Fractional Matrix : A new Eigenvalue method on Spectral Analysis.

Biswanath Rath(*)

Department of Physics, North Orissa University, Takatpur, Baripada -757003, Odisha, INDIA(* E.mail:biswanathrath10@gmail.com).

A new member in Matrix representation has been introduced under the name **Fractional – Matrix** and defined properly. Further we show how one can address spectral analysis using this **Fractional – Matrix**. Interesting examples have been considered.

PACS 02.10.Sp,03.65.Ge

Key words - fractional matrix, inverse matrix, eigenvalue, matrix diagonalisation method, differential equation.

**I.Introduction.**

Mathematics has been a beautiful subject for some one, who understands it in a fruitful way. Mathematics has been a tool to understand physical systems in a more specialised way [1]. In the context of differential equation

\[ \ddot{x} + f(x)\dot{x}^2 + g(x) = 0 \]  

(1)
or in eigenvalue relation

\[ A|\Psi \rangle = \lambda |\Psi_n \rangle \]  \hspace{1cm} (2)

its role is outstanding. In the above \( f(x), g(x) \) represent function of co-ordinate 'x' and the derivative is w.r.t time. In the context eigenvalue relation \( A \) stands for a standard matrix preferably symmetric and non-singular in its behaviour i.e \( \text{Det}(A) \neq 0 \). In this context we would like to state that for physical application matrix 'A' must be symmetric. Further we would like to state that all the literature on matrix analysis appears to be incomplete as no where it has been discussed "Fractional - Matrix" hence forward (FM). Hence aim of this communication is to introduce the FM and suggest how one can utilise it understanding the eigenvalues of a physical system.

II. Fractional - Matrix: Definition, Calculation and Eigenvalue relation.

Let us consider a matrix 'A' defined by the symbol

\[ A = a_{i,j} \]  \hspace{1cm} (3)

where \( i, j = 1 \ldots n \). In other word we consider (nxn) matrix. Let the inverse of matrix A is defined by the standard relation(1)

\[ A^{-1} = \text{Inv}(A) = \frac{\text{adj}(A)}{\text{det}(A)} \]  \hspace{1cm} (4)

Here the "adj(A)" i.e adjoint matrix is written in terms of co-factors.
Now we define the 'Fractional-Matrix' as

\[ B = \frac{\alpha + A}{A} = I + \alpha A^{-1} \]  

(5)

where \( I \) stands for an unit matrix of same dimension as that of \( A \). Let the eigenvalue relation of \( B \) is

\[ B|\Psi >= \beta|\Psi_n > \]  

(6)

then the eigenvalue of original matrix \( A \) can be written as

\[ \lambda = \frac{\alpha}{(\beta - 1)} \]  

(7)

In derieving the above relation we have taken standard relation (for symmetric matrix) as [2]9 for detail proof see [2])

\[ A\Psi = \lambda \Psi \rightarrow A^{-1}\psi = \frac{1}{\lambda} \Psi \]  

(8)

III. Example for Mathematical Sciences: Simple example of (2x2) matrix.

Let us consider a simple (2x2) matrix as

\[
A = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\]  

(9)

having inverse as

\[
A^{-1} = \begin{bmatrix}
1 & 0 \\
0 & 0.5
\end{bmatrix}
\]  

(10)
Now we fine fractional matrix $B_{\alpha=1,2,3,4}$ as for $\alpha = 1$

$$B_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$$ (11)

for $\alpha = 2$ as

$$B_2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$ (12)

for $\alpha = 3$ as

$$B_3 = \begin{bmatrix} 4 & 0 \\ 0 & 2.5 \end{bmatrix}$$ (13)

for $\alpha = 4$ as

$$B_4 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$ (14)

for $\alpha = 5$ as

$$B_5 = \begin{bmatrix} 6 & 0 \\ 0 & 3.5 \end{bmatrix}$$ (15)

Now we calculate the eigenvalues of original matrix $A$ through the formula

$$\lambda_n = \frac{\alpha}{(\beta_n - 1)}$$ (16)

and tabulate in table-1.
Table -I : Eigenvalues of Fractional- Matrix $B_\alpha$ and comparison with $A$

<table>
<thead>
<tr>
<th>. $A$</th>
<th>$\alpha$</th>
<th>$B_\alpha$</th>
<th>$A$ from $B_\alpha$</th>
<th>remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>same</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>2</td>
<td></td>
<td>same</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>same</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>2</td>
<td>2</td>
<td>same</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>same</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>2.5</td>
<td>2</td>
<td>same</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>same</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>3</td>
<td>2</td>
<td>same</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>same</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>3.5</td>
<td>2</td>
<td>same</td>
</tr>
</tbody>
</table>

IV. Example for Physical Sciences

Let us consider the differential equation as stated above [3] with $f(x) = \frac{x}{(1+x^2)}$ and $g(x) = \frac{x}{(1+x^2)}$. The corresponding Hamiltonian can be written as

$$H = \frac{1}{2} \left[ x^2 + \frac{p^2}{1 + x^2} \right]$$

(17)

The second term is not not invariant under exchane of momentum [4], so we consider the Hamiltonian as

$$H = \frac{1}{2} \left[ p^2 \frac{1}{1 + x^2} p + x^2 \right]$$

(18)
which is invariant under exchange of momentum. Using matrix diagonalisation method [5] we compute the eigenvalues of $H$ and $B$ (for $\alpha = 5$ and tabulate them in table-II as follows.

<table>
<thead>
<tr>
<th>n</th>
<th>$H$</th>
<th>$B \rightarrow \alpha = 5$</th>
<th>$H$ from $B$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.355 026 280 726</td>
<td>15.083 464 440 362</td>
<td>0.355 026 28</td>
<td>same</td>
</tr>
<tr>
<td>1</td>
<td>1.226 397 537 460</td>
<td>5.076 981 441 394</td>
<td>1.226 397 53</td>
<td>same</td>
</tr>
<tr>
<td>2</td>
<td>1.846 999 994 984</td>
<td>3.707 092 589 916</td>
<td>1.846 999 99</td>
<td>same</td>
</tr>
<tr>
<td>3</td>
<td>2.445 481 397 035</td>
<td>3.044 587 215 450</td>
<td>2.445 481 39</td>
<td>same</td>
</tr>
<tr>
<td>4</td>
<td>2.977 016 520 362</td>
<td>2.679 533 843 967</td>
<td>2.977 016 52</td>
<td>same</td>
</tr>
</tbody>
</table>

V. Discussion

We have introduced a new member in the theory of Linear Algebra under the section ”Matrix - Analysis” as ”Fractional Matrix”. The illustrated examples will motivate many to add new information to this new member. Last but not the least, the present method is a self-checking method on eigen-
References


