight line unless acted upon by a force, and gravity is not a force.

Appendix E: Other curvature

It takes twenty values to describe curvature at a point in four dimensions.. Ten of these values form the Einstein tensor, the other ten values form the Weyl tensor. We experience the Einstein curvature as mass, momentum and stress, we experience the Weyl curvature as gravity. Two stones dropped from the same height both fall towards the centre of the earth, so as well as accelerating towards the earth, they accelerate towards each other. The acceleration towards the earth is an artefact of the model, but the acceleration towards each other is an effect of the Weyl curvature. If two stones one above the other are dropped at the same time, gravitational acceleration will be greater for the lower stone, so the two stones will draw apart. This is another effect of the Weyl curvature. The tides are yet another.

The mathematical definition of curvature is complicated because it does not make any assumption about the distortions in the model we use when we think about space/time. We can simplify the ideas if we look at one small volume, lasting only a small time, and make the model undistorted over that small volume of space/time. This is the equivalent of mapping a part of the earth's surface which is small enough for us to treat it as flat. This is always possible as long as there are no creases or corners in space/time.[15]

Imagine yourself in deep space, where there is no gravity, or at least, none expected. Now 'mark out' a cube of space by putting eight pebbles at the corners of a one metre cube. If the pebbles are placed carefully and are not moving, then we expect them to stay in the same place and they will continue to mark out a one metre cube. But if there is any space/time curvature within the cube the pebbles will start to move, gradually accelerating from their original positions.

If the cube changes volume as it distorts, then the cube contains curvature described by the Einstein tensor. If the cube distorts but keeps the same volume, then it contains curvature described by the Weyl tensor. [16]

Now imagine we placed more pebbles, forming a lattice of cubes, with one cube being 'our' cube, the original cube. If our cube changes in volume, then the surrounding cubes will be distorted. If the surrounding cubes contain no Einstein curvature, then they must contain Weyl curvature, since they will distort but without changing their volume. They in turn mean the more distant cubes distort, and they too must contain Weyl curvature.

Wheeler famously said 'Spacetime tells matter how to move; matter tells spacetime how to curve.'[17] This suggests mass causes the Einstein curvature, and the Einstein curvature causes the surrounding Weyl curvature, but that is slightly misleading. The curvature is there from the start: the mass no more causes the curvature than the curvature causes the mass. However, we can say that roughly speaking, inertia defines the Einstein tensor, and the Weyl tensor is the distortion forced on space time by the Einstein tensor. The scientific laws tell us how matter evolves, and hence how the Einstein tensor evolves. If we know how the Einstein tensor evolves, we can calculate how the Weyl tensor evolves.[18]

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[15]A crease or corner in space/time is called a singularity.

[16]Mathematically, it possible that a cube containing only Einstein curvature will distort without changing volume, but it is believed that flavour of Einstein curvature is not found in reality, or at least, not in detectable strength.

[17]This is the wording according to https://en.wikiquote.org/wiki/John_Archibald_Wheeler.

[18]I shall argue in a later paper that due to the quantum nature of matter we do not know **exactly** how matter evolves, so it is possible that the Weyl tensor is to some very small extent responsible for the evolution of the Einstein tensor.

Appendix F: a model of gravity on a universal scale.

A general formula

A pebble at a distance R from the sun has acceleration towards the sun equal to A/R^2 where A is a positive constant (of $length^3/time^2$) that we will evaluate later. So writing R” for d^2R/dt^2 and R' for dR/dt:

$R''=-A/R^2$
$R'dR'/dR=-A/R^2$
$R'^2/2=A/R+B$ where B is another constant (of $length^2/time^2$).

Let R_i and R''_i be the distance of the pebble from the sun and its acceleration towards the sun when it starts to drop (hence when $R=R_i$ and $R'=0$). Then $BR_i=-A$, so:

$R'^2/2=A/R-A/R_i$

To solve this, start by solving $(dx/dy)^2=(1/x -1)$. Given that solution, $R=R_ix$ and $t=\sqrt{(R_i^3/2A)}y$ is a solution to $R'^2/2=A/R-A/R_i$. [19] So $dx/\sqrt{(x/(1-x))}=dy$

Make the further substitution of $x=\cos^2(a)$, so:

$dx=-2\cos(a)\sin(a)da$ and
$\sqrt{(x/(1-x))}=\pm\cos(a)/\sin(a)$
$dx.\sqrt{(x/(1-x))} = 2\sin^2(a)da=[1-\cos(2a)]\ da$
$y=a-\sin(2a)/2 + \text{constant}$

Choose to set y=0 (and hence t=0) at x'=0 (and hence at x=1, a=0 and $R=R_i$), then:

$y=a-\sin(2a)/2$

When R=0, x=0, $\cos^2(a)=0$, $a=\pi/2$, $\sin(2a)=0$, so $y=\pi/2$, $t=(R_i^3/2A)^{0.5}\pi/2$

If the velocity of the earth in its orbit around the sun is V, then its acceleration towards the sun is V^2/R_i, so $-A/R_i^2=-V^2/R_i$ and so $A=V^2R_i$.

Time T to collapse = $(R_i^3/2A)^{0.5}\pi/2 = (R_i^2/2V^2)^{0.5}\pi/2 = (R_i/V)\pi/(2\sqrt{2})$

Applying the formula

$V=2\pi R_i/P$, where P is one year, so $T=P/(4\sqrt{2})$=roughly 64.5 days.

This ignores the effect of pressure on the collapse. The following calculates the time for any slightly earlier event.

When $R=R_i\delta$ where $\delta<<1$, $x=\delta=\cos^2(a)$,
$\cos(a)=\sqrt{\delta}=\sin(\pi/2-a)\approx\pi/2-a \Rightarrow a\approx\pi/2-\sqrt{\delta}$
$\sin(2a)=\sin(\pi-2\sqrt{\delta})=\sin(2\sqrt{\delta})\approx2\sqrt{\delta}$
$y=a-\sin(2a)/2\approx\pi/2-2\sqrt{\delta}=\pi/2(1-4\sqrt{\delta}/\pi)$

At $R=R_i/600$, the pebbles will just about be touching, and the pressure is about to escalate with incredible violence. This is after $64.5(1-4/\pi\sqrt{600})\approx61$ days. The pressure on the spacecraft will build up very little before this.

[19] Verifying this. If $R=R_ix$ then $dR=R_idx$, and if $t=\sqrt{(R_i^3/2A)}y$ then $dt=\sqrt{(R_i^3/2A)}dy$

$(dR/dt)^2/2=R_i^2/(2R_i^3/2A)\ (dx/dy)^2 =(A/R_i)(dx/dy)^2 = (A/R_i)(1/x -1)=A/R-A/R_i$

The number of ball bearings having the same mass as the sun.

The density of stainless steel is roughly 8,000 Kg/m^3, A sphere of radius r has volume $4\pi r^3/3$, so a sphere of steel radius 0.5cm has volume 0.52E-6m^3 so has mass roughly 4E-3Kg.

The number of ball bearings required to equal the mass of the sun is 2E30/4E-3=0.5E33.

The initial spacing of the ball bearings.

A sphere 1.5E8 Km in radius has volume 1.4E25 Km^3, so there are 0.5E33/1.4E25, roughly 0.35E8, ball bearings per cubic kilometre,. There is one ball bearing every $(0.35E8)^{-1/3}$Km, which is roughly 1 every 3 metres.

The Einstein tensor for this 'universe'.

The model contains only stationary mass, so only T^{tt} and G^{tt} are non-zero.

A unit volume of the model has proper volume $1/R^3$, so if the unit volume has mass M, the proper density, ϱ, will be M/R^3. Newton's law of gravitation says R" is proportional to $-1/R^2$, so ϱ is proportional to -R"/R. Hence if $G^{ab}=kT^{ab}$, then G^{tt} is proportional to -R"/R. [20]

Space/time <u>is</u> curved.

To recap, it was difficult to see how Newton's law of gravitation could be applied to an infinite universe that was roughly the same everywhere. On the one hand, choose any large sphere, and gravity would pull everything towards the centre of the sphere. On the other, if everywhere was the same, everywhere should behave the same, and if the centre of the sphere did not move, nothing should move. There were other attempts to address this problem before the advent of special relativity. One proposal suggested that the universe was not infinite, and that the mass of the universe was rotating so that the centripetal force of rotation balanced the gravitational attraction. For that reason it is worth noting that we can predict from special relativity that space/time is undoubtedly curved in the earth's gravitational field.[21]

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[20] The route to this result via tensor calculus runs as follows.

The universe described is a spherically curved space whose scale varies with time, so the metric may be expressed as diagonal {1, $-F^2(1,1,1)$ }, with indices t,x,y,z,
where F = C(r) R(t), $r^2=x^2+y^2+z^2$, $C=1/(1+a\,r^2)$, and 'a' is a constant.

From this we can calculate that the mixed Einstein tensor is diagonal, with:

$G^0_0=(3R'R'+12a)/R^2$ and

$G^x_x=G^y_y=G^z_z=(2RR''+R'R'+4a)/R^2$

= 0 since there is no pressure.

So $R'R'+4a=2RR''$, and $G^0_0=6R''/R$

[21] Here is a mathematical justification for that assertion.

If we assume the space about the earth is spherically symmetrical then it can be described by a coordinate system with line element $ds^2=e^{\nu}dt^2-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)$ and we can show that $R^t_{rtr}=-0.5r.e^{-\lambda}d\nu/dr$. Since gravitational red shift implies that $d\nu/dr$ is non-zero, and since e^{λ} is close to unity, at least one component of the Riemann tensor is non-zero in this coordinate system and so space/time is not flat.

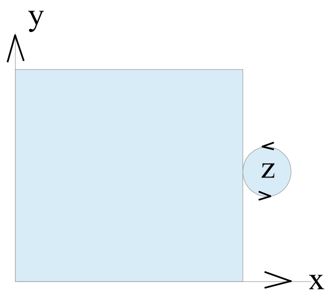
Appendix G: The stress energy tensor for the pebbles, and variations thereof.

The movement of a body is described by a four velocity that can be expressed as $\mathbf{v}=a_v\{1,v_x,v_y,v_z\}$, where $\mathbf{v}=\{v_x,v_y,v_z\}$ is the speed of the body in each of the 3 directions in space, and a_v is the dilation of the bodies proper time.[22] If the rest mass of the body is m_0 then the four momentum is $m_0\mathbf{v}=\{M,p_x,p_y,p_z\}$ where M is the mass when moving with velocity $\mathbf{v}$, and $\{p_x,p_y,p_z\}$ is the momentum in each direction. But a moving body contracts to a fraction a_v of its rest volume, so the density of mass and momentum is $\rho_0 a_v^2\{1,v_x,v_y,v_z\}$, where ρ_0 is the rest density of the body. This is the top line (or first column) of a matrix $T^{ab}=\rho_0\mathbf{v}^a\mathbf{v}^b$, meaning (for example) that element $T^{tx}=\rho_0\mathbf{v}^t\mathbf{v}^x=\rho_0 a_v^2 v_x$, and element $T^{xy}=\rho_0\mathbf{v}^x\mathbf{v}^y=\rho_0 a_v^2 v_x v_y$.

We may think of these densities as being smoothed out. The matter is concentrated in small volumes, like the pebbles in space discussed earlier, or like the molecules in water or air, but the density is the average density over some small volume. The variations in density and hence of curvature are there, but can be ignored for some purposes.[23]

The element T^{xx} is the pressure in the x direction. This is more obvious if we calculate the result of mixing two streams of particles moving in opposite directions, so $T^{ab}=\rho_0\mathbf{v}^a\mathbf{v}^b+\rho_0(-\mathbf{v}^a)(-\mathbf{v}^b)$. If $\mathbf{v}=a_v\{1,v_x,0,0\}$, then the only non-zero elements of T^{ab} are T^{tt} and T^{xx}, and we are describing a mass which has no net movement but is under pressure in the x direction. If we rotate the axes so that $\mathbf{v}=a_v\{1,v_x,v_y,0\}$ then it becomes apparent that T^{xy} is a shear or torque stress.[24]

The Newtonian laws of conservation and of motion can be applied to the values of T^{ab} and justify my asserting that some elements represent pressure, and some torque. These laws can be expressed as four differential equations, one for each row of T^{ab}. The first row expresses conservation of mass (or equivalently, of energy). If the momentum increases with distance as you cross a point, then the density of mass must be decreasing with time at that point unless there is a compensating external force. The remaining rows express conservation of momentum for each of the three directions. For example, unless there is a compensating external force the momentum in the x direction at a point will change over time if the pressure in the x direction changes as you cross the point in the x direction, or if the shear force in the y direction changes as you cross the point in the z direction, or if the shear force in the z direction changes as you cross the point in the y direction.



The effect of torque or shear.

The circle illustrates a positive torque in the z direction. If this torque increases in the y direction, then the result is a force in the x direction. If the same torque increases in the x direction, then the result is a force in the -y direction.

In tensor notation, the four equations are written as one: $T^{ab}{}_{;b}=F^a$ where F^a is the external four force. But if there is an external force acting on one form of matter, then there must be other types of

[22] $a_v=1/\sqrt{(1-v_x^2-v_y^2-v_z^2)}$

[23] This smoothing requires that the volume can be chosen large enough so that adjacent volumes do not differ significantly due to local variation, and yet small enough that neither do they differ significantly due to large scale variation. For example, the density of air decreases with elevation, the so the volume chosen when measuring the density must not be so large that the density of the chosen volume is significantly larger than the density of the volume immediately above or below it. The meaning of 'significantly' depends on the calculations we wish to make.

[24] I think of shear stress as being a stress that can be turned into pressure stress by a suitable choice of coordinate directions, as is the case here. A torque stress cannot be eliminated in this way. In tensor notation, a shear stress does not contribute to the value of the scalar $T^{ab}T_{ab}$ but a torque stress does.

matter at the point each with its associated stress energy tensor: if ΣT^{ab} is the sum of all the stress energies at a point, the external forces cancel and $\Sigma T^{ab}{}_{;b}=0$.

If the mass at a point is associated with curvature at the point, then the curvature must represented by a similar tensor that transforms in the same way, so the curvature will have the form $X^{ab}=kT^{ab}$, though there may be other forms of curvature at the same point. And the curvature X^{ab} must obey the same conservation laws, so $X^{ab}{}_{;b}=0$. That is sufficient to identify the curvature, and X^{ab} is known as the Einstein tensor, and usually written G^{ab}.

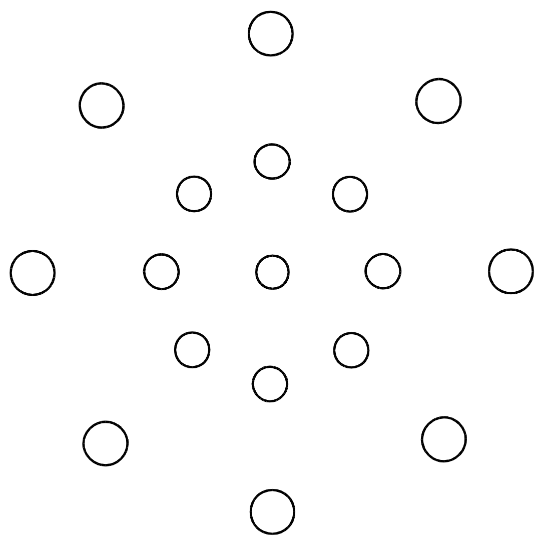
Appendix H: General relativity and cosmology.

Modelling a spherical universe

The simplest assumption is that the universe is the same everywhere, or rather, that it is possible to represent the universe by a model for which, with reasonable smoothing, space is the same everywhere at any instant of model time. Such a space is said to be spherical, because at any instant of model time, any extrinsically flat slice through the space will be intrinsically spherical.

Mapping the surface of a sphere.

It is not possible to make a map of a spherical surface to a constant scale, but it is possible to make one where the only distortion at any point is a change of scale, as in the map below. The scale of the map at a distance 'r' from the centre is $1/(1+ar^2)$.



This is a map of the surface of a sphere. The circles on the map are Tissot's indicatrix: they represent circles of a constant size drawn on the sphere.

At a distance 'r' from the centre of the map the scale of the map is $1/(1+ar^2)$. So if a short line on the map at a distance r from the centre is of length x, it represents a line on the sphere of length $x/(1+ar^2)$.

The circumference of the sphere is $\pi/2\sqrt{a}$, so you can perhaps see that our map represents a bit less than half the sphere.

Modelling a spherically curved space.

In the same way, we can design a model of a spherical space where the only distortion is a change of scale, and the scale at a distance 'r' from the centre of the model is $1/(1+ar^2)$.

Adding time.

The space can expand or contract over time, but at any instant of model time the space remains spherically curved. Thus the scale of the model becomes $R(t)/(1+ar^2)$: R gets larger as the space gets larger, and smaller as the space gets smaller. When I calculated the behaviour of a universe filled with pebbles, I was able to use Newton's law of gravitation to calculate $R''=d^2R/dt^2$, but only as long as there was no pressure.

The implications of symmetry.

The symmetry of spherically curved space means that the smoothed Einstein tensor, and hence the smoothed stress energy tensor, have only two parameters, the mass density ρ and the pressure density p. Any momentum, torque, or asymmetry of pressure has been smoothed away, as has any Weyl curvature. So (after converting to a locally 'natural' model[25]) $G^{tt}=k\rho$, and $G^{xx}=G^{yy}=G^{zz}=kp$.

The effect of pressure.

The mathematics of curvature allows us to determine that in a natural model:

$$G^{tt}=(3R'^2+12a)/R^2=k\rho \qquad \text{where } R'=dR/dt$$

and

$$G^{xx}=G^{yy}=G^{zz}=(2RR''+R'^2+4a)/R^2=kp$$

so

$$R''/R = k(\rho+3p)$$

Thus pressure increases the gravity effect in the expanding or collapsing universe.

The size and curvature of space.

If the scale of the model of the universe is $R/(1+ar^2)$ then if $a>0$ the space is positively curved and finite. A straight line in the universe of length $R\pi\sqrt{a}$ would go all the way across the universe and end up in the same place, much as a great circle through Paris (say) goes all the way across the surface of the earth and ends up at Paris again.[26]

If $a\leq 0$ then space is infinite; if $a=0$ then space is flat, if $a<0$ then space is negatively curved. The current thinking is that space is probably flat.[27]

Cosmology.

Einstein's initial assumption was that space contained mass but negligible pressure, implying the universe either started with a big bang or will end with a big crunch, or both. This seemed too improbable and so Einstein tried to imagine what else might contribute to the Einstein tensor. He guessed there might be something which had the value $T^{00}=-T^{xx}=-T^{yy}=-T^{zz}=\Lambda$ everywhere, and he called Λ the **cosmological constant**.[28] Later, when evidence was found that the universe did indeed start with a big bang, Einstein said that proposing the cosmological constant had been his biggest mistake.

However, there is (it seems) a lot of Einstein curvature in the universe that does not correspond to any inertia that we are aware of. On the scale of the solar system the match is good, but observation of galaxies suggests they contain a lot more mass than expected: the extra mass is referred to as “dark matter”. On an even larger scale the universe appears to be increasing its rate of expansion, suggesting there is a component of the Einstein tensor that may be similar to Einstein's cosmological constant. The matter driving this accelerated expansion is referred to as dark energy.

When it was first realised that the motion of stars within galaxies seemed inconsistent with the amount of mass in the galaxies, it was assumed that Newton's gravitational constant (G) was not in fact a universal constant. However, the postulated dark matter and dark energy are both consistent with G being constant.[29]

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25 By a 'natural' model I mean one where angles are true, and time is unbiased, and the scale is relativistic and unity and unchanging in all directions at the point under consideration. In mathematical terms, I mean the converting to a locally Minkowski metric whose first derivative is zero.

26 Length of line = $R\int_{-\infty}^{\infty} [dr/(1+ar^2)]$. Let $r=\tan(\theta)/\sqrt{a}$, $(1+ar^2)=(1+\tan^2(\theta))=\sec^2(\theta)$, $dr=\sec^2(\theta)d\theta/\sqrt{a}$, $dr/(1+ar^2)=d\theta/\sqrt{a}$. So length = $R\int_{-\pi/2}^{\pi/2} d\theta/\sqrt{a} = R\pi/\sqrt{a}$.

27 http://www.skyandtelescope.com/astronomy-resources/curvature-of-spacece/

28 That mix of inertia has the property that it appears the same in all reference frames. It can be thought of as an even density of mass together with a relatively gigantic tension.

29 There has been at least one theory that implied that G varied, proposed by Brans, Dicke and Jordan, but it adds a lot of complexity without yielding a better explaining of practical observations.