Duplex Fraction Method To Compute The Determinant Of A 4 × 4 Matrix

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Abstract. In this paper, We present a new method to compute the determinant of a 4 × 4 matrix, that is very simplest than previous methods in this subject. This method is obtained by a new definition of fraction and also by using the Dodgson's condensation method and Salihu's method.

Keywords: 4 × 4 matrix, determinant, Dodgson's condensation, Salihu's method

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1 Introduction

In the matrix theory and linear algebra, determinant of a square matrix very important. By [4], the basic formula to compute the determinant of a square matrix of order *n*, such as $A = \{a_{ij}\}$, is equal to

$$D(A) = det(A) = |A| = \sum_{j_1 \dots j_n \in s_n} sgn(j_1 \dots j_n)\, a_{1j_1} \dots a_{nj_n}$$

Where $sgn(j_1 \dots j_n) = \begin{cases} +1 & , if\ j_1 \dots j_n \ is\ an\ even\ permutation \\ -1 & , if\ j_1 \dots j_n \ is\ an\ odd\ permutation \end{cases}$.

Also Dodgson in 1866 [1] and Salihu in 2012 [3], offered two methods for compute the determinant of square matrix of order *n* that we using of their methods for proof of the new method in this article. whereas for compute the determinant of a square matrix, always we used of simple methods, therefore in this article we present a simple method for compute the square matrix of order 4. For this work we offer a new definition of fraction that we named it the duplex fraction. In the next section you can see this definition. Afterward using the duplex fraction and Dodgson's condensation method and Salihu's method we obtain a new method for compute the square matrix of order 4.

2 The main definitions and lemmas

To prove the main theorem we need the following definitions and lemmas.

Definition 2.1. *Let* $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *and* $B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. *If* $det(B) \neq 0$ *and* $b_{ij} \neq 0$ $(\forall\, i, j = 1,2)$ *then the duplex fraction (duplex division) of determinant A on B is defined as follows*

$$\frac{|A|}{|B|} = \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}} = \frac{\begin{vmatrix} \frac{a_{11}}{b_{11}} & \frac{a_{12}}{b_{12}} \\ \frac{a_{21}}{b_{21}} & \frac{a_{22}}{b_{22}} \end{vmatrix}}{\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}} \qquad (1)$$

Definition 2.2. *Let* $B = \{b_{ij}\}$ *is a* $n \times n$ *matrix. Dodgson's condensation of matrix B is a* $(n-1) \times (n-1)$ *matrix that defined as follows*

$$DC(B) = \begin{bmatrix} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} & \cdots & \begin{vmatrix} b_{1(n-1)} & b_{1n} \\ b_{2(n-1)} & b_{2n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} b_{(n-1)1} & b_{(n-1)2} \\ b_{n1} & b_{n2} \end{vmatrix} & \cdots & \begin{vmatrix} b_{(n-1)(n-1)} & b_{(n-1)n} \\ b_{n(n-1)} & b_{nn} \end{vmatrix} \end{bmatrix} \tag{2}$$

Definition 2.3. *Let* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$. *If Dodgson's condensation of matrix A ,is equal to*

$$DC(A) = \begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{bmatrix} \tag{3}$$

Then, Twice Dodgson's condensation of matrix A is defined as follows

$$TDC(A) = DC\big(DC(A)\big) = \begin{bmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{bmatrix} \tag{4}$$

Dodgson's condensation for the first time used in the compute the determinant of a $n \times n$ matrix by Dodgson [1].

Lemma 2.1(Dodgson's method). *Determinant of matrix* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, *with assuming* $a_{22} \neq 0$, *is equal to*

$$|A| = \frac{1}{a_{22}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} \tag{5}$$

Proof. See Dodgson's condensation method [1,2].

Lemma 2.2(Salihu's method). *Determinant of matrix* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ *is equal to*

$$|A| = \frac{1}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \times \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \tag{6}$$

If $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0.$

Proof. See Salihu's method [3].

3 A new method

In the following theorem we offer a new method, just to compute the determinant of a 4 × 4 matrix. This method is obtained of Dodgson's condensation method and Salihu's method.

Theorem 3.1. *Determinant of matrix* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ *is equal to*

$$|A| = \frac{|\,TDC(A)\,|}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \tag{7}$$

If $a_{22}, a_{23}, a_{32}, a_{33} \neq 0$ *and* $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0.$

Proof. If $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0$, by lemma 2.2 we have

$$|A| = \frac{1}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \times \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \end{vmatrix}$$

On the other hand if $a_{22}, a_{23}, a_{32}, a_{33} \neq 0$, by lemma 2.1 we have

$$|A| = \frac{1}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \times \begin{vmatrix} \frac{1}{a_{22}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{23}} \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{32}} \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{33}} \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix}$$

Now, using the definition 2.1, we can write

$$|A| = \frac{\begin{Vmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{Vmatrix}}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}$$

That by definition 2.3, we know

$$|TDC(A)| = \begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix}$$

Therefore, we have

$$|A| = \frac{|\,TDC(A)\,|}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}$$

And proof is complete.

Example . The determinant of $A = \begin{bmatrix} 2 & 3 & 7 & 1 \\ 4 & 5 & 10 & 0 \\ 6 & 3 & 2 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}$, is obtain as follows:

$$\xrightarrow{\text{DC (A)}} \begin{bmatrix} -2 & -5 & -10 \\ -18 & -20 & 0 \\ 9 & 1 & 4 \end{bmatrix} \xrightarrow{\text{DC (DC (A))}} \begin{bmatrix} -50 & -200 \\ 162 & -80 \end{bmatrix}$$

$$|\text{A}| = \frac{\begin{vmatrix} -50 & -200 \\ 162 & -80 \end{vmatrix}}{\begin{vmatrix} 5 & 10 \\ 3 & 2 \end{vmatrix}} = \frac{\begin{vmatrix} \frac{-50}{5} & \frac{-200}{10} \\ \frac{162}{3} & \frac{-80}{2} \end{vmatrix}}{\begin{vmatrix} 5 & 10 \\ 3 & 2 \end{vmatrix}} = -74$$

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