

# An early contribution to vector maximisation of De Finetti\*

Ate Nieuwenhuis<sup>†</sup>

First version April 6, 2016; this version September 4, 2016

An achievement of De Finetti for which he has received little recognition thus far is his contribution to the field of vector maximisation in two articles published in 1937, *Problemi di "optimum"* and *Problemi di "optimum" vincolato*. The speech will put his contribution in historical perspective and will discuss its importance for economic theory.

## Introduction

It is a great honour for me to be invited to deliver a speech at this event in memory of Bruno de Finetti.

The achievements of De Finetti are many. He is best known for his contributions to the theory of probability (of which he liked to say: "Probability does not exist."), contributions for which he has gained recognition internationally during his lifetime. But other work has gone largely unnoticed for a long time. Only rather recently has it become widely known that in an article published in 1940, *Il problema dei "pieni"*, De Finetti has anticipated to a great extent the results in the Nobel-Prize-winning article of Markowitz on *Portfolio selection*, published in 1952. In the language of financial economics, the problem is to select a portfolio of securities so as to maximise expected return and to minimise the variance of return (as a measure of the riskiness of the portfolio). De Finetti has studied a mathematically equivalent problem in a different setting, that of determining the optimal way of reinsuring parts of a portfolio of insurances according to the same maxim. Please note that we here have a problem with two objectives, maximal expected return and minimal variance of return, to be achieved with the same set of instruments. It is what we nowadays call a problem of *vector optimisation*, also called *multiple-objective optimisation* or *multicriteria optimisation*. De Finetti knew how to deal with it. In fact, he had given "... a sketch of what might be the systematic and general treatment of <such> problems" in two articles published in 1937. The articles are

*Problemi di "optimum"* and *Problemi di "optimum" vincolato*,

where "optimum" between parentheses denotes what we nowadays call *Pareto optimum*. As far as I know, the articles contain the first systematic treatment of the vector optimisation problem ever. The reason for me standing here is that I have translated them into English.

---

\*Speech delivered at a seminar in memory of Bruno de Finetti, Sapienza Università di Roma (Dipartimento di Metodi e Modelli per l'Economia, il Territorio e la Finanza), April 6, 2016.

<sup>†</sup>E-mail address: ate\_1949@hotmail.com

As a personal note, allow me to add that I have arrived at the articles not through *Il problema dei "pieni,"* but along a different route. At some time, my work at CPB Netherlands Bureau for Economic Policy Analysis (where I spent my working life) excited my interest in the oligopoly problem. During my search of the literature I stumbled upon the English translation of a paper of Emilio Zaccagnini, *Simultaneous maxima in pure economics*. Zaccagnini recognized the oligopoly problem as a vector maximisation problem and treated it along the lines sketched by De Finetti, to whose articles he referred. My desire to go back to the sources was fulfilled not long afterwards when, through an Italian connection, I got hold of copies of the articles. It took a long time, learning how little traces they had left in the literature, until I decided to translate them. Only last year did I learn that an English translation of the second article was available even before I started my work. It is

*Problems of constrained "optimum,"*

translated by Loredana Gustin and Renato Palessoni of the university of Triëste (at least at the time), and published in *Italian Economic Papers*, Volume 3, 1998, edited by Luigi Pasinetti.

## Vector optimisation

Having set the record straight, let me return to the subject of today. Vector optimisation originated independently in three areas:

1. Economic equilibrium and welfare theories,
2. Game theory,
3. Pure mathematics.

De Finetti was motivated by the work of Pareto on economic theory. As I already mentioned, *Pareto optimum* is the common name for the solution of a vector optimisation problem. De Finetti denotes it with "optimum" between parentheses. Other names for vector optimisation are *multi-objective optimisation* and *multicriteria optimisation*.

De Finetti's style of exposition in the two articles is admirable. He starts with some very simple examples, passes on to more complicated ones, and illustrates them with figures. When he finally gets to a mathematical treatment of the general problem, he focuses on the essential and avoids technicalities. I have next to no knowledge of his other work, but from what I have read about it I have gathered that this style is characteristic of his writings generally and of the way he taught.

In the first example the choice set is discrete. The problem is this:

A traveler may choose between nine roads along which to go to his destination. The routes are, in order of decreasing panoramic beauty according to his taste,

**A   B   C   D   E   F   G   H   I**

and have the lengths in kilometres of respectively

**57   59   55   43   50   45   42   48   42**

If this traveler wants both minimal length and maximal panoramic beauty, he may quickly discard the routes B, E, F, H and I. Why? Because they are preceded in panoramic beauty by shorter ones. He is thus left with a choice between the routes **A, C,**

**D** and **G**. These routes constitute the solution set of the problem as stated. The choice of one particular route requires a trade-off between length and panoramic beauty; absent such a trade-off, the solution remains indeterminate to some extent.

Immediately in the first example De Finetti drove home this important point of indeterminacy, which he feared might not be clear from Pareto's work.

Game theory as constructed by Von Neumann and Morgenstern also deals with discrete choice sets, the strategies in the reduced form of a game. Strategies are generally vectors of possibly considerable length. There are other differences as well:

- Whereas in the example above there is just one decision maker with more than one objective, in game theory there are at least two decision makers or players, each with his own objective.
- Moreover, each player is in command of only a subset of the strategies (the subsets do not overlap).
- Finally, players may form coalitions to further their interests (when there are at least three players).

All this does not detract from the fact that a game is a vector optimisation problem, albeit a considerably complicated one. Hence it causes no surprise that the solution of a game as defined by Von Neumann and Morgenstern is generally set-valued. We will see an example in a little while.

In the second example De Finetti switches to continuous decision variables and differentiable objective functions with continuous first derivatives. The problem is to choose a point in some plane with minimal Euclidean distance to two given points  $M_1$  and  $M_2$ . The two sets of concentric circles in Figure 1 are the level curves of the objective functions. The solutions are all points of the line segment  $M_1M_2$ . In any point  $P$  not on the line  $m$  through  $M_1$  and  $M_2$  a circle of one bundle intersects a circle of the other bundle; all points in the lens-shaped area inside both circles are closer to  $M_1$  and  $M_2$  than point  $P$  is. On the line  $m$ , by contrast, a circle of one bundle is tangent to a circle of the other bundle. Still, in points outside of the segment  $M_1M_2$  one may get closer to both  $M_1$  and  $M_2$  (the tangency condition is necessary, not sufficient).

The very same example was used by the mathematician and Fields medalist Smale, who in the 1970s ventured into economics. Inspired by Pareto's notion of optimum, just

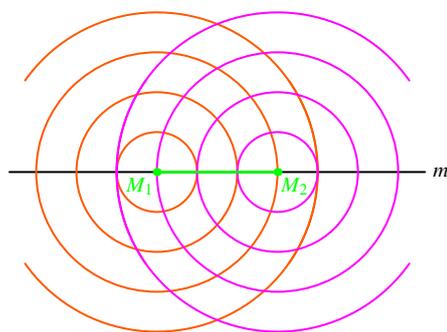


Figure 1: Simultaneous maximality

like De Finetti, he independently reinvented vector optimisation almost four decades later. In a series of articles he applied vector optimisation to the study of general economic equilibrium using a calculus approach. In my opinion his work is still underrated.

De Finetti gives several other examples in the first article, and he mentions that vector optimisation problems occur also in the technical sciences. As a matter of fact, multi-objective optimisation has been an active field of research since the 1970s, with applications in engineering, medicine, and many other areas. Stadler, who is a pioneer of this line of applied mathematics, writes:

Multicriteria optimization is what one attempts to carry out in every applied decision process.

In his characteristic way De Finetti deliberately keeps the mathematical treatment simple, giving merely "... a sketch of what might be the systematic and general treatment ... " of the problem. Had he strived for mathematical rigor, he might have scooped Kuhn and Tucker, who in their 1951-article *Nonlinear programming* devote a section to vector optimisation subject to inequality constraints. This section is the place where vector optimisation has emerged in pure mathematics. Still there is an economic connection: Kuhn and Tucker seem to have been inspired by the concept of vector maximum in the contemporary paper of Koopmans with the title *Analysis of production as the efficient combination of activities*.

De Finetti postpones an example in economics till after his treatment of the problem of vector optimisation subject to equality constraints in the second article. The special case of one maximand and one constraint is of course familiar to every student of economics. Probably the first such problem he encounters is that of the budget-constrained consumer who wants maximal utility. With just two goods we get Figure 2. The quantities of the two goods are measured along the axes. The figure shows three (out of infinitely many) isoquants and the line representing the given budget (the prices are constant parameters). The optimal choice is at the tangent point of the budget line with one of the isoquants.

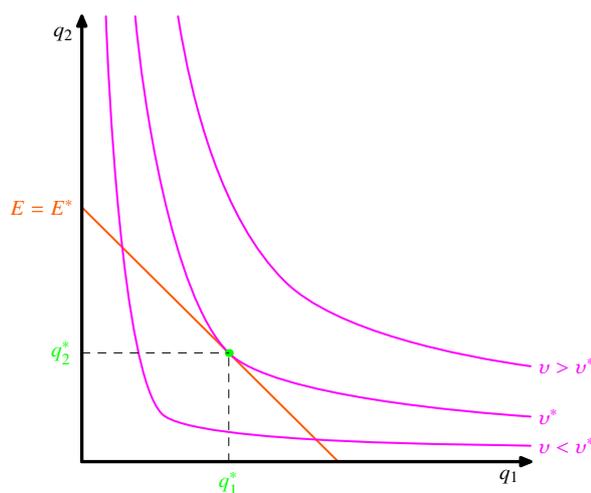


Figure 2: Maximal utility  $v^*$  at given expenditure  $E^*$

Figure 3 is for the dual problem of minimising expenditure at a given level of utility. The figure shows the isoquant corresponding to the given level of utility and three (out of infinitely many) parallel budget lines. The optimal choice is at the tangent point of the isoquant with one of the budget lines.

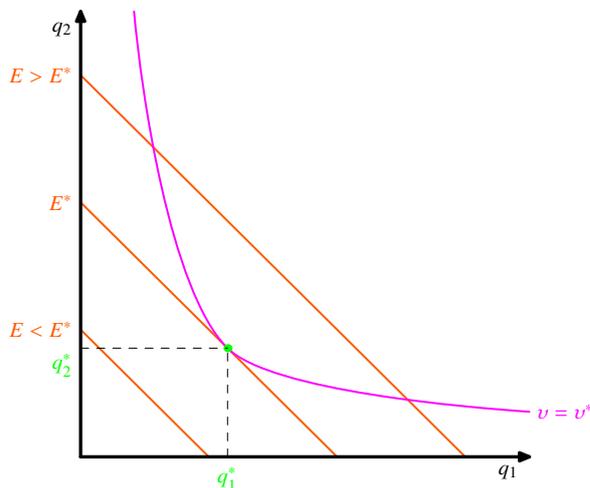


Figure 3: Minimal expenditure  $E^*$  at given utility  $v^*$

And if we want maximal utility at minimal expenditure – still at constant prices – we get Figure 4.

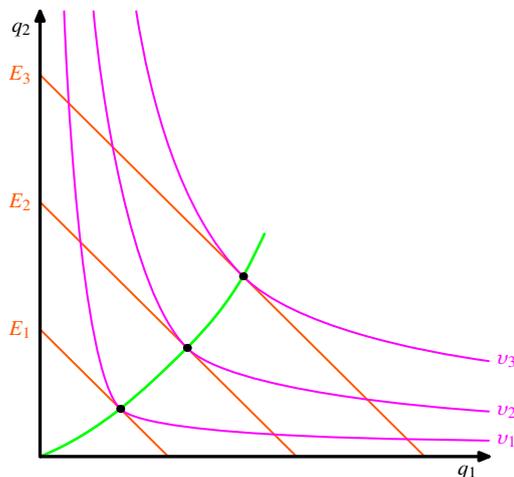


Figure 4: Maximal utility at minimal expenditure

The green curve is the set of Pareto-optimal points, where an isoquant and a budget line are tangent.

The last three figures show the close relationship between vector optimisation and constrained optimisation. In practice, a popular way of dealing with a complex multi-objective optimisation problem is by repeatedly solving a number of associated constrained optimisation problems.

A more interesting case of constrained vector optimisation, taken by De Finetti as an example, is what one might call the "allocation problem." Fixed quantities of  $m$  goods must be allocated to  $n$  individuals so as to maximise their utilities. Let us take the simplest case of two individuals and two goods. Then again we can represent the problem in a figure:

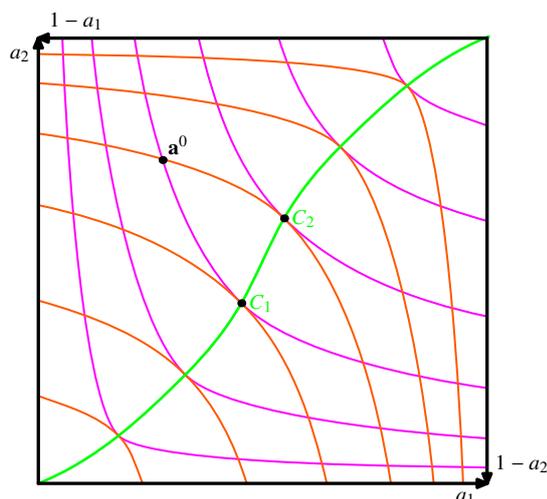


Figure 5: The contract curve

The first individual receives the share  $a_1$  of the first good, measured along the horizontal axis at the bottom, and the share  $a_2$  of the second good, measured along the vertical axis to the left. The shares allocated to the second individual are measured in the opposite directions along the two other axes. We see isoquants of the first (second) individual, the curves convex to the Lower Left (Top Right) corners. The green curve is the Pareto optimum.

Now you will ask, "Where are the constraints in this problem?" Well, I have used them to eliminate the shares of the second individual from the problem, knowing that for each good the shares sum to one. With just two individuals elimination is the easiest way to deal with the problem; with more than two individuals the method of De Finetti is to be preferred. In fact my reason for including the problem is not to illustrate the treatment of De Finetti, it lies elsewhere: with its help I can elucidate the link between vector optimisation and game theory.

To that end, let us add some elements to the problem and interpret it as a game.

1. There is an initial allocation of the goods to the individuals, yielding them certain initial levels of utility.
2. The individuals may exchange some quantities of the goods among themselves to better their positions.
3. Exchange is voluntary, that is, neither individual may be forced to enter an exchange that lowers his utility.

The problem is a very simple game indeed. The strategies of the players are scalars (the shares of the goods). And when there are only two players with opposed interests, the issue of coalition formation does not arise. The solution of the problem is indicated

in the figure. It is a subset of the Pareto optimum, the segment  $C_1C_2$ . The points of the green curve outside the segment  $C_1C_2$  are not acceptable to one of the individuals, because there his utility is lower than in the initial situation. By assumption he is able to block the exchanges involved. Only points of the segment  $C_1C_2$ , which lies within the lens-shaped area demarcated by the isoquants through the initial point, are acceptable to both individuals. At any point of this area not on the green curve there is an incentive for further exchange.

The segment  $C_1C_2$  is the famous Contract Curve, which in this game coincides with the "solution" as defined by Von Neumann and Morgenstern.

The whole figure itself is called the Edgeworth Box. In fact the Contract Curve was first derived by Edgeworth and the box was first drawn by Pareto. Anyhow, both Edgeworth and Pareto are credited for their early application of the notion of vector maximum.

Edgeworth and Pareto both wrote on the duopoly problem, which is a vector maximisation problem, without applying the notion of Pareto optimum. This raises doubts as to whether they were fully aware of the notion's significance and general applicability, in economics and other sciences. De Finetti definitely was. In any case, he deserves credit for being the first to give (a sketch of) the systematic and general treatment of vector maximisation.

## Incorrect conditioning

I would now like to say a few words on the current state of economic theory as I see it. To clarify my view, consider the following imaginary history of mathematical statistics.

- Initially statisticians had no notion of dependent random variables (*random quantities*, De Finetti would have preferred to say). Whenever they needed the joint distribution of two random variables, they simply took the product of the conditional distributions of the individual random variables.
- One day a clever guy pointed out that the correct way was to multiply the conditional distribution of one variable with the marginal distribution of the other one.
- The community of statisticians was full of admiration. But after an initial surge of interest, they continued to use the product of the conditional distributions in their studies.

In my view the actual history of economic theory closely matches this imaginary history of mathematical statistics. Let me explain why.

- Initially economists had no notion of vector maximisation. Whenever they needed a multi-person economic model, they first derived the first-order conditions of the conditional maximum problem of each agent, treating only the agent's own decision variables as endogenous. Next they assembled all first-order conditions into one system of equations, usually adding some new (!) constraints (the market equilibrium conditions).
- Later some clever guys pointed out that the individual maximum problems were in fact interdependent and had to be treated as a whole from the start.
- The community of economists was full of admiration. In spite of the initial surge of interest, they never really stopped building their models in the old way.

The essential message of Von Neumann and Morgenstern (and of De Finetti a decade earlier), I think, is that multi-person economic models based on the rationality postulate are vector maximisation problems and must be treated as such. (Their use of discrete choice sets and emphasis on coalition formation may have hidden the message.)

According to Nash, each agent considers the – endogenous – actions of the other players as given if they are unable to cooperate. Mathematically the assumption replaces the vector maximisation problem by the old set of conditional maximum problems. Thus the work of Nash provides the "justification" for the old practice.

Regrettably, non-cooperative game theory has come to dominate economic theory. It seems that the meaning and importance of vector maximisation are still not understood and appreciated by many at their proper value. I have written a paper on the subject, which I recommend to your attention.

## **Closing remarks**

During the preparations for my speech I have read some articles about De Finetti, the man, his life, his work. I appreciate that he was a concerned citizen, and that his interest in economics derived from his desire to further the common good and to enhance social justice. As I read along, a picture of De Finetti arose before my eyes that reminded me of Tinbergen. I see a great affinity between them.

In The Netherlands, Tinbergen, through his social concern, moral standards, the choice of subjects he took up, and the high quality of his work has set an example that has inspired many economics students, of which I am one. I presume that likewise in Italy De Finetti has inspired many students. Correct me if I am wrong.

We honour and commemorate such men.

Let me say no more. Today I merely wanted to point out to you that an item may be added to the list of achievements of De Finetti: He made an early contribution to vector maximisation.

## References

- De Finetti, B., 1937a, Problemi di “optimum,” *Giornale dell’Istituto Italiano degli Attuari*, vol. 8, no. 1, pp. 48–67.
- De Finetti, B., 1937b, Problemi di “optimum” vincolato, *Giornale dell’Istituto Italiano degli Attuari*, vol. 8, no. 2, pp. 112–26.
- De Finetti, B., 1940, Il problema dei “pieni,” *Giornale dell’Istituto Italiano degli Attuari*, vol. 9, no. 1, pp. 1–88.
- Edgeworth, F.Y., 1881, *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences*, C. Kegan Paul & Co., London.
- Koopmans, T. C., 1951, Analysis of production as an efficient combination of activities, in T.C. Koopmans, ed., *Activity Analysis of Production and Allocation*, Cowles Commission Monograph 13, Wiley, New York.
- Kuhn, H.W. and A.W. Tucker., 1950, Nonlinear programming, in *Proceedings of the Second Berkley Symposium on Mathematical Statistics and Probability* (held in 1951), pp. 481–92, University of California Press, Berkley.
- Markowitz, H., 1952, Portfolio selection, *Journal of Finance*, vol. 6, pp. 77–91.
- Nash, J., 1951, Non-cooperative games, *Annals of Mathematics*, vol. 54, no. 2, pp. 286–95.
- Nieuwenhuis, A., 2011, *Simultaneous maximisation in economic theory*, translations of De Finetti (1937a,b), available at <http://www.brunodefinetti.it> and at <http://vixra.org/abs/1309.0119>.
- Nieuwenhuis, A., 2013, Reconsidering Nash: the Nash equilibrium is inconsistent, available at <http://vixra.org/abs/1309.0118> and at [https://www.dropbox.com/s/ctoxntaocy9tgz2/NotNash\\_vixra.pdf](https://www.dropbox.com/s/ctoxntaocy9tgz2/NotNash_vixra.pdf).
- Pareto, V., 1909, *Manuel d’Economie Politique*, Giard et Brière, Paris.
- Smale, S., 1975, Optimizing several functions, in *Manifolds, Tokyo 1973: Proceedings of the International Conference on Manifolds and Related Topics in Topology*, pp. 69–75, University of Tokyo Press, Tokyo.
- Stadler, W., 1987, Initiators of multicriteria optimization, in J. Jahn and W. Krabs, eds., *Recent Advances and Historical Development of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, No. 294, Springer-Verlag, Berlin.
- Von Neumann, J. and O. Morgenstern, 1947, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, New Jersey, Second ed.
- Zaccagnini, E., 1947, Massimi simultanei in economia pura, *Giornale degli Economisti e Annali di Economia*, vol. 6 (Nuova serie), pp. 258–92.
- Zaccagnini, E., 1951, Simultaneous maxima in pure economics (translation by R. Orlando of Zaccagnini, 1947), in A.T. Peacock, H.S. Houthakker, R. Turvey, F. Lutz and E. Henderson, eds., *International Economic Papers*, vol. 1 of *Translations prepared for the International Economic Association*, pp. 208–44, Macmillan and Company Ltd and The Macmillan Company, London and New York.