A Model for Topological Manipulations on Manifolds

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Abstract

The definition of an essential n-dimensional compact manifold is given. A tensor-like object used as an invariant for compact n-dimensional spaces and a model for describing topological manipulations based on the above invariants is proposed. The proposed theory is not fully proven and therefore this paper rather then being a proper formal paper has to be considered a note that gives some hints for further formal mathematical research.

Key Words: compact manifold, manifold decomposition.

1 Introduction

There are many way to decompose compact n-dimensional manifolds in simpler elements. A possible way is to decompose it as a connected sum of prime[1] compact manifolds. For example, for 2-dimensional manifolds, any compact manifold can be obtained by connected sum of 2-real projective planes (cross-caps) and 2-tori (handles) to a 2-sphere (see [1]).

However, it is possible to obtain any compact 2-dimensional manifold by adding and removing projective planes from a 2-sphere without using handles. For example you can obtain a tours by the following procedure:

- Add three cross-caps to a sphere by means of connected sums.
- Combine two of the three cross-caps on the sphere in order to obtain a Klein Bottle attached to the sphere.
- Readjust the Klein bottle so that it can be seen as an handle with one end attached to one side of the sphere and the other end attached to the opposite side of the sphere (see Fig. 1a).
- Drag one end of the handle (Klein Bottle) inside to the third cross-cap (see Fig 1a), and move it in order to make it emerge from the cross-cap attached to the opposite side of the sphere (see Fig. 1b). Now you have a proper handle (2-torus).

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[1]A prime manifold is an n-manifold that cannot be expressed as a non-trivial connected sum of two n-manifolds. Non-trivial means that neither of the two is an n-sphere

- Remove the third cross-cap. Now you have eventually a 2-torus.

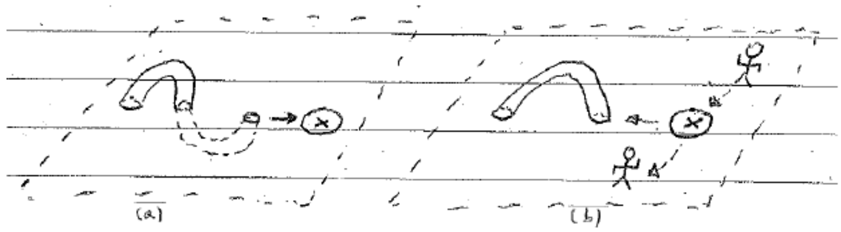

Figure 1: From Klein Bottle to Torus

From the example above we see that projective planes are somehow more fundamental then tori and, in this paper, we will call such kind of fundamental spaces "Essential Spaces".

Aim of this paper is to extend the idea introduced above and to make interesting considerations on it.

2 Strip configuration in 2-Dimensions

We start defining our the idea from the 2-dimensional case. Suppose we attach several cross-caps to a sphere. We want to characterise the surface we obtain by means of loops, in a similar manner of Homotopy and Homology, but in our case rather then using 1-dimensional loops, we will use 2-dimensional strips (see Fig. 2).

We see that we may associate a Möbius Strip linked to each cross-cap in such a way it cannot be unlinked and float away on the sphere (this is obvious since a Möbius Strip cannot be embedded in an oriented surface such the sphere). The two strips are homotopic.

We may add a third strip linking the two cross-cap which is twisted by the first cross cap and untwisted by the second one (it may also be twisted twice but we are not interested in this kind of strips) resulting in a final untwisted strip. This strip is, once again, homotopic to the first two described above, but we are really interested in configurations of strips with a number of elements equal to the number of generators of the fundamental groups, chosen as simple loops, and going around them. For the above reason, before adding the new untwisted strip, we remove one of the two twisted one.

We have therefore two possible configurations (see definitions in Appendix A.1) of strips going with our topological space and, since strips are not homotopic, the only way to go from on configuration to another, is by cutting the strips, moving them around the space and joining them again. For example, for the first configuration described above, we may cut the first strip so that we open the loop and we drag one end of the strip into the other cross-cap in such a way that, when it emerge from it, it is twisted with respect of the previous configuration (see Fig. 1b). By doing so, we may reattach the two ends of the strip and get a regular untwisted strip.

We will call the operation described above a a strip rearrangement (see definitions in Appendix A.1).

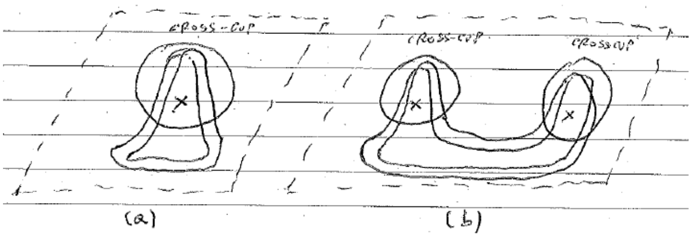

Figure 2: Strips on a Surface

We note that the second configuration has a Möbius Strip around one of the cross-caps and a regular untwisted strip linking the two cross-caps. This 2 strips may be easily associated to the relevant cycles ($\mathbb{Z}$ and $\mathbb{Z}_2$) of the H_1 Homology group of the Klein Bottle in the obvious way. It turns out that the two above described strip configurations, taken in account symmetries, are the only ones possible on a Klein bottle.

3 A Model for Manipulating the Strips

We want to find a model for describing the strip rearrangement operation, defined in the paragraph above. To each strip configuration we associate an object that, although misleading, we will call the tensor associated to it (see definitions in Appendix A.1). The term tensor is used because we will define, later on in this paper, a tensor-like product to combine simpler strip configurations. However, this kind of objects do not come with covariant and contravariant component and have noting to do with regular tensors.

We will associate to the S^2-strip (i.e. the untwisted strip) the tensor r_0=[1 1] and to the RP^2-strip (i.e. the twisted strip) the tensor r_1=[1 -1] (or [1 -1] both representations are good). In this notation the positive and negative signs are associated to the verse with which the two far ends of the strip are glued. According to this convention it is legal to associate also [-1 -1] to the S^2-strip although we will not use it.

Given k completely entangled strips (i.e. they do not cross or they cross an even number of times, see definitions in Appendix A.1), we will associate a tensor of rank k to it by means of a tensor-like product. For examples in a configuration of two entangled strips linked to 2 cross-caps attached to a sphere we would have the following 2-dimensional tensor associated:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1}$$

with 2 negative 1s on one of the diagonal depending on which representation we choose for each of the cross-caps.

A strip configuration is characterised by the number of strips, the way they are twisted and the way they are entangled (i.e. they cross each other). A good tensor representation of a strip configuration should be able to carry all this information.

Given a tensor T of order k it should be possible to associate a specific configuration C of k strips on an underlying space M to it. If the tensor cannot be factorised it means that the strips are entangled each other (i.e. they cross in a non trivial way, for example they cross an odd number of times). Given two sub-configurations C_1 and C_2 of $k_1 + k_2 = k$ strips, the level of entanglement of the two sub-configurations (i.e. an index of the number of times the strips cross) is given by the number of elements of the tensor that have to change sign in order to have T factorizable in two tensor T_1 and T_2 such that $T = T_1 \otimes T_2$. In general by evaluating the level of entanglement of each couple of sub-configurations, it should be possible to determine in the general case which strips cross. However, this task is not an easy task.

We want to find a way to perform the strip rearranged operation on the tensors rather then the strips themselves (see Appendix A.1). We define two elements A and B of the tensor to be equivalent if it is possible to move A to the position previously taken by B by means of moves consisting in rotating some of the tensor layers around one of the tensor axis just like in a Rubik's Cube. The analogy with the Rubik's cube is particularly evident when working on rank 3 tensors. For example in a 3-dimensional rank 3 tensor (i.e. basically a cube), all vertex elements are equivalent, all edge elements are equivalent, all center face elements are equivalent and the element at the very center of the cube is equivalent only to itself.

Our hypothesis, still to be proven, is that this manipulation can be done swapping equivalent elements. This operation corresponds to readjusting the strip configuration (i.e. cutting some strips and twisting them around a prime/essential space or to do the opposite operation) without changing the topology of the underlying space.

For example, with the tensor above we may go for the following moves:

$$\underbrace{\begin{bmatrix} 1 & -1^{\downarrow} \\ -1 & 1_{\uparrow} \end{bmatrix}}_{1} \rightarrow \underbrace{\begin{bmatrix} 1^{\downarrow} & 1 \\ -1_{\uparrow} & -1 \end{bmatrix}}_{2} \rightarrow \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}}_{3} \tag{2}$$

where $\rightarrow$ corresponds to one of our strip manipulations.

When we go from tensor 1 to tensor 2 (by swapping the two elements marked with an arrow), tensor 1 corresponds to a configuration with 2 strips linked to 2 cross-caps while tensor 2 corresponds to an untwisted strip and a twisted strip linked to 2 cross-cap (i.e. the twisted and the untwisted loop of a Klein Bottle). When we go from tensor 2 to tensor 3 we do the opposite strip rearrangement and we go back to the original configuration.

Now we add a third cross-cap to our manifold with the relevant strip and we evaluate the tensor configuration:

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{array}\right] \tag{3}$$

We perform our strip manipulation:

$$\left[\begin{array}{cc|cc} 1 & -1 & -1 & 1 \\ -1 & 1_{\rightarrow} & 1 & -1_{\leftarrow} \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{array}\right] \tag{4}$$

and, as a final step, we factorize the new strip configuration in factors having only 1s and -1s as elements:

$$\left[\begin{array}{cc|cc} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{array}\right] = \begin{bmatrix} -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{5}$$

We can summarize what we have found with the following equivalence:

$$\begin{bmatrix} -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cong \begin{bmatrix} -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{6}$$

In the equivalence above we have two factors in each side and this means that our manifold can be represented by two separated spaces attached by a connected sum to a sphere (see theorems in Appendix A.1).

The first factor in each side is relevant to a cross-cap. The second 2 by 2 factor in the left hand side of the equivalence is relevant to a Klein Bottle. The second 2 by 2 factor in the right hand side of the equivalence is no further factorisable in tensors which elements are only 1s and -1s and therefore corresponds to a configuration of 2 strips linked to a space with a level of entanglement 1 (they cross once) that cannot be separated in further prime spaces. Since there are no configurations of two essential space with this tensor representation, this must be a prime space. From what we have said in the previous paragraph we can easily see that this space is the 2-torus.

4 The 3-Dimensional Case

The method described above can be extended to space of any dimension. In this paragraphs we will try to extend it to 3-dimension compact manifolds.

Given a tetrahedron, we can couple its four faces in two pairs and identify them to get a compact 3-dimensional manifold. What we get is not a Δ-complex but what is known as a generalised triangulation (see [2]). Me may identify the two faces in each couple without torsion (and we assign 1 to it) or with a torsion (and we assign-1 to it). Finally, we record the way we identify faces by means of a row of 1s and -1s where the sign of the first identification is given by the sign transition between the first element of the row and the second one and the sign of the second identification is given by the sign transition between the second element of the row and the third one. We have three choices, for identifying faces (see Fig. 3):

- Choice [1 1 1]. This is a 3-sphere S^3. Also [-1 -1 -1] is a possible notation for it although we will not use it.
- Choice [1 1 -1]. This is the Lens space $L(4,1)$. Also [1 -1 -1], [-1 -1 1] and [-1 1 1] are possible notations for it.
- Choice [1 -1 1]. This is the Lens space $L(5,2)$. Also [-1 1 -1] is a possible notation for it.

We believe that the two Lens spaces defined above are good candidates to be our essential spaces for the 3-dimensional version of our theory and for brevity we will call then $E_1 = L(4,1)$ and $E_2 = L(5,2)$.

We need now the probe strips that, in this case, will be 3-dimensional object with boundaries. To get the 2-dimensional strips we could have used a square where two opposite edges of the square can be used as boundary of the strip, and the other two can be identified in according or opposite orientation. In this way we get the relevant 2-D probe Strips.

For the 3-D case we will use a cube instead of the square. We leave two opposite faces of the cube for the boundary of the probe strips and we identify

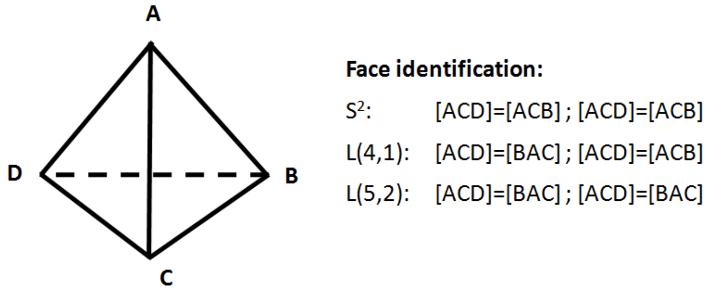

Figure 3: Essential 3D Spaces

the four remaining faces of the cube two by two. We get in this way, once again, three chooses for identifying the orientations of the two couples of faces of the cube and we get three probe strips for which we will use, as tensor notation, the same as for the relevant space the strip can be linked to. We have:

- S^3-strip. With tensor notation: p_0=[1 1 1].
- E_1-strip. With tensor notation: p_1=[1 1 -1].
- E_2-strip. With tensor notation: p_2=[1 -1 1].

where, as before, there is more then one tensor notation for each strip only one of which is reported above.

We can see the strips as fibre bundles with base space B made up of a strip and fibre F made by the unit interval I. For the S^3-strip, B is a plain untwisted strip and the fibre is not twisted. For the E_1-strip, B is a plain untwisted strip and the fibre is twisted or, which is the same, B is a twisted strip (Möbius Strip) with an untwisted bundle. The E_2-strip is a twisted strip (Möbius Strip) with a twisted bundle.

Basically, the probe strips defined above are solid loops carrying 2 axis with twisted status (see definitions in Appendix A.1). When a strip goes thought an E_1 space, either of the two axis switches its twisted status, when a strip goes through an E_2 space, both of the two axis switch their twisted status. In this way an essential space can change the strip twisted status just as happened in the 2D case.

Now we have all we need to extend our theory to 3-dimensional compact space in a straight forward way.

5 Prime Spaces in Dimensions Higher then 2

By performing the tensor product of two rank 1 tensors, and taking into account all the possible symmetries that do not change the strip configuration we get 51 strip configurations (class of tensors equivalent for symmetries with respect to the axis, rotations and change of sign of all elements). These configurations can be grouped in 25 classes of tensors equivalent with respect to the strip rearrangement operation (see Tab. from A2.4 to A2.6).

With this 25 classes of tensors we can write several tensor product equivalences of the form $p_i \otimes q_j \cong p_n \otimes q_m$ (see Tab. A2.3 for the definition of the

$p_{i,j}$). The equivalences have to be intended in the class of strip rearrangement relations for the final 3 rank tensor on the two sides of the equivalence. It is easy to show that if the equivalence is valid for one element of the class it is working with all of them and therefore we deal with classes more then with elements.

By checking all the possible combinations, we find that all the possible equivalences are the ones listed in the following table:

	Equivalence	Note
1	$p_0 \otimes q_2 \cong p_1 \otimes q_{14}$	$p_0 \otimes p_0 \otimes p_1$
2	$p_0 \otimes q_{13} \cong p_2 \otimes q_{14}$	$p_0 \otimes p_0 \otimes p_2$
3	$p_0 \otimes q_1 \cong p_1 \otimes q_2$	$p_0 \otimes p_1 \otimes p_1$
4	$p_2 \otimes q_1 \cong p_1 \otimes q_4$	$p_1 \otimes p_1 \otimes p_2$
5	$p_2 \otimes q_4 \cong p_1 \otimes q_{19}$	$p_1 \otimes p_2 \otimes p_2$
6	$p_2 \otimes q_{13} \cong p_0 \otimes q_{19}$	$p_0 \otimes p_2 \otimes p_2$
7	$p_2 \otimes q_2 \cong p_0 \otimes q_4 \cong p_1 \otimes q_{13}$	$p_0 \otimes p_1 \otimes p_2$

Table 1 : 3D Space Rank 3 Tensors Equivalences

All this equivalences are trivial and do not lead to any new prime space (unlike to what happened with Eq.(6) for the 2-dimensional case).

We may see what happen with rank 4 tensors. In this cases we can write equivalence of the form $q_i \otimes q_j \cong q_n \otimes q_m$. By checking all the possible combinations, we find that all the possible equivalences are the ones listed in the following table:

	Equivalence	Note
1	$q_1 \otimes q_{13} \cong q_2 \otimes q_4$	$p_0 \otimes p_1 \otimes p_1 \otimes p_2$
2	$q_1 \otimes q_{14} \cong q_2 \otimes q_2$	$p_0 \otimes p_0 \otimes p_1 \otimes p_1$
3	$q_1 \otimes q_{19} \cong q_4 \otimes q_4$	$p_1 \otimes p_1 \otimes p_2 \otimes p_2$
4	$q_2 \otimes q_{13} \cong q_4 \otimes q_{14}$	$p_0 \otimes p_0 \otimes p_1 \otimes p_2$
5	$q_2 \otimes q_{19} \cong q_4 \otimes q_{13}$	$p_0 \otimes p_1 \otimes p_2 \otimes p_2$
6	$q_{13} \otimes q_{13} \cong q_{14} \otimes q_{19}$	$p_0 \otimes p_0 \otimes p_2 \otimes p_2$

Table 2 : 3D Space Rank 4 Tensors Equivalences

Once again all this equivalences are trivial and do not lead to any new prime space (unlike to what happened with Eq.(6) for the 2-dimensional case). Although we should prove it, from what we see for the rank 3 and 4 case tensors, it looks like for 3D compact manifolds all the prime spaces are also essential and there is no way to turn any prime space into a new one by using the procedure described in the paragraphs above.

However, when we turn or attention to 4D compact manifolds we see that things change. To find an example we do not even need to run a brutal force exercise and analyse all the configurations. From Eq. (6) and substituting tensor elements with 2x2 and 2x1 blocks we have:

$$\begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix} \otimes \begin{bmatrix} + & + & - & - \\ + & + & - & - \\ - & - & + & + \\ - & - & + & + \end{bmatrix} \cong \begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix} \otimes \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & - & - \\ + & + & - & - \end{bmatrix} \tag{7}$$

If we name F_1 the underlying space of the rank 1 tensor on both sides of the equivalence (i.e. the F_1-Strip $[++--]$) then the left hand side represents a string configuration with underlying space $F_1\#F_1\#F_1$ because the rank 2 tensor on that side of the equivalence is the tensor product of the F_1-strip with itself. However, if we name F_2 the underlying space of the rank 2 tensor on the right hand side of the equivalence, since this tensor is not factorisable, this side represent a strip configuration with underlying space $F_1\#F_2$.

In analogy to the Klein Bottle to Torus transformation described in the previous chapters, if our theory is correct, the rank 2 tensor on the left hand side of the equivalence can be associated to a prime space F_2 which is not essential.

6 Conclusions

In this paper we have defined he concept of essential spaces. It is only a definition but it is probably the only rigorous mathematical things present in the paper. However, the idea to use solid strips to probe n-dimensional manifolds and define classes of tensor-like object as topological invariants for manifolds is interesting and, according to us, deserves further consideration.

We believe that a good hint showing that, once further detailed, the proposed theory is worth to be further studied, would be proving that in four dimensions there are prime spaces which are not essential (and we believe that this is true and easy to be proven) and that in three dimension all prime spaces are also essential as suggested by our theory.

Appendix

A.1 Theorems and Definitions

In this section we will provide some definitions and theorems in an attempt to give a slightly more formal structure to our theory.

Definition 1. *We define an n dimensional strip the space given by the product $I^{n-1} \times I$ where I is the unit interval and two faces I^{n-1} at the two ends of the second interval I are identified in various way. A n dimensional strips carries $n-1$ twist status. Two strips are equivalent if they have the same number of twisted and untwisted status.*

For a better understanding of the definition above, see the description of strips given in the previous paragraph.

Definition 2. *Given a compact n-dimensional manifold M, a strip configuration is a collection of k strips on the manifold such that each strip goes along the path of a simple loop generator (of the fundamental group) and all the strip are homotopic to each other.*

Where a simple loop, is a loop that goes around a circular path only once.

Definition 3. *Given a compact n-dimensional manifold M, and a strip configuration on it, this strip configuration is called maximal if and only if it is not possible to add any other strip to the configuration along the path of simple loop generator, which is homotopic to all the other strips already present in in it.*

Proposition 1. *Given a compact n-dimensional manifold M, all the maximal strip configurations on it are composed by the same number of strips.*

The above proposition is trivial.

Definition 4. *Given a compact n-dimensional manifold M, and a maximal strip configuration on it, any other maximal strip configuration on the same manifold it is called a strip rearrangement on M.*

Definition 5. *Given a compact n-dimensional manifold M, we define the rank of a M to be the number of strips present in each maximal strip configuration on M.*

Definition 6. *Given a compact n-dimensional manifold M, and two n-dimensional strips strips on it, then the two strips are said to be entangled if and only if they intersect an odd number of times and are said not to be entangled if they intercept an even number of times or they do not intersect. Moreover two sub-configurations of strips on the same manifold are said to be not entangled if and only if each strips of the first sub-configuration is not entangled to any strip of the second sub-configuration.*

For example, in a two dimensional torus the two strips going around the two main homotopy loops intersect in one point and are entangled.

Proposition 2. *Given two compact n-dimensional manifold M_1 and M_2 of rank k_1 and k_2, then the compact manifold $M_1 \# M_2$, connected sum of M_1 and M_2 has rank $k_1 + k_2$.*
The above proposition is trivial.

Proposition 3. *Given a compact n-dimensional manifold M of rank k then, it is possible to associate, in an unique way, each maximal strip configuration on it to a n-dimensional tensor T of rank k such that:*

- *if the manifold is the connected sum of two manifolds M_1 and M_2 and T is associated to a strip configuration C which is composed of two maximal non entangled sub-configurations C_1 and C_2 which are maximal configurations on M_1 and M_2 with tensor representation T_1 and T_2, then $T = T_1 \otimes T_2$.*
- *if the tensor T can be decomposed in two factors $T = T_1 \otimes T_2$, then M is the connect sum of two spaces M_1 and M_2 and the two factors T_1 and T_2 are the representation of maximal strip configurations on M_1 and M_2*
- *if f_i is a class of functions such that $T_i = f_1(T)$ where T_i is the tensor associated to a separate maximal strip configuration on M. Then the f_i are algebraic.*

where algebraic means that the f_i, which take tensors associated to maximal configurations on a manifold to tensors associated to maximal configurations on the same manifold (or at least homeomorfic to it), can be defined as an algebraic manipulation on the elements of T (including elements permutations) with no knowledge of the topological structure of the underlying space but based only on the dimension and rank of the tensor.

This is the only non trivial and most important theorem of our theory. The all point of this paper is about it. Unfortunately we do not have a proof of this theorem.

NOTE 1: We have proposed a possible definition of the f_i in the paragraphs above but our definition is a pure guess with no evidences. Further studies are necessary to prove that the above theorem is true and to find the correct definition of the f_i.

NOTE 2: If M is a compact manifold of rank k, all maximal strip configurations have a unique tensor representation of rank k. On the contrary there are tensors of rank k which represent a configuration of strips on M but do not represent any (maximal) strip configuration on M (as defined above). For example, in the 2-dimensional case, a rank 2 tensor which has all four elements equal to $+1$ represents a configuration of two untwisted strips. The underlying manifold of this configuration, at least as a compact surfaces, does not exist.

NOTE 3: Two different strip configurations may have the same tensor representation. For example, the Klein Bottle has three trip configurations (two non entangled twisted strips, two non entangled strips one twisted and one untwisted and two entangled strips one twisted and one untwisted) but only two tensor representations (see Tab. A2.2).

Definition 7. *Let M be a compact n-dimensional manifold and let T be the tensor representation of a maximal strip configuration on it. In analogy with the topological object we call T a (maximal) strip configuration on M.*

Definition 8. *In analogy with the topological case, we call the function f_i of the proposition above a strip rearrangement on the manifold M.*

A.2 Tensor Classes

In this paragraph we summarise all the possible tensor classes for both 2-dimensional and 3-dimensional spaces. Class elements are given taking into account the obvious symmetries (tensors equivalent for symmetries with respect to the axis, rotations and change of sign of all elements).

Class	Configurations	Space
r_1	$\begin{bmatrix} + & + \end{bmatrix}$	Untwisted Strip
r_2	$\begin{bmatrix} + & - \end{bmatrix}$	Möbius Strip

Table A2.1 : 2 Dimension Strip Classes

Class	Configurations	Space
s_1	$\begin{bmatrix} + & + \\ + & + \end{bmatrix}$	N/A
s_2	$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ and $\begin{bmatrix} - & - \\ + & + \end{bmatrix}$	$RP^2 \# RP^2 \cong K$
s_3	$\begin{bmatrix} + & + \\ + & - \end{bmatrix}$	T^2

Table A2.2: 2 Dimension Strip Classes

Class	Configurations	Space
p_1	$\begin{bmatrix} + & + & + \end{bmatrix}$	Untwisted Strip
p_2	$\begin{bmatrix} + & - & - \end{bmatrix}$	E_1-Strip
p_3	$\begin{bmatrix} + & - & + \end{bmatrix}$	E_2-Strip

Table A2.3: 3 Dimension Strip Classes

Class	Configurations	Space
q_1 $(p_1 \otimes p_1)$	$\begin{bmatrix} + & + & + \\ + & + & - \\ - & - & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ + & + & - \\ - & - & + \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & + \\ + & - & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & + \\ - & - & + \end{bmatrix}$ $\begin{bmatrix} + & + & + \\ - & + & - \\ - & + & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & - \\ + & + & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & - \\ - & + & + \end{bmatrix}$	$E_1 \# E_1$

Table A2.4 : 7 Element Classes

Class	Configurations	Space
q_2 $(p_0 \otimes p_1)$	$\begin{bmatrix} + & + & + \\ + & + & + \\ - & - & - \end{bmatrix}$; $\begin{bmatrix} + & + & + \\ + & + & - \\ - & + & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ + & + & + \\ - & - & + \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & + \\ + & + & - \end{bmatrix}$	?
q_3	$\begin{bmatrix} + & + & + \\ + & + & - \\ + & - & - \end{bmatrix}$; $\begin{bmatrix} + & + & + \\ + & + & - \\ - & - & + \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & + \\ + & - & + \end{bmatrix}$; $\begin{bmatrix} + & + & + \\ - & + & - \\ + & + & - \end{bmatrix}$	?
q_4 $(p_1 \otimes p_2)$	$\begin{bmatrix} + & + & + \\ - & + & - \\ - & - & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & - \\ + & - & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & - \\ - & - & + \end{bmatrix}$; $\begin{bmatrix} + & - & + \\ - & + & - \\ - & + & - \end{bmatrix}$	$E_1 \# E_2$
q_5	$\begin{bmatrix} + & + & - \\ + & + & - \\ - & - & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & + \\ - & - & - \end{bmatrix}$; $\begin{bmatrix} + & - & - \\ - & + & + \\ - & + & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & - \\ - & + & - \end{bmatrix}$	?

Table A2.4 : 4 Element Classes

Class	Configurations	Space
q_6	$\begin{bmatrix} + & + & + \\ + & + & + \\ + & - & - \end{bmatrix}$; $\begin{bmatrix} + & + & + \\ + & + & - \\ - & + & + \end{bmatrix}$	?
q_7	$\begin{bmatrix} + & + & + \\ + & + & + \\ - & + & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ + & + & + \\ - & + & + \end{bmatrix}$	?
q_8	$\begin{bmatrix} + & + & + \\ + & + & - \\ + & - & + \end{bmatrix}$; $\begin{bmatrix} + & + & + \\ - & + & - \\ + & + & + \end{bmatrix}$	?
q_9	$\begin{bmatrix} + & + & + \\ - & + & - \\ + & - & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & - \\ + & - & + \end{bmatrix}$	?
q_{10}	$\begin{bmatrix} + & + & - \\ + & + & + \\ - & - & - \end{bmatrix}$; $\begin{bmatrix} + & + & - \\ - & + & + \\ - & + & - \end{bmatrix}$	?
q_{11}	$\begin{bmatrix} + & + & - \\ - & + & - \\ - & - & - \end{bmatrix}$; $\begin{bmatrix} + & - & - \\ - & + & + \\ - & - & - \end{bmatrix}$	?
q_{12}	$\begin{bmatrix} + & - & + \\ - & + & - \\ - & - & - \end{bmatrix}$; $\begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix}$	?
q_{13} $(p_0 \otimes p_2)$	$\begin{bmatrix} - & + & - \\ + & + & - \\ - & - & - \end{bmatrix}$; $\begin{bmatrix} - & + & - \\ - & + & - \\ - & + & - \end{bmatrix}$	?

Table A2.5 : 2 Element Classes

Class	Configurations	Space	Class	Configurations	Space
q_{14} $(p_0 \otimes p_0)$	$\begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}$	N/A	q_{15}	$\begin{bmatrix} + & + & + \\ + & + & + \\ + & + & - \end{bmatrix}$	?
q_{16}	$\begin{bmatrix} + & + & + \\ + & + & + \\ + & - & + \end{bmatrix}$	?	q_{17}	$\begin{bmatrix} + & + & + \\ - & + & - \\ + & - & + \end{bmatrix}$	?
q_{18}	$\begin{bmatrix} + & + & - \\ + & + & + \\ - & + & - \end{bmatrix}$	?	q_{19} $(p_2 \otimes p_2)$	$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$	$E_2 \# E_2$
q_{20}	$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & - \end{bmatrix}$	?	q_{21}	$\begin{bmatrix} + & - & - \\ - & + & - \\ - & - & - \end{bmatrix}$	?
q_{22}	$\begin{bmatrix} - & + & - \\ + & + & + \\ - & + & - \end{bmatrix}$	?	q_{23}	$\begin{bmatrix} - & + & - \\ + & + & + \\ - & - & - \end{bmatrix}$	?
q_{24}	$\begin{bmatrix} - & + & - \\ - & + & - \\ - & - & - \end{bmatrix}$	?	q_{25}	$\begin{bmatrix} - & - & - \\ - & + & - \\ - & - & - \end{bmatrix}$	?

Table A2.6 : 1 Element Classes

References

[1] J. Gallier, D. Xu. *A Guide to the Classification Theorem for Compact Surfaces.* Springer (2013)

[2] V. Nardozza. *Homology Classes of Generalised Triangulations Made up of a Small Number of Simplexes.* www.vixra.org/abs/1409.0124 (2014)