Surface Formulations of the Electromagnetic-Power-based Characteristic Mode Theory for Material Bodies

Renzun Lian

***Abstract*—Recently, a volume formulation of the ElectroMagnetic-Power-based Characteristic Mode Theory (CMT) for Material bodies (Mat-EMP-CMT) is built by expressing various electromagnetic powers as the functions of the total fields in material bodies, so it can be simply called as Vol-Mat-EMP-CMT. As a companion to the Vol-Mat-EMP-CMT, several Surface formulations of the Mat-EMP-CMT (Surf-Mat-EMP-CMT) are established in this paper by expressing various electromagnetic powers as the functions of the surface equivalent sources on the boundaries of material bodies.**

The physical essence of Surf-Mat-EMP-CMT is the same as the Vol-Mat-EMP-CMT, i.e., to construct the various power-based Characteristic Mode (CM) sets for material bodies, but the former is more advantageous than the latter in some aspects. For example, the former saves computational resources; the former avoids to compute the modal scattering field in source region; the field-based definitions for the impedance and admittance of material bodies can be easily introduced into the former.

***Index Terms*—Characteristic Mode (CM), Electromagnetic Power, Input Admittance, Input Impedance, Input Power, Interaction, Material Body, Output Power, Surface Equivalent Principle.**

I. INTRODUCTION

THE Characteristic Mode Theory (CMT), which was firstly established by Robert J. Garbacz in 1965 [1], is a new modal theory which is different from the traditional Eigen-Mode Theory (EMT) [2] in classical mathematical physics. In 1971, Roger F. Harrington and Joseph R. Mautz refined Garbacz's CMT by placing the CMT into the Integral Equation-based MoM (IE-MoM) framework, and established a Surface EFIE-based CMT for PEC systems (PEC-SEFIE-CMT) [3]. Subsequently, many variants of the PEC-SEFIE-CMT were developed under the IE-MoM framework, such as the Volume Integral Equation-based CMT for Material bodies (Mat-VIE-CMT) [4], the PMCHWT-based CMT for Material bodies (Mat-PMCHWT-CMT) [5], the Surface MFIE-based CMT for PEC systems (PEC-SMFIE-CMT) [6], and the Surface CFIE-based CMT for PEC systems (PEC-SCFIE-CMT) [7]. A very comprehensive review for the CMT and its applications can be found in [8]. In addition, the Poynting's theorem-based interpretations for the PEC-SEFIE-CMT and Mat-PMCHWT-CMT are respectively given in [8] and [9], such that the physical pictures of these two IE-MoM-based CMTs become clearer.

R. Z. Lian is with the School of Electronic Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China. (e-mail: rzlian@vip.163.com).

Recently, some ElectroMagnetic-Power-based CMTs (EMP-CMTs), such as the EMP-CMT for PEC systems (PEC-EMP-CMT) [10] and the EMP-CMT for Material bodies (Mat-EMP-CMT) [11] are introduced. Not only some classical Characteristic Mode (CM) sets, such as the CM set derived from PEC-SEFIE-CMT and the CM set derived from the Mat-VIE-CMT for lossless non-magnetic material bodies, are generalized, but also many new power-based CM sets, such as the radiated power CM set which has ability to optimize the radiation of electromagnetic system, are constructed under the EMP-CMT framework.

In [11], the optimization problems for various power functionals were transformed into the matrix characteristic value problems by expressing various objective electromagnetic powers as the functions of the total field in material body. Based on this, the Mat-EMP-CMT developed in [11] is specifically called as the Volume formulation of the Mat-EMP-CMT, and simply denoted as Vol-Mat-EMP-CMT. As a companion to the Vol-Mat-EMP-CMT, several Surface formulations of the Mat-EMP-CMT (Surf-Mat-EMP-CMT) are developed in this paper by expressing various objective electromagnetic powers as the functions of the surface equivalent sources on the boundary of material body.

The physical essence of Surf-Mat-EMP-CMT is the same as the Vol-Mat-EMP-CMT, i.e., to construct various power-based CM sets for material bodies, which have abilities to depict the inherent characteristics of material bodies to utilize various electromagnetic energies. The applicable range of Surf-Mat-EMP-CMT is similar to the Vol-Mat-EMP-CMT, for example, the Surf-Mat-EMP-CMT is valid for the lossy material bodies placed in any electromagnetic environment like the Vol-Mat-EMP-CMT, and any kind of objective power optimized by the Vol-Mat-EMP-CMT can be selected as the object to be optimized by the Surf-Mat-EMP-CMT.

In addition, the Surf-Mat-EMP-CMT is more advantageous than the Vol-Mat-EMP-CMT in some aspects. For example, the former saves the computational resources; the former avoids to compute the modal scattering field in source region [12]; the field-based definitions for the impedance and admittance introduced in [10] can be easily generalized to the former.

Although many electromagnetic powers can be selected as the objects to be optimized by Surf-Mat-EMP-CMT, only the theory and formulations corresponding to the input/output power given in [10]-[11] are explicitly provided in this paper because of its notable importance as explained in [10]-[11]. The CM sets derived by optimizing other objective powers can be similarly obtained, and they will not be specifically provided in this paper.

In what follows, the $e^{j\omega t}$ convention is used throughout.

II. Interaction, Output Power, and Input Power

The material body is simply called as scatterer in this paper. When an external excitation field $\bar{F}^{inc}$ incidents on the scatterer, the scattering sources will be excited on the scatterer V, and then the scattering field $\bar{F}^{sca}$ is generated in whole space $\mathbb{R}^3$, as illustrated in Fig. 1. The summation of $\bar{F}^{inc}$ and $\bar{F}^{sca}$ is the total field, and it is denoted as $\bar{F}^{tot}$, i.e., $\bar{F}^{tot} = \bar{F}^{inc} + \bar{F}^{sca}$, here $F = E, H$.

When the conductivity of scatterer is not infinity, the scattering sources include the volume ohmic electric current $\bar{J}^{vo}$ and the related electric charges $\{\rho^{vo}, \rho^{so}\}$ due to the conduction phenomenon, the volume polarized electric current $\bar{J}^{vp}$ and the related electric charges $\{\rho^{vp}, \rho^{sp}\}$ due to the polarization phenomenon, and the volume magnetic current $\bar{M}^{vm}$ and the related magnetic charges $\{\rho_m^{vm}, \rho_m^{sm}\}$ due to the magnetization phenomenon [13]-[15]. The $\{\rho^{vo}, \rho^{vp}, \rho_m^{vm}\}$ are the volume charges, and the $\{\rho^{so}, \rho^{sp}, \rho_m^{sm}\}$ are the surface charges on the boundary of scatterer. The various charges are related to the corresponding currents by current continuity equations, so it is sufficient to only use the scattering currents to determine the scattering field [13]-[15].

A. *The interaction between incident field and scatterer.*

The interaction between the incident field and scatterer is just the interaction between the $\{\bar{E}^{inc}, \bar{H}^{inc}\}$ and $\{\bar{J}^{vop}, \bar{M}^{vm}\}$, and its mathematical expression is as follows

$$\begin{aligned} \mathcal{I} &= (1/2)\langle \bar{J}^{vop}, \bar{E}^{inc} \rangle_V + (1/2)\langle \bar{H}^{inc}, \bar{M}^{vm} \rangle_V \\ &= (1/2)\langle \bar{J}^{vop}, \bar{E}^{tot} \rangle_V + (1/2)\langle \bar{H}^{tot}, \bar{M}^{vm} \rangle_V \\ &\quad -(1/2)\langle \bar{J}^{vop}, \bar{E}^{sca} \rangle_V - (1/2)\langle \bar{H}^{sca}, \bar{M}^{vm} \rangle_V \end{aligned} \tag{1}$$

In (1), the inner product is defined as $\langle \bar{g}, \bar{h} \rangle_\Omega \triangleq \int_\Omega \bar{g}^* \cdot \bar{h} \, d\Omega$, and the symbol " $*$ " represents the complex conjugate of relevant quantity, and the symbol " $\cdot$ " is the scalar product for field vectors. The second equality in (1) is due to that $\bar{F}^{inc} = \bar{F}^{tot} - \bar{F}^{sca}$.

Inserting the (A-2) into the second line in (1), and inserting the (A-4) into the last line in (1), the interaction $\mathcal{I}$ can be rewritten as follows

$$\mathcal{I} = P^{sca,rad} + P^{tot,loss} + j\left(P^{sca,react,vac} + P^{tot,react,mat}\right) \tag{2}$$

here the $P^{sca,rad}$ is the radiated power carried by scattering field [16]; the $P^{sca,react,vac}$ is the reactively stored scattering power in vacuum [16]; the $P^{tot,loss}$ is the ohmic loss power due to the interaction between the total electric field $\bar{E}^{tot}$ and scatterer [16]; the $P^{tot,react,mat}$ is the reactive power due to the polarization and magnetization originating from the interaction between the total fields $\{\bar{E}^{tot}, \bar{H}^{tot}\}$ and scatterer [16].

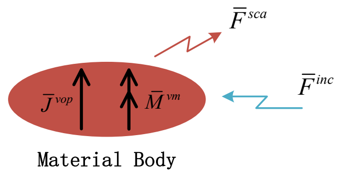

Fig. 1. The material body excited by incident field.

The mathematical expressions for the various powers mentioned above are given as follows [16]

$$P^{sca,rad} = (1/2)\oiint_{S_\infty}\left[\bar{E}^{sca} \times \left(\bar{H}^{sca}\right)^*\right] \cdot d\bar{S} \tag{3.1}$$

$$P^{sca,react,vac} = 2\omega\left(W_m^{sca,vac} - W_e^{sca,vac}\right) \tag{3.2}$$

and

$$P^{tot,loss} = (1/2)\langle \sigma \bar{E}^{tot}, \bar{E}^{tot} \rangle_V \tag{4.1}$$

$$P^{tot,react,mat} = 2\omega\left(W_m^{tot,mat} - W_e^{tot,mat}\right) \tag{4.2}$$

In (3.2), the $W_m^{sca,vac}$ and $W_e^{sca,vac}$ are respectively the magnetic and electric energies stored in scattering field; in (4.2), the $W_m^{tot,mat}$ and $W_e^{tot,mat}$ are respectively the total magnetized and polarized energies stored in matter due to the interaction between the total field and scatterer, and their mathematical expressions are as follows [16]

$$W_m^{sca,vac} = (1/4)\langle \bar{H}^{sca}, \mu_0 \bar{H}^{sca} \rangle_{\mathbb{R}^3} \tag{5.1}$$

$$W_e^{sca,vac} = (1/4)\langle \varepsilon_0 \bar{E}^{sca}, \bar{E}^{sca} \rangle_{\mathbb{R}^3} \tag{5.2}$$

and

$$W_m^{tot,mat} = (1/4)\langle \bar{H}^{tot}, \Delta\mu \bar{H}^{tot} \rangle_V \tag{6.1}$$

$$W_e^{tot,mat} = (1/4)\langle \Delta\varepsilon \bar{E}^{tot}, \bar{E}^{tot} \rangle_V \tag{6.2}$$

The meanings of various material parameters appearing in (5) and (6) can be found in Appendix A.

B. *Output power.*

Based on the physical meanings of the various powers in (2) and the discussions in the Appendix B, it is easily found out that the interaction $\mathcal{I}$ equals to the output power P^{out} generated by material scatterer, i.e.,

$$\begin{aligned} P^{out} &= P^{sca,rad} + P^{tot,loss} + j\left(P^{sca,react,vac} + P^{tot,react,mat}\right) \\ &= (1/2)\langle \bar{J}^{vop}, \bar{E}^{tot} \rangle_V + (1/2)\langle \bar{H}^{tot}, \bar{M}^{vm} \rangle_V \\ &\quad -(1/2)\langle \bar{J}^{vop}, \bar{E}^{sca} \rangle_V - (1/2)\langle \bar{H}^{sca}, \bar{M}^{vm} \rangle_V \end{aligned} \tag{7}$$

C. Input power.

If the input power from the external excitation to the material scatterer is denoted as symbol P^{inp}, the following (8) can be derived from the conservation law of energy [17].

$$P^{inp} = P^{out} \tag{8}$$

Based on the (1), (7), and (8), it can be found out that the physical essence of the interaction $\mathcal{I}$ is just the input power P^{inp} which is the power done by the incident fields $\{\bar{E}^{inc},\bar{H}^{inc}\}$ on the scattering currents $\{\bar{J}^{vop},\bar{M}^{vm}\}$, i.e.,

$$P^{inp} = (1/2)\langle \bar{J}^{vop},\bar{E}^{inc}\rangle_V + (1/2)\langle \bar{H}^{inc},\bar{M}^{vm}\rangle_V = \mathcal{I} \tag{9}$$

III. The Surface Equivalent Source-based Expressions for Output Power

Based on the conclusions given in Appendix C, the $\bar{F}^{sca}$ on $\mathbb{R}^3$, the $\bar{F}^{tot}$ on V, the $\bar{F}^{inc}$ on V, and the $\{\bar{J}^{vop},\bar{M}^{vm}\}$ on V can be written as the following linear operator forms.

$$\begin{aligned}\bar{F}^{sca}(\bar{r}) &= \bar{F}^{sca}(\bar{J}^{SE},\bar{M}^{SE};\bar{r})\\ &= \begin{cases}\bar{F}^{sca}_-(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) &, \quad (\bar{r}\in V)\\ \bar{F}^{sca}_+(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) &, \quad (\bar{r}\in \mathbb{R}^3\setminus V)\end{cases}\end{aligned} \tag{10}$$

$$\bar{F}^{tot}_-(\bar{r}) = \bar{F}^{tot}_-(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) \quad , \quad (\bar{r}\in V) \tag{11}$$

$$\bar{F}^{inc}_-(\bar{r}) = \bar{F}^{inc}_-(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) \quad , \quad (\bar{r}\in V) \tag{12}$$

$$\bar{J}^{vop}(\bar{r}) = \bar{J}^{vop}(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) \quad , \quad (\bar{r}\in V) \tag{13.1}$$

$$\bar{M}^{vm}(\bar{r}) = \bar{M}^{vm}(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) \quad , \quad (\bar{r}\in V) \tag{13.2}$$

here the $\bar{J}^{SE}$ and $\bar{M}^{SE}$ are the surface equivalent sources defined in (C-17) and (C-18).

A. The operator form I of output power.

By inserting the (10), (11), and (13) into the second equality in (7), the output power P^{out} can be expressed as the function of surface equivalent sources $\bar{J}^{SE}$ and $\bar{M}^{SE}$ as follows

$$\begin{aligned}P^{out} = &\ (1/2)\langle \bar{J}^{vop}(\bar{J}^{SE},\bar{M}^{SE}),\bar{E}^{tot}_-(\bar{J}^{SE},\bar{M}^{SE})\rangle_V\\ &+(1/2)\langle \bar{H}^{tot}_-(\bar{J}^{SE},\bar{M}^{SE}),\bar{M}^{vm}(\bar{J}^{SE},\bar{M}^{SE})\rangle_V\\ &-(1/2)\langle \bar{J}^{vop}(\bar{J}^{SE},\bar{M}^{SE}),\bar{E}^{sca}_-(\bar{J}^{SE},\bar{M}^{SE})\rangle_V\\ &-(1/2)\langle \bar{H}^{sca}_-(\bar{J}^{SE},\bar{M}^{SE}),\bar{M}^{vm}(\bar{J}^{SE},\bar{M}^{SE})\rangle_V\end{aligned} \tag{14.1}$$

here the position vectors $\bar{r}$ in various operators have been omitted to simplify the symbolic system of this paper. In fact, the (14.1) can also be simply rewritten as the following operator form.

$$P^{out} = P^{out}_1(\bar{J}^{SE},\bar{M}^{SE}) \tag{14.2}$$

B. The operator form II of output power.

By inserting the (10) into the (A-8)-(A-10), and inserting the (11) into the (4) and (6), and utilizing the relation (A-13), and employing the first equality in (7), the output power P^{out} can be expressed as the function of surface equivalent sources $\bar{J}^{SE}$ and $\bar{M}^{SE}$ as follows

$$\begin{aligned}P^{out} = &\ (1/2)\oint\!\!\!\oint_{\partial V}\{\bar{E}^{sca}_+(\bar{J}^{SE},\bar{M}^{SE})\times[\bar{H}^{sca}_+(\bar{J}^{SE},\bar{M}^{SE})]^*\}\cdot d\bar{S}\\ &+(1/2)\langle \sigma\bar{E}^{tot}_-(\bar{J}^{SE},\bar{M}^{SE}),\bar{E}^{tot}_-(\bar{J}^{SE},\bar{M}^{SE})\rangle_V\\ &+j\,2\omega\Big[(1/4)\langle \bar{H}^{sca}_-(\bar{J}^{SE},\bar{M}^{SE}),\mu_0\bar{H}^{sca}_-(\bar{J}^{SE},\bar{M}^{SE})\rangle_V\\ &\qquad-(1/4)\langle \varepsilon_0\bar{E}^{sca}_-(\bar{J}^{SE},\bar{M}^{SE}),\bar{E}^{sca}_-(\bar{J}^{SE},\bar{M}^{SE})\rangle_V\Big]\\ &+j\,2\omega\Big[(1/4)\langle \bar{H}^{tot}_-(\bar{J}^{SE},\bar{M}^{SE}),\Delta\mu\bar{H}^{tot}_-(\bar{J}^{SE},\bar{M}^{SE})\rangle_V\\ &\qquad-(1/4)\langle \Delta\varepsilon\bar{E}^{tot}_-(\bar{J}^{SE},\bar{M}^{SE}),\bar{E}^{tot}_-(\bar{J}^{SE},\bar{M}^{SE})\rangle_V\Big]\end{aligned} \tag{15.1}$$

and the (15.1) can also be simply rewritten as the following operator form.

$$P^{out} = P^{out}_2(\bar{J}^{SE},\bar{M}^{SE}) \tag{15.2}$$

C. The operator form III of output power.

By inserting the (12) and (13) into the (9) and employing the relation (8), the output power P^{out} can be expressed as the function of surface equivalent sources $\bar{J}^{SE}$ and $\bar{M}^{SE}$ as follows

$$\begin{aligned}P^{out} = P^{inp} = &\ (1/2)\langle \bar{J}^{vop}(\bar{J}^{SE},\bar{M}^{SE}),\bar{E}^{inc}_-(\bar{J}^{SE},\bar{M}^{SE})\rangle_V\\ &+(1/2)\langle \bar{H}^{inc}_-(\bar{J}^{SE},\bar{M}^{SE}),\bar{M}^{vm}(\bar{J}^{SE},\bar{M}^{SE})\rangle_V\end{aligned} \tag{16.1}$$

and the (16.1) can also be simply rewritten as the following operator form.

$$P^{out} = P^{out}_3(\bar{J}^{SE},\bar{M}^{SE}) \tag{16.2}$$

D. The uniform form for the output power operators (14.2), (15.2), and (16.2).

It is obvious that

$$P^{out}_1(\bar{J}^{SE},\bar{M}^{SE}) = P^{out}_2(\bar{J}^{SE},\bar{M}^{SE}) = P^{out}_3(\bar{J}^{SE},\bar{M}^{SE}) \tag{17}$$

though the mathematical expressions of the operators in (14.2), (15.2), and (16.2) are different from each other. In the following discussions, the above three output power operators are uniformly written as the following (18).

$$P^{out} = P^{out}(\bar{J}^{SE},\bar{M}^{SE}) \tag{18}$$

In fact, all other powers which are intrinsically related to the material scatterer as discussed in Appendixes A and B can also be expressed as the functions of surface equivalent sources $\bar{J}^{SE}$

and $\bar{M}^{SE}$, and they are not repeated here.

IV. The Matrix Forms for Output Power

In this section, several matrix forms for the output power P^{out} are provided.

A. The matrix forms for output power.

The surface equivalent source $\bar{C}^{SE}$ is expanded in terms of the basis function set $\{\bar{b}_\xi^C\}_{\xi=1}^{\Xi^C}$ as follows

$$\bar{C}^{SE} = \sum_{\xi=1}^{\Xi^C} a_\xi^C \bar{b}_\xi^C = \bar{B}^C \cdot \bar{a}^C \tag{19}$$

here $C = J, M$, and the symbol " $\cdot$ " in (19) represents matrix multiplication, and

$$\bar{B}^C = \begin{bmatrix} \bar{b}_1^C & , & \bar{b}_2^C & , & \cdots & , & \bar{b}_{\Xi^C}^C \end{bmatrix} \tag{20.1}$$

$$\bar{a}^C = \begin{bmatrix} a_1^C & , & a_2^C & , & \cdots & , & a_{\Xi^C}^C \end{bmatrix}^T \tag{20.2}$$

In (20), the superscript " T " represents the transpose of related matrix.

Inserting the (19) into (18), the output power P^{out} can be written as the following matrix form.

$$P^{out} = \bar{a}^H \cdot \bar{\bar{P}}^{out} \cdot \bar{a} \tag{21}$$

here

$$\bar{a} = \begin{bmatrix} \bar{a}^J \\ \bar{a}^M \end{bmatrix} \tag{22.1}$$

$$\bar{\bar{P}}^{out} = \begin{bmatrix} \bar{\bar{P}}_{JJ}^{out} & \bar{\bar{P}}_{JM}^{out} \\ \bar{\bar{P}}_{MJ}^{out} & \bar{\bar{P}}_{MM}^{out} \end{bmatrix} \tag{22.2}$$

and

$$\bar{\bar{P}}_{C'C''}^{out} = \left[p_{C'C'',\xi\zeta}^{out} \right]_{\Xi^{C'} \times \Xi^{C''}} \tag{23}$$

here $C', C'' = J, M$, and $\xi = 1, 2, \cdots, \Xi^{C'}$, and $\zeta = 1, 2, \cdots, \Xi^{C''}$.

In the (23),

$$\begin{aligned} p_{C'C'',\xi\zeta}^{out} = & (1/2)\left\langle \bar{J}^{vop}\left(\bar{b}_\xi^{C'}\right), \bar{E}_-^{tot}\left(\bar{b}_\zeta^{C''}\right)\right\rangle_V \\ & + (1/2)\left\langle \bar{H}_-^{tot}\left(\bar{b}_\xi^{C'}\right), \bar{M}^{vm}\left(\bar{b}_\zeta^{C''}\right)\right\rangle_V \\ & - (1/2)\left\langle \bar{J}^{vop}\left(\bar{b}_\xi^{C'}\right), \bar{E}_-^{sca}\left(\bar{b}_\zeta^{C''}\right)\right\rangle_V \\ & - (1/2)\left\langle \bar{H}_-^{sca}\left(\bar{b}_\xi^{C'}\right), \bar{M}^{vm}\left(\bar{b}_\zeta^{C''}\right)\right\rangle_V \end{aligned} \tag{24.1}$$

for the operator form I given in (14), or

$$\begin{aligned} p_{C'C'',\xi\zeta}^{out} = & (1/2)\oiint_{\partial V}\left\{\bar{E}_+^{sca}\left(\bar{b}_\zeta^{C''}\right)\times\left[\bar{H}_+^{sca}\left(\bar{b}_\xi^{C'}\right)\right]^*\right\}\cdot d\bar{S} \\ & + (1/2)\left\langle \sigma\bar{E}_-^{tot}\left(\bar{b}_\xi^{C'}\right), \bar{E}_-^{tot}\left(\bar{b}_\zeta^{C''}\right)\right\rangle_V \\ & + j\,2\omega\Big[(1/4)\left\langle \bar{H}_-^{sca}\left(\bar{b}_\xi^{C'}\right), \mu_0\bar{H}_-^{sca}\left(\bar{b}_\zeta^{C''}\right)\right\rangle_V \\ & \qquad - (1/4)\left\langle \varepsilon_0\bar{E}_-^{sca}\left(\bar{b}_\xi^{C'}\right), \bar{E}_-^{sca}\left(\bar{b}_\zeta^{C''}\right)\right\rangle_V\Big] \\ & + j\,2\omega\Big[(1/4)\left\langle \bar{H}_-^{tot}\left(\bar{b}_\xi^{C'}\right), \Delta\mu\bar{H}_-^{tot}\left(\bar{b}_\zeta^{C''}\right)\right\rangle_V \\ & \qquad - (1/4)\left\langle \Delta\varepsilon\bar{E}_-^{tot}\left(\bar{b}_\xi^{C'}\right), \bar{E}_-^{tot}\left(\bar{b}_\zeta^{C''}\right)\right\rangle_V\Big] \end{aligned} \tag{24.2}$$

for the operator form II given in (15), or

$$p_{C'C'',\xi\zeta}^{out} = \frac{1}{2}\left\langle \bar{J}^{vop}\left(\bar{b}_\xi^{C'}\right), \bar{E}_-^{inc}\left(\bar{b}_\zeta^{C''}\right)\right\rangle_V + \frac{1}{2}\left\langle \bar{H}_-^{inc}\left(\bar{b}_\xi^{C'}\right), \bar{M}^{vm}\left(\bar{b}_\zeta^{C''}\right)\right\rangle_V \tag{24.3}$$

for the operator form III given in (16). In (24.1), $\bar{J}^{vop}\left(\bar{b}_\xi^J\right) = \bar{J}^{vop}\left(\bar{b}_\xi^J, 0; \bar{r}\right)$, and $\bar{J}^{vop}\left(\bar{b}_\xi^M\right) = \bar{J}^{vop}\left(0, \bar{b}_\xi^M; \bar{r}\right)$, and the other symbols can be similarly explained.

B. The improved matrix forms for output power.

Because the scattering sources don't distribute on the boundary ∂V as explained in Appendix C, the surface equivalent sources satisfy the following surface electric field integral equation for any $\bar{r} \in \partial V$.

$$\left[\bar{E}_+^{sca}\left(\bar{J}^{SE}, \bar{M}^{SE}; \bar{r}_+\right)\right]_{\bar{r}_+ \to \bar{r} \in \partial V}^{\tan} = \left[\bar{E}_-^{sca}\left(\bar{J}^{SE}, \bar{M}^{SE}; \bar{r}_-\right)\right]_{\bar{r}_- \to \bar{r} \in \partial V}^{\tan} \tag{25}$$

here $\bar{r}_+ \in \mathbb{R}^3 \setminus V$, and $\bar{r}_- \in V$; the superscript " $\tan$ " and the subscript " $\bar{r}_\pm \to \bar{r} \in \partial V$ " represent that the above equation is satisfied by the tangential field component on the boundary ∂V , and the relevant operators in (25) are defined in (10).

Of course, a similar equation can be established for the scattering magnetic field.

Inserting (19) into (25), the following equation is derived.

$$\begin{aligned} & \left[\sum_{\zeta=1}^{\Xi^J} a_\zeta^J \bar{E}_+^{sca}\left(\bar{b}_\zeta^J, 0; \bar{r}_+\right) + \sum_{\zeta=1}^{\Xi^M} a_\zeta^M \bar{E}_+^{sca}\left(0, \bar{b}_\zeta^M; \bar{r}_+\right)\right]_{\bar{r}_+ \to \bar{r} \in \partial V}^{\tan} \\ = & \left[\sum_{\zeta=1}^{\Xi^J} a_\zeta^J \bar{E}_-^{sca}\left(\bar{b}_\zeta^J, 0; \bar{r}_-\right) + \sum_{\zeta=1}^{\Xi^M} a_\zeta^M \bar{E}_-^{sca}\left(0, \bar{b}_\zeta^M; \bar{r}_-\right)\right]_{\bar{r}_- \to \bar{r} \in \partial V}^{\tan} \end{aligned} \tag{26}$$

By making the inner products of (26) with $\bar{b}_\xi^\Phi$ over ∂V , here $\xi = 1, 2, \cdots, \Xi^\Phi$, the following matrix equation is constructed.

$$\bar{\bar{E}}_{+;\Phi J}^{sca} \cdot \bar{a}^J + \bar{\bar{E}}_{+;\Phi M}^{sca} \cdot \bar{a}^M = \bar{\bar{E}}_{-;\Phi J}^{sca} \cdot \bar{a}^J + \bar{\bar{E}}_{-;\Phi M}^{sca} \cdot \bar{a}^M \tag{27}$$

here $\Phi = M$ or J , and

$$\bar{\bar{E}}_{\pm;\Phi C}^{sca} = \left[e_{\pm;\Phi C;\xi\zeta}^{sca}\right]_{\Xi^\Phi \times \Xi^C} \tag{28.1}$$

in which $C = J, M$, and

$$e_{\pm;\Phi C;\xi\zeta}^{sca} = \left\langle \bar{b}_\xi^\Phi(\bar{r}), \bar{E}_\pm^{sca}\left(\bar{b}_\zeta^C; \bar{r}_\pm\right)\right\rangle_{\partial V} \tag{28.2}$$

In (28.2), $\bar{r}_{\pm} \to \bar{r} \in \partial V$; $\bar{E}^{sca}_{\pm}\left(\bar{b}^{J}_{\zeta};\bar{r}_{\pm}\right) = \bar{E}^{sca}_{\pm}\left(\bar{b}^{J}_{\zeta},0;\bar{r}_{\pm}\right)$, and $\bar{E}^{sca}_{\pm}\left(\bar{b}^{M}_{\zeta};\bar{r}_{\pm}\right) = \bar{E}^{sca}_{\pm}\left(0,\bar{b}^{M}_{\zeta};\bar{r}_{\pm}\right)$.

From (27), the following transformation relation is derived.

$$\bar{a}^{\Phi} = \bar{\bar{T}}^{C\to\Phi} \cdot \bar{a}^{C} \tag{29}$$

here $(\Phi,C) = (M,J)$ or (J,M) , and it depends on whether $\Phi = M$ or $\Phi = J$ in (27). In (29),

$$\bar{\bar{T}}^{C\to\Phi} = \left(\bar{\bar{E}}^{sca}_{+;\Phi\Phi} - \bar{\bar{E}}^{sca}_{-;\Phi\Phi}\right)^{-1} \cdot \left(\bar{\bar{E}}^{sca}_{-;\Phi C} - \bar{\bar{E}}^{sca}_{+;\Phi C}\right) \tag{30}$$

in which the superscript " -1 " represents the inverse of matrix.

The relation (30) is valuable for the surface equivalent source-based MoM formulation of material scattering problem, such as the PMCHWT-based MoM [5], [14]-[15], because the (30) makes the unknowns be reduced to the half of original. In fact, the relation (30) is also indispensable for the surface equivalent source-based CMT for material bodies, such as the Mat-PMCHWT-CMT [5], [8]-[9] and the Surf-Mat-EMP-CMT developed in this paper, and the reason can be found in [8]-[9] and the following parts of this section.

By inserting the (29) into (21), the P^{out} can be rewritten as

$$P^{out}\left(\bar{a}^{C}\right) = \left(\bar{a}^{C}\right)^{H} \cdot \bar{\bar{P}}^{out}_{C} \cdot \bar{a}^{C} \tag{31}$$

here $C = J$ or $C = M$, and it depends on whether $\Phi = M$ or $\Phi = J$ in (27). In (31),

$$\bar{\bar{P}}^{out}_{J} = \begin{bmatrix} \bar{\bar{I}} \\ \bar{\bar{T}}^{J\to M} \end{bmatrix}^{H} \cdot \bar{\bar{P}}^{out} \cdot \begin{bmatrix} \bar{\bar{I}} \\ \bar{\bar{T}}^{J\to M} \end{bmatrix} \tag{32.1}$$

$$\bar{\bar{P}}^{out}_{M} = \begin{bmatrix} \bar{\bar{T}}^{M\to J} \\ \bar{\bar{I}} \end{bmatrix}^{H} \cdot \bar{\bar{P}}^{out} \cdot \begin{bmatrix} \bar{\bar{T}}^{M\to J} \\ \bar{\bar{I}} \end{bmatrix} \tag{32.2}$$

The $\bar{\bar{I}}$ in (32) is identity matrix.

C. The Hermitian decomposition for matrix $\bar{\bar{P}}^{out}_{C}$.

Similarly to the papers [10] and [11], if the matrix $\bar{\bar{P}}^{out}_{C}$ is decomposed as follows

$$\bar{\bar{P}}^{out}_{C} = \bar{\bar{P}}^{out}_{C;+} + j\,\bar{\bar{P}}^{out}_{C;-} \tag{33}$$

here

$$\bar{\bar{P}}^{out}_{C;+} = \frac{1}{2}\left[\bar{\bar{P}}^{out}_{C} + \left(\bar{\bar{P}}^{out}_{C}\right)^{H}\right] \tag{34.1}$$

$$\bar{\bar{P}}^{out}_{C;-} = \frac{1}{2j}\left[\bar{\bar{P}}^{out}_{C} - \left(\bar{\bar{P}}^{out}_{C}\right)^{H}\right] \tag{34.2}$$

then [11]

$$\begin{aligned} \mathrm{Re}\left\{P^{out}\left(\bar{a}^{C}\right)\right\} &= P^{sca,rad}\left(\bar{a}^{C}\right) + P^{tot,loss}\left(\bar{a}^{C}\right) \\ &= \left(\bar{a}^{C}\right)^{H} \cdot \bar{\bar{P}}^{out}_{C;+} \cdot \bar{a}^{C} \end{aligned} \tag{35.1}$$

$$\begin{aligned} \mathrm{Im}\left\{P^{out}\left(\bar{a}^{C}\right)\right\} &= P^{sca,react,vac}\left(\bar{a}^{C}\right) + P^{tot,react,mat}\left(\bar{a}^{C}\right) \\ &= \left(\bar{a}^{C}\right)^{H} \cdot \bar{\bar{P}}^{out}_{C;-} \cdot \bar{a}^{C} \end{aligned} \tag{35.2}$$

In fact, if the decomposition (33) is directly applied to the matrix $\bar{\bar{P}}^{out}$ in the absence of relation (29), neither the positive definiteness nor semi-definiteness of matrix $(1/2)\left[\bar{\bar{P}}^{out} + (\bar{\bar{P}}^{out})^{H}\right]$ can be guaranteed, and the reason can be found in [11]. However, it must be clearly pointed out here that the relation (29), which is used to relate the $\bar{J}^{SE}$ with $\bar{M}^{SE}$ in this paper, is essentially different from the relation used in [8] and [9].

V. SURF-MAT-EMP-CMT

The Output CM (OutCM) set and corresponding modal expansion method are provided in this section. The fundamental principles and procedures to construct the OutCM set are similar to the Vol-Mat-EMP-CMT [11], so only some important conclusions and formulations corresponding to the surface equivalent source-based OutCM set are simply given as follows.

A. Output power CM (OutCM) set.

When the matrix $\bar{\bar{P}}^{out}_{C;+}$ is positive definite at frequency f , the OutCM set can be obtained by solving the following generalized characteristic equation [10]-[11], [20].

$$\bar{\bar{P}}^{out}_{C;-}(f) \cdot \bar{a}^{C}_{\xi}(f) = \lambda_{\xi}(f)\,\bar{\bar{P}}^{out}_{C;+}(f) \cdot \bar{a}^{C}_{\xi}(f) \tag{36}$$

When the matrix $\bar{\bar{P}}^{out}_{C;+}$ is positive semi-definite at frequency f_0 , the modal vectors can be obtained by using the following limitations [10]-[11].

$$\bar{a}^{C}_{\xi}(f_0) = \lim_{f\to f_0} \bar{a}^{C}_{\xi}(f) \tag{37}$$

for any $\xi = 1,2,\cdots,\Xi^{C}$.

The surface equivalent modal currents are as follows for any $\xi = 1,2,\cdots,\Xi^{C}$.

$$\bar{J}^{SE}_{\xi} = \bar{B}^{J} \cdot \bar{a}^{J}_{\xi} \tag{38.1}$$

$$\bar{M}^{SE}_{\xi} = \bar{B}^{M} \cdot \bar{\bar{T}}^{J\to M} \cdot \bar{a}^{J}_{\xi} \tag{38.2}$$

when $\Phi = M$ in (27), or

$$\bar{J}^{SE}_{\xi} = \bar{B}^{J} \cdot \bar{\bar{T}}^{M\to J} \cdot \bar{a}^{M}_{\xi} \tag{39.1}$$

$$\bar{M}^{SE}_{\xi} = \bar{B}^{M} \cdot \bar{a}^{M}_{\xi} \tag{39.2}$$

when $\Phi = J$ in (27). The scattering modal currents corresponding to above surface equivalent modal currents are

$$\bar{J}^{vop}_{\xi}(\bar{r}) = \bar{J}^{vop}\left(\bar{J}^{SE}_{\xi},\bar{M}^{SE}_{\xi};\bar{r}\right) \quad , \quad (\bar{r} \in V) \tag{40.1}$$

$$\bar{M}^{vm}_{\xi}(\bar{r}) = \bar{M}^{vm}\left(\bar{J}^{SE}_{\xi},\bar{M}^{SE}_{\xi};\bar{r}\right) \quad , \quad (\bar{r} \in V) \tag{40.2}$$

and relevant operators are defined as (13). Various modal fields corresponding to the above modal currents are as follows

$$\bar{F}_{\xi}^{sca}(\bar{r}) = \bar{F}^{sca}(\bar{J}_{\xi}^{SE},\bar{M}_{\xi}^{SE};\bar{r}) \quad , \quad (\bar{r}\in\mathbb{R}^3) \tag{41}$$

$$\bar{F}_{-,\xi}^{tot}(\bar{r}) = \bar{F}_{-}^{tot}(\bar{J}_{\xi}^{SE},\bar{M}_{\xi}^{SE};\bar{r}) \quad , \quad (\bar{r}\in V) \tag{42}$$

$$\bar{F}_{-,\xi}^{inc}(\bar{r}) = \bar{F}_{-}^{inc}(\bar{J}_{\xi}^{SE},\bar{M}_{\xi}^{SE};\bar{r}) \quad , \quad (\bar{r}\in V) \tag{43}$$

here the relevant operators are defined as (10)-(12).

Above modal currents and modal fields satisfy the following power orthogonality [11].

$$\begin{aligned} P_{\xi}^{out}\delta_{\xi\zeta} &= P_{\xi\zeta}^{out} \\ &= P_{\xi\zeta;\,sca,\,rad}^{out} + P_{\xi\zeta;\,tot,\,loss}^{out} + j\left(P_{\xi\zeta;\,sca,\,react,\,vac}^{out} + P_{\xi\zeta;\,tot,\,react,\,mat}^{out}\right) \end{aligned} \tag{44}$$

In (44), the $\delta_{\xi\zeta}$ is Kronecker delta symbol, and

$$\begin{aligned} P_{\xi}^{out} &= \operatorname{Re}\{P_{\xi}^{out}\} + j\operatorname{Im}\{P_{\xi}^{out}\} \\ &= P_{\xi;\,sca,\,rad}^{out} + P_{\xi;\,tot,\,loss}^{out} + j\left(P_{\xi;\,sca,\,react,\,vac}^{out} + P_{\xi;\,tot,\,react,\,mat}^{out}\right) \end{aligned} \tag{45}$$

and

$$P_{\xi\zeta}^{out} = \frac{1}{2}\left\langle \bar{J}_{\xi}^{vop},\bar{E}_{-,\zeta}^{inc}\right\rangle_V + \frac{1}{2}\left\langle \bar{H}_{-,\xi}^{inc},\bar{M}_{\zeta}^{vm}\right\rangle_V \tag{46.1}$$

$$P_{\xi\zeta;\,sca,\,rad}^{out} = \frac{1}{2}\oint\!\!\!\oint_{S_\infty}\left[\bar{E}_{\zeta}^{sca}\times\left(\bar{H}_{\xi}^{sca}\right)^*\right]\cdot d\bar{S} \tag{46.2}$$

$$P_{\xi\zeta;\,tot,\,loss}^{out} = \frac{1}{2}\left\langle \sigma\bar{E}_{-,\xi}^{tot},\bar{E}_{-,\zeta}^{tot}\right\rangle_V \tag{46.3}$$

$$P_{\xi\zeta;\,sca,\,react,\,vac}^{out} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}_{\xi}^{sca},\mu_0\bar{H}_{\zeta}^{sca}\right\rangle_{\mathbb{R}^3} - \frac{1}{4}\left\langle \varepsilon_0\bar{E}_{\xi}^{sca},\bar{E}_{\zeta}^{sca}\right\rangle_{\mathbb{R}^3}\right] \tag{46.4}$$

$$P_{\xi\zeta;\,tot,\,react,\,mat}^{out} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}_{-,\xi}^{tot},\Delta\mu\bar{H}_{-,\zeta}^{tot}\right\rangle_V - \frac{1}{4}\left\langle \Delta\varepsilon\bar{E}_{-,\xi}^{tot},\bar{E}_{-,\zeta}^{tot}\right\rangle_V\right] \tag{46.5}$$

In (45), $P_{\xi;sca,rad}^{out}=P_{\xi\xi;sca,rad}^{out}$, $P_{\xi;tot,loss}^{out}=P_{\xi\xi;tot,loss}^{out}$, $P_{\xi;sca,react,vac}^{out}=P_{\xi\xi;sca,react,vac}^{out}$, and $P_{\xi;tot,react,mat}^{out}=P_{\xi\xi;tot,react,mat}^{out}$.

B. OutCM-based modal expansion method.

Because of the completeness of the OutCM set [10]-[11], [20], the surface equivalent currents $\{\bar{J}^{SE},\bar{M}^{SE}\}$, the scattering currents $\{\bar{J}^{vop},\bar{M}^{vm}\}$, the scattering fields $\{\bar{E}^{sca},\bar{H}^{sca}\}$ in $\mathbb{R}^3$, and the fields $\{\bar{E}^{inc},\bar{H}^{inc}\}$ and $\{\bar{E}^{tot},\bar{H}^{tot}\}$ in V can be expanded in terms of the OutCM set as follows

$$\left.\begin{aligned} \bar{J}^{SE}(\bar{r}) &= \sum_{\xi=1}^{\Xi^C} c_\xi \bar{J}_{\xi}^{SE}(\bar{r}) \\ \bar{M}^{SE}(\bar{r}) &= \sum_{\xi=1}^{\Xi^C} c_\xi \bar{M}_{\xi}^{SE}(\bar{r}) \end{aligned}\right. \quad , \quad (\bar{r}\in\partial V) \tag{47}$$

and

$$\left.\begin{aligned} \bar{J}^{vop}(\bar{r}) &= \sum_{\xi=1}^{\Xi^C} c_\xi \bar{J}_{\xi}^{vop}(\bar{r}) \\ \bar{M}^{vm}(\bar{r}) &= \sum_{\xi=1}^{\Xi^C} c_\xi \bar{M}_{\xi}^{vm}(\bar{r}) \end{aligned}\right. \quad , \quad (\bar{r}\in V) \tag{48}$$

and

$$\bar{F}^{sca}(\bar{r}) = \sum_{\xi=1}^{\Xi^C} c_\xi \bar{F}_{\xi}^{sca}(\bar{r}) \quad , \quad (\bar{r}\in\mathbb{R}^3) \tag{49}$$

and

$$\left.\begin{aligned} \bar{F}_{-}^{inc}(\bar{r}) &= \sum_{\xi=1}^{\Xi^C} c_\xi \bar{F}_{-,\xi}^{inc}(\bar{r}) \\ \bar{F}_{-}^{tot}(\bar{r}) &= \sum_{\xi=1}^{\Xi^C} c_\xi \bar{F}_{-,\xi}^{tot}(\bar{r}) \end{aligned}\right. \quad , \quad (\bar{r}\in V) \tag{50}$$

here $F=E,H$; $\Xi^C=\Xi^J$ or $\Xi^C=\Xi^M$, and it depends on whether $\Phi=M$ or $\Phi=J$ in (27). Based on the power orthogonality for OutCM set, the output power can be expanded as

$$\begin{aligned} P^{out} &= \sum_{\xi=1}^{\Xi^C}|c_\xi|^2 P_{\xi}^{out} \\ &= \sum_{\xi=1}^{\Xi^C}|c_\xi|^2\operatorname{Re}\{P_{\xi}^{out}\} + j\sum_{\xi=1}^{\Xi^C}|c_\xi|^2\operatorname{Im}\{P_{\xi}^{out}\} \\ &= \left(\sum_{\xi=1}^{\Xi^C}|c_\xi|^2 P_{\xi;\,sca,\,rad}^{out} + \sum_{\xi=1}^{\Xi^C}|c_\xi|^2 P_{\xi;\,tot,\,loss}^{out}\right) \\ &\quad + j\left(\sum_{\xi=1}^{\Xi^C}|c_\xi|^2 P_{\xi;\,sca,\,react,\,vac}^{out} + \sum_{\xi=1}^{\Xi^C}|c_\xi|^2 P_{\xi;\,tot,\,react,\,mat}^{out}\right) \end{aligned} \tag{51}$$

In (51), the terms corresponding to loss will disappear, if the material scatterer is lossless.

C. Expansion coefficients.

When the excitation $\bar{F}^{inc}$ is given, the input power P^{inp} and output power P^{out} can be respectively written as the following (52.1) and (52.2) based on the (9), (16), (19), and (29).

$$\begin{aligned} P^{inp} &= \mathcal{I}(\bar{C}^{SE}) \\ &= \frac{1}{2}\left\langle \bar{J}^{vop}(\bar{C}^{SE}),\bar{E}^{inc}\right\rangle_V + \frac{1}{2}\left\langle \bar{H}^{inc},\bar{M}^{vm}(\bar{C}^{SE})\right\rangle_V \end{aligned} \tag{52.1}$$

$$P^{out} = P_3^{out}(\bar{C}^{SE}) \tag{52.2}$$

In (52), $C=J$ or $C=M$, and it depends on whether $\Phi=M$ or $\Phi=J$ in (27). The $\bar{E}^{inc}$ and $\bar{H}^{inc}$ in (52.1) are known.

Based on the (52), the conservation law of energy (8), and the variational principle [21], the $\bar{C}^{SE}$ will make the following functional be zero and stationary.

$$\mathcal{F}(\bar{C}^{SE}) = \mathcal{I}(\bar{C}^{SE}) - P_3^{out}(\bar{C}^{SE}) \tag{53}$$

Inserting the (47) into (53) and employing the Ritz's procedure [22], the following simultaneous equations for the expansion coefficients $\{c_\xi\}_{\xi=1}^{\Xi^C}$ are derived for any $\xi=1,2,\cdots,\Xi^C$.

$$\begin{aligned} &\frac{1}{2}\left\langle \bar{J}_{\xi}^{vop},\bar{E}^{inc}\right\rangle_V + \frac{1}{2}\left\langle \bar{H}^{inc},\bar{M}_{\xi}^{vm}\right\rangle_V \\ &= \frac{1}{2}\left\langle \bar{J}_{\xi}^{vop},\sum_{\zeta=1}^{\Xi^C} c_\zeta \bar{E}_{-,\zeta}^{inc}\right\rangle_V + \frac{1}{2}\left\langle \bar{H}_{-,\xi}^{inc},\sum_{\zeta=1}^{\Xi^C} c_\zeta \bar{M}_{\zeta}^{vm}\right\rangle_V \\ &\quad + \frac{1}{2}\left\langle \sum_{\zeta=1}^{\Xi^C} c_\zeta \bar{J}_{\zeta}^{vop},\bar{E}_{-,\xi}^{inc}\right\rangle_V + \frac{1}{2}\left\langle \sum_{\zeta=1}^{\Xi^C} c_\zeta \bar{H}_{-,\zeta}^{inc},\bar{M}_{\xi}^{vm}\right\rangle_V \end{aligned} \tag{54.1}$$

and

$$\begin{aligned} & -\frac{1}{2}\left\langle \bar{J}_{\xi}^{vop},\bar{E}^{inc}\right\rangle_{V}+\frac{1}{2}\left\langle \bar{H}^{inc},\bar{M}_{\xi}^{vm}\right\rangle_{V} \\ = & -\frac{1}{2}\left\langle \bar{J}_{\xi}^{vop},\sum_{\zeta=1}^{\Xi^{C}}c_{\zeta}\bar{E}_{-,\zeta}^{inc}\right\rangle_{V}-\frac{1}{2}\left\langle \bar{H}_{-,\xi}^{inc},\sum_{\zeta=1}^{\Xi^{C}}c_{\zeta}\bar{M}_{\zeta}^{vm}\right\rangle_{V} \\ & +\frac{1}{2}\left\langle \sum_{\zeta=1}^{\Xi^{C}}c_{\zeta}\bar{J}_{\zeta}^{vop},\bar{E}_{-,\xi}^{inc}\right\rangle_{V}+\frac{1}{2}\left\langle \sum_{\zeta=1}^{\Xi^{C}}c_{\zeta}\bar{H}_{-,\zeta}^{inc},\bar{M}_{\xi}^{vm}\right\rangle_{V} \end{aligned} \tag{54.2}$$

By solving the (54), the coefficient $\{c_\xi\}_{\xi=1}^{\Xi^C}$ can be determined. If the orthogonality in (46) is utilized in (54), the coefficient $\{c_\xi\}_{\xi=1}^{\Xi^C}$ can be concisely written as the following (55) for any $\xi=1,2,\cdots,\Xi^C$.

$$c_{\xi}=\begin{cases} \frac{1}{P_{\xi}^{out}}\cdot\frac{1}{2}\left\langle \bar{J}_{\xi}^{vop},\bar{E}^{inc}\right\rangle_{V} & ,\ (\Delta\mu=0,\Delta\varepsilon_{c}\neq 0) \\ \left[\frac{1}{P_{\xi}^{out}}\cdot\frac{1}{2}\left\langle \bar{H}^{inc},\bar{M}_{\xi}^{vm}\right\rangle_{V}\right]^{*} & ,\ (\Delta\mu\neq 0,\Delta\varepsilon_{c}=0) \\ \frac{1}{P_{\xi}^{out}}\cdot\frac{1}{2}\left\langle \bar{J}_{\xi}^{vop},\bar{E}^{inc}\right\rangle_{V}=\left[\frac{1}{P_{\xi}^{out}}\cdot\frac{1}{2}\left\langle \bar{H}^{inc},\bar{M}_{\xi}^{vm}\right\rangle_{V}\right]^{*} & ,\ (\Delta\mu,\Delta\varepsilon_{c}\neq 0) \\ 0 & ,\ (\Delta\mu,\Delta\varepsilon_{c}=0) \end{cases} \tag{55}$$

VI. The Impedance and Admittance of Material Electromagnetic Systems

Following the ideas of [10], the surface equivalent source can be normalized as follows

$$\tilde{\tilde{J}}^{SE}(\bar{r}) \triangleq \frac{\bar{J}^{SE}(\bar{r})}{\left(1/\sqrt{2}\right)\left\langle \bar{J}^{SE},\bar{J}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in\partial V) \tag{56}$$

if $\Phi=M$ in (27), or as follows

$$\underset{\sim}{\bar{M}}^{SE}(\bar{r}) \triangleq \frac{\bar{M}^{SE}(\bar{r})}{\left(1/\sqrt{2}\right)\left\langle \bar{M}^{SE},\bar{M}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in\partial V) \tag{57}$$

if $\Phi=J$ in (27).

Based on the (56), the various fields, the scattering currents, and the system output power are automatically normalized as the following (58)-(62) because of the (10)-(13) and (18).

$$\tilde{\tilde{F}}^{sca}(\bar{r}) = \frac{\bar{F}^{sca}\left(\bar{J}^{SE};\bar{r}\right)}{\left(1/\sqrt{2}\right)\left\langle \bar{J}^{SE},\bar{J}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in\mathbb{R}^3) \tag{58}$$

$$\tilde{\tilde{F}}_{-}^{tot}(\bar{r}) = \frac{\bar{F}_{-}^{tot}\left(\bar{J}^{SE};\bar{r}\right)}{\left(1/\sqrt{2}\right)\left\langle \bar{J}^{SE},\bar{J}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in V) \tag{59}$$

$$\tilde{\tilde{F}}_{-}^{inc}(\bar{r}) = \frac{\bar{F}_{-}^{inc}\left(\bar{J}^{SE};\bar{r}\right)}{\left(1/\sqrt{2}\right)\left\langle \bar{J}^{SE},\bar{J}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in V) \tag{60}$$

$$\tilde{\tilde{J}}^{vop}(\bar{r}) = \frac{\bar{J}^{vop}\left(\bar{J}^{SE};\bar{r}\right)}{\left(1/\sqrt{2}\right)\left\langle \bar{J}^{SE},\bar{J}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in V) \tag{61.1}$$

$$\tilde{\tilde{M}}^{vm}(\bar{r}) = \frac{\bar{M}^{vm}\left(\bar{J}^{SE};\bar{r}\right)}{\left(1/\sqrt{2}\right)\left\langle \bar{J}^{SE},\bar{J}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in V) \tag{61.2}$$

and

$$\tilde{P}^{out} = \frac{P^{out}\left(\bar{J}^{SE}\right)}{(1/2)\left\langle \bar{J}^{SE},\bar{J}^{SE}\right\rangle_{\partial V}} \tag{62}$$

In (58)-(62), the $\bar{M}^{SE}$ has been expressed as the function of $\bar{J}^{SE}$ based on (19) and (29), so only the $\bar{J}^{SE}$ explicitly appears.

Based on the (57), the various fields, the scattering currents, and the system output power are automatically normalized as the following (63)-(67) because of the (10)-(13) and (18).

$$\underset{\sim}{\bar{F}}^{sca}(\bar{r}) = \frac{\bar{F}^{sca}\left(\bar{M}^{SE};\bar{r}\right)}{\left(1/\sqrt{2}\right)\left\langle \bar{M}^{SE},\bar{M}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in\mathbb{R}^3) \tag{63}$$

$$\underset{\sim}{\bar{F}}_{-}^{tot}(\bar{r}) = \frac{\bar{F}_{-}^{tot}\left(\bar{M}^{SE};\bar{r}\right)}{\left(1/\sqrt{2}\right)\left\langle \bar{M}^{SE},\bar{M}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in V) \tag{64}$$

$$\underset{\sim}{\bar{F}}_{-}^{inc}(\bar{r}) = \frac{\bar{F}_{-}^{inc}\left(\bar{M}^{SE};\bar{r}\right)}{\left(1/\sqrt{2}\right)\left\langle \bar{M}^{SE},\bar{M}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in V) \tag{65}$$

$$\underset{\sim}{\bar{J}}^{vop}(\bar{r}) = \frac{\bar{J}^{vop}\left(\bar{M}^{SE};\bar{r}\right)}{\left(1/\sqrt{2}\right)\left\langle \bar{M}^{SE},\bar{M}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in V) \tag{66.1}$$

$$\underset{\sim}{\bar{M}}^{vm}(\bar{r}) = \frac{\bar{M}^{vm}\left(\bar{M}^{SE};\bar{r}\right)}{\left(1/\sqrt{2}\right)\left\langle \bar{M}^{SE},\bar{M}^{SE}\right\rangle_{\partial V}^{1/2}} \quad , \quad (\bar{r}\in V) \tag{66.2}$$

and

$$\underset{\sim}{P}^{out} = \frac{P^{out}\left(\bar{M}^{SE}\right)}{(1/2)\left\langle \bar{M}^{SE},\bar{M}^{SE}\right\rangle_{\partial V}} \tag{67}$$

In the (63)-(67), the $\bar{J}^{SE}$ has been expressed as the function of $\bar{M}^{SE}$ based on (19) and (29), so only the $\bar{M}^{SE}$ appears explicitly.

Obviously, the dimension of $\tilde{P}^{out}\left(\bar{J}^{SE}\right)$ is Ohm, and the dimension of $\underset{\sim}{P}^{out}\left(\bar{M}^{SE}\right)$ is Siemens. Based on this, the impedance and admittance of material system can be defined as the following (68) and (69) respectively.

$$Z\left(\bar{J}^{SE}\right) \triangleq \tilde{P}^{out}\left(\bar{J}^{SE}\right) \tag{68}$$

$$Y\left(\bar{M}^{SE}\right) \triangleq \underset{\sim}{P}^{out}\left(\bar{M}^{SE}\right) \tag{69}$$

The real and imaginary parts of $Z\left(\bar{J}^{SE}\right)$ are called as system resistance and reactance, and the real and imaginary parts of $Y\left(\bar{M}^{SE}\right)$ are called as system conductance and susceptance, and they are respectively denoted as follows

$$R\left(\bar{J}^{SE}\right) = \operatorname{Re}\left\{Z\left(\bar{J}^{SE}\right)\right\} \tag{70.1}$$

$$X\left(\bar{J}^{SE}\right) = \operatorname{Im}\left\{Z\left(\bar{J}^{SE}\right)\right\} \tag{70.2}$$

and

$$G\left(\bar{M}^{SE}\right) = \operatorname{Re}\left\{Y\left(\bar{M}^{SE}\right)\right\} \tag{71.1}$$

$$B\left(\bar{M}^{SE}\right) = \operatorname{Im}\left\{Y\left(\bar{M}^{SE}\right)\right\} \tag{71.2}$$

In fact, the modal currents, fields, and powers can be similarly normalized as system currents, fields, and powers in (58)-(67), and the modal impedance (resistance and reactance) and admittance (conductance and susceptance) can be similarly introduced as (68)-(71).

VII. Conclusions

Just like the previous PEC-EMP-CMT and Vol-Mat-EMP-CMT, the CM sets derived from Surf-Mat-EMP-CMT can reveal the system's inherent characteristics to utilize various electromagnetic energies.

The physical effectiveness of Surf-Mat-EMP-CMT is the same as the Vol-Mat-EMP-CMT. For example, the Surf-Mat-EMP-CMT is applicable not only to the material bodies placed in vacuum, but also to the ones surrounded by any electromagnetic environment; any kind of power-based CM set constructed in Vol-Mat-EMP-CMT can also be derived from Surf-Mat-EMP-CMT. However, the Surf-Mat-EMP-CMT is more advantageous than the Vol-Mat-EMP-CMT in some aspects, such as saving computational resources and avoiding to compute the modal scattering field in source region etc.

An efficient way to establish the relation between the surface equivalent electric and magnetic currents on the boundary of material body is provided in this paper. The relation is valuable for the surface equivalent source-based MoM for material scattering problem, because the unknowns are reduced to the half of original. The relation is also indispensable for the surface equivalent source-based formulation of the CMT for material bodies, because the unrelated surface equivalent electric and magnetic currents will lead to unphysical fields.

In addition, a surface equivalent source-based variational formulation for the scattering problem of material bodies is provided based on the conservation law of energy, and the field-based definitions for the impedance and admittance of material systems are introduced in this paper.

Appendixes

In this section, some valuable principles and formulations related to the theory developed in this paper are provided. Some conclusions given in this section can be found in many classical textbooks on electromagnetics, and to specifically list them here is for the convenience of the discussions in this paper.

A. Poynting's theorem.

The Maxwell's equations for the scattering fields $\{\bar{E}^{sca},\bar{H}^{sca}\}$ generated by material scatterer are as follows [13]-[14]

$$\nabla\times\bar{H}^{sca} = \bar{J}^{vop} + j\omega\varepsilon_0\bar{E}^{sca} \tag{A-1.1}$$

$$\nabla\times\bar{E}^{sca} = -\bar{M}^{vm} - j\omega\mu_0\bar{H}^{sca} \tag{A-1.2}$$

here

$$\bar{J}^{vop} = \bar{J}^{vo} + \bar{J}^{vp} \tag{A-2.1}$$

$$\bar{M}^{vm} = j\omega\,\Delta\mu\bar{H}^{tot} \tag{A-2.2}$$

in which $\bar{J}^{vo} = \sigma\bar{E}^{tot}$, and $\bar{J}^{vp} = j\omega\Delta\varepsilon\bar{E}^{tot}$, so $\bar{J}^{vop} = j\omega\Delta\varepsilon_c\bar{E}^{tot}$. In the (A-1) and (A-2), $\Delta\mu = \mu - \mu_0$, $\Delta\varepsilon = \varepsilon - \varepsilon_0$, and $\Delta\varepsilon_c = \varepsilon_c - \varepsilon_0$; the $\varepsilon_c = \varepsilon + \sigma/j\omega$ is complex permittivity; the ε and ε_0 are the permitivities in scatterer and vacuum; the μ and μ_0 are the permeabilities in scatterer and vacuum; the σ is the electric conductivity in scatterer, and its vacuum version is zero. The $\omega = 2\pi f$ is angle frequency, and the f is frequency.

Multiplying the complex conjugate of (A-1.1) with $\bar{E}^{sca}$ and doing some necessary simplifications, the following relation is obtained [15].

$$\begin{aligned} &-(1/2)\left(\bar{J}^{vop}\right)^*\cdot\bar{E}^{sca} - (1/2)\left(\bar{H}^{sca}\right)^*\cdot\bar{M}^{vm} \\ &= (1/2)\nabla\cdot\left[\bar{E}^{sca}\times\left(\bar{H}^{sca}\right)^*\right] \\ &\quad + j\,2\omega\left[(1/4)\left(\bar{H}^{sca}\right)^*\cdot\mu_0\bar{H}^{sca} - (1/4)\varepsilon_0\left(\bar{E}^{sca}\right)^*\cdot\bar{E}^{sca}\right] \end{aligned} \tag{A-3}$$

here the symbol "$\times$" is the cross product for field vectors.

If the (A-3) is integrated on whole space $\mathbb{R}^3$, the following Poynting's theorem can be obtained [15].

$$-\frac{1}{2}\left\langle\bar{J}^{vop},\bar{E}^{sca}\right\rangle_V - \frac{1}{2}\left\langle\bar{H}^{sca},\bar{M}^{vm}\right\rangle_V = P^{sca,rad} + j\,P^{sca,react,vac} \tag{A-4}$$

here

$$P^{sca,rad} = (1/2)\oint\!\!\!\oint_{S_\infty}\left[\bar{E}^{sca}\times\left(\bar{H}^{sca}\right)^*\right]\cdot d\bar{S} \tag{A-5.1}$$

$$P^{sca,react,vac} = 2\omega\left(W_m^{sca,vac} - W_e^{sca,vac}\right) \tag{A-5.2}$$

and

$$W_m^{sca,vac} = (1/4)\left\langle\bar{H}^{sca},\mu_0\bar{H}^{sca}\right\rangle_{\mathbb{R}^3} \tag{A-6.1}$$

$$W_e^{sca,vac} = (1/4)\left\langle\varepsilon_0\bar{E}^{sca},\bar{E}^{sca}\right\rangle_{\mathbb{R}^3} \tag{A-6.2}$$

and the S_∞ is a closed spherical surface at infinity.

If the (A-3) is integrated over whole scatterer V and the divergence theorem is employed, the following relation is derived.

$$\begin{aligned} &-(1/2)\left\langle\bar{J}^{vop},\bar{E}^{sca}\right\rangle_V - (1/2)\left\langle\bar{H}^{sca},\bar{M}^{vm}\right\rangle_V \\ &= (1/2)\oint\!\!\!\oint_{\partial V}\left[\bar{E}^{sca}\times\left(\bar{H}^{sca}\right)^*\right]\cdot d\bar{S} + j\,P_V^{sca,react,vac} \end{aligned} \tag{A-7}$$

here

$$P_V^{sca,react,vac} = 2\omega\left(W_{m,V}^{sca,vac} - W_{e,V}^{sca,vac}\right) \quad \text{(A-8)}$$

and

$$W_{m,V}^{sca,vac} = (1/4)\left\langle \bar{H}^{sca}, \mu_0 \bar{H}^{sca} \right\rangle_V \quad \text{(A-9.1)}$$

$$W_{e,V}^{sca,vac} = (1/4)\left\langle \varepsilon_0 \bar{E}^{sca}, \bar{E}^{sca} \right\rangle_V \quad \text{(A-9.2)}$$

Comparing the (A-4) with (A-7) and considering of that the $P^{sca,rad}$, $P^{sca,react,vac}$, and $P_V^{sca,react,vac}$ are real numbers, it is immediately found out that [15]

$$\mathrm{Re}\left\{(1/2)\oint_{\partial V}\left[\bar{E}^{sca}\times\left(\bar{H}^{sca}\right)^*\right]\cdot d\bar{S}\right\} = P^{sca,rad} \quad \text{(A-10.1)}$$

$$\mathrm{Im}\left\{(1/2)\oint_{\partial V}\left[\bar{E}^{sca}\times\left(\bar{H}^{sca}\right)^*\right]\cdot d\bar{S}\right\} = P_{\mathbb{R}^3\setminus V}^{sca,react,vac} \quad \text{(A-10.2)}$$

here

$$P_{\mathbb{R}^3\setminus V}^{sca,react,vac} = 2\omega\left(W_{m,\mathbb{R}^3\setminus V}^{sca,vac} - W_{e,\mathbb{R}^3\setminus V}^{sca,vac}\right) \quad \text{(A-11)}$$

and

$$W_{m,\mathbb{R}^3\setminus V}^{sca,vac} = (1/4)\left\langle \bar{H}^{sca}, \mu_0 \bar{H}^{sca} \right\rangle_{\mathbb{R}^3\setminus V} \quad \text{(A-12.1)}$$

$$W_{e,\mathbb{R}^3\setminus V}^{sca,vac} = (1/4)\left\langle \varepsilon_0 \bar{E}^{sca}, \bar{E}^{sca} \right\rangle_{\mathbb{R}^3\setminus V} \quad \text{(A-12.2)}$$

The symbol "$\mathbb{R}^3\setminus V$" is the whole space except the V.

In addition, it is obvious that

$$P^{sca,react,vac} = P_V^{sca,react,vac} + P_{\mathbb{R}^3\setminus V}^{sca,react,vac} \quad \text{(A-13)}$$

B. Various electromagnetic powers related to material bodies.

The various electromagnetic powers related to the material scatterer can be divided into the following four categories: the lossy powers, the radiated powers carried by radiative fields, the reactive powers due to the energies stored in non-radiative fields (simply called as reactively stored powers in fields), and the reactive powers due to the energies stored in matter (simply called as reactively stored powers in matter) [16]-[18]. The former two kinds are collectively referred to as active powers, and the latter two kinds are collectively referred to as reactive powers.

If the material scatterer is regarded as a whole object, there exist only two kinds of fields in $\mathbb{R}^3$, that are the $\bar{F}^{sca}$ generated by scatterer and the $\bar{F}^{inc}$ generated by external sources. The $\bar{F}^{sca}$ and $\bar{F}^{inc}$ respectively contribute all kinds of powers mentioned above, and they are detailedly listed as below.

1) The active powers

(1.1) The radiated powers include the $P^{sca,rad}$ carried by $\bar{F}^{sca}$, the $P^{inc,rad}$ carried by $\bar{F}^{inc}$, and the $P^{coup,rad}$ corresponding to the coupling between $\bar{F}^{sca}$ and $\bar{F}^{inc}$ on the surface S_∞. The $P^{sca,rad}$ is given in (A-5.1), and the mathematical expressions for $P^{inc,rad}$ and $P^{coup,rad}$ are expressed as follows

$$P^{inc,rad} = \frac{1}{2}\oint_{S_\infty}\left[\bar{E}^{inc}\times\left(\bar{H}^{inc}\right)^*\right]\cdot d\bar{S} \quad \text{(B-1.1)}$$

$$P^{coup,rad} = \frac{1}{2}\oint_{S_\infty}\left[\bar{E}^{sca}\times\left(\bar{H}^{inc}\right)^*\right]\cdot d\bar{S} + \frac{1}{2}\oint_{S_\infty}\left[\bar{E}^{inc}\times\left(\bar{H}^{sca}\right)^*\right]\cdot d\bar{S} \quad \text{(B-1.2)}$$

(1.2) The lossy powers include the $P^{sca,loss}$ dissipated by $\bar{F}^{sca}$, the $P^{inc,loss}$ dissipated by $\bar{F}^{inc}$, and the $P^{coup,loss}$ corresponding to the coupling between $\bar{F}^{sca}$ and $\bar{F}^{inc}$. Their mathematical expressions are as follows

$$P^{sca,loss} = (1/2)\left\langle \sigma\bar{E}^{sca}, \bar{E}^{sca}\right\rangle_V \quad \text{(B-2.1)}$$

$$P^{inc,loss} = (1/2)\left\langle \sigma\bar{E}^{inc}, \bar{E}^{inc}\right\rangle_V \quad \text{(B-2.2)}$$

$$P^{coup,loss} = (1/2)\left\langle \sigma\bar{E}^{sca}, \bar{E}^{inc}\right\rangle_V + (1/2)\left\langle \sigma\bar{E}^{inc}, \bar{E}^{sca}\right\rangle_V \quad \text{(B-2.3)}$$

It is obvious that

$$P^{tot,loss} = P^{sca,loss} + P^{inc,loss} + P^{coup,loss} \quad \text{(B-3)}$$

here the $P^{tot,loss}$ is given in (4.1).

2) The reactive powers

(2.1) The reactively stored powers in various fields include the $P^{sca,react,vac}$ in (A-5.2), the $P^{inc,react,vac}$ which is the reactively stored power in $\bar{F}^{inc}$, and the $P^{coup,react,vac}$ corresponding to the coupling between $\bar{F}^{sca}$ and $\bar{F}^{inc}$ in $\mathbb{R}^3$. The mathematical expressions for $P^{inc,react,vac}$ and $P^{coup,react,vac}$ are as follows

$$P^{inc,react,vac} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{inc}, \mu_0\bar{H}^{inc}\right\rangle_{\mathbb{R}^3} - \frac{1}{4}\left\langle \varepsilon_0\bar{E}^{inc}, \bar{E}^{inc}\right\rangle_{\mathbb{R}^3}\right] \quad \text{(B-4.1)}$$

$$\begin{aligned} P^{coup,react,vac} = 2\omega\Big[&\frac{1}{4}\left\langle \bar{H}^{sca}, \mu_0\bar{H}^{inc}\right\rangle_{\mathbb{R}^3} - \frac{1}{4}\left\langle \varepsilon_0\bar{E}^{sca}, \bar{E}^{inc}\right\rangle_{\mathbb{R}^3} \\ &+\frac{1}{4}\left\langle \bar{H}^{inc}, \mu_0\bar{H}^{sca}\right\rangle_{\mathbb{R}^3} - \frac{1}{4}\left\langle \varepsilon_0\bar{E}^{inc}, \bar{E}^{sca}\right\rangle_{\mathbb{R}^3}\Big] \end{aligned} \quad \text{(B-4.2)}$$

(2.2) The reactively stored powers in matter include the $P^{sca,react,mat}$ due to the interaction between $\bar{F}^{sca}$ and scatterer, the $P^{inc,react,mat}$ due to the interaction between $\bar{F}^{inc}$ and scatterer, and the $P^{coup,react,mat}$ due to the coupling between $\bar{F}^{sca}$ and $\bar{F}^{inc}$. The mathematical expressions for these reactive powers are given as follows

$$P^{sca,react,mat} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{sca}, \Delta\mu\bar{H}^{sca}\right\rangle_V - \frac{1}{4}\left\langle \Delta\varepsilon\bar{E}^{sca}, \bar{E}^{sca}\right\rangle_V\right] \quad \text{(B-5.1)}$$

$$P^{inc,react,mat} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{inc}, \Delta\mu\bar{H}^{inc}\right\rangle_V - \frac{1}{4}\left\langle \Delta\varepsilon\bar{E}^{inc}, \bar{E}^{inc}\right\rangle_V\right] \quad \text{(B-5.2)}$$

$$\begin{aligned} P^{coup,react,mat} = 2\omega\Big[&\frac{1}{4}\left\langle \bar{H}^{sca}, \Delta\mu\bar{H}^{inc}\right\rangle_V - \frac{1}{4}\left\langle \Delta\varepsilon\bar{E}^{sca}, \bar{E}^{inc}\right\rangle_V \\ &+\frac{1}{4}\left\langle \bar{H}^{inc}, \Delta\mu\bar{H}^{sca}\right\rangle_V - \frac{1}{4}\left\langle \Delta\varepsilon\bar{E}^{inc}, \bar{E}^{sca}\right\rangle_V\Big] \end{aligned} \quad \text{(B-5.3)}$$

It is obvious that

$$P^{tot,react,mat} = P^{sca,react,mat} + P^{inc,react,mat} + P^{coup,react,mat} \quad \text{(B-6)}$$

here the $P^{tot,react,mat}$ is given in (4.2).

Obviously, the $P^{sca,rad}$, $P^{sca,loss}$, $P^{inc,loss}$, $P^{coup,loss}$, $P^{sca,react,vac}$, $P^{sca,react,mat}$, $P^{inc,react,mat}$, and $P^{coup,react,mat}$ are intrinsically related to the scatterer. However, the $P^{inc,rad}$, $P^{coup,rad}$, $P^{inc,react,vac}$, and $P^{coup,react,vac}$ are not intrinsically related to the scatterer [11].

C. The surface equivalent principle for material bodies.

In any environment Ω whose material parameters are $\{\breve{\varepsilon},\breve{\mu}\}$, the electromagnetic fields $\{\bar{E},\bar{H}\}$ related to the currents $\{\bar{J},\bar{M}\}$ satisfy the following Maxwell's equations.

$$\begin{aligned}\nabla\times\bar{H}(\bar{r}) &= \bar{J}(\bar{r})+j\omega\breve{\varepsilon}\bar{E}(\bar{r})\\ \nabla\times\bar{E}(\bar{r}) &= -\bar{M}(\bar{r})-j\omega\breve{\mu}\bar{H}(\bar{r})\end{aligned}\quad,\quad(\bar{r}\in\Omega)\qquad\text{(C-1)}$$

Various electromagnetic dyadic Green's functions are defined as follows [19]

$$\begin{aligned}\nabla\times\bar{\bar{G}}^{JH}(\bar{r},\bar{r}') &= \bar{\bar{I}}\delta(\bar{r}-\bar{r}')+j\omega\breve{\varepsilon}\bar{\bar{G}}^{JE}(\bar{r},\bar{r}')\\ \nabla\times\bar{\bar{G}}^{JE}(\bar{r},\bar{r}') &= -j\omega\breve{\mu}\bar{\bar{G}}^{JH}(\bar{r},\bar{r}')\end{aligned}\qquad\text{(C-2.1)}$$

for the Green's functions corresponding to electric-type unity dyadic source, and

$$\begin{aligned}\nabla\times\bar{\bar{G}}^{MH}(\bar{r},\bar{r}') &= j\omega\breve{\varepsilon}\bar{\bar{G}}^{ME}(\bar{r},\bar{r}')\\ \nabla\times\bar{\bar{G}}^{ME}(\bar{r},\bar{r}') &= -\bar{\bar{I}}\delta(\bar{r}-\bar{r}')-j\omega\breve{\mu}\bar{\bar{G}}^{MH}(\bar{r},\bar{r}')\end{aligned}\qquad\text{(C-2.2)}$$

for the Green's functions corresponding to magnetic-type unity dyadic source. In (C-2), the $\delta(\bar{r}-\bar{r}')$ is Dirac delta function, and $\bar{r},\bar{r}'\in\Omega$.

Inserting $\bar{P}=\bar{E}(\bar{r})$ and $\bar{\bar{Q}}=\bar{\bar{G}}^{JE}(\bar{r},\bar{r}')$ into the following vector-dyadic Green's theorem [19].

$$\begin{aligned}&\iiint_\Omega\left[\bar{P}\cdot\left(\nabla\times\nabla\times\bar{\bar{Q}}\right)-\left(\nabla\times\nabla\times\bar{P}\right)\cdot\bar{\bar{Q}}\right]d\Omega\\ =&\oiint_{\partial\Omega}\left\{\hat{n}_{\to\Omega}\cdot\left[\bar{P}\times\left(\nabla\times\bar{\bar{Q}}\right)+\left(\nabla\times\bar{P}\right)\times\bar{\bar{Q}}\right]\right\}dS\end{aligned}\qquad\text{(C-3)}$$

and employing the (C-1) and (C-2), the following integral expression for the electric field $\bar{E}$ at any position $\bar{r}$ in domain Ω is derived.

$$\begin{aligned}\bar{E}_\Omega(\bar{r}) =& +\iiint_\Omega\bar{J}(\bar{r}')\cdot\bar{\bar{G}}^{JE}_\Omega(\bar{r}',\bar{r})d\Omega'\\ &-\iiint_\Omega\bar{M}(\bar{r}')\cdot\bar{\bar{G}}^{JH}_\Omega(\bar{r}',\bar{r})d\Omega'\\ &+\oiint_{\partial\Omega}\left[\hat{n}_{\to\Omega}\times\bar{H}_\Omega(\bar{r}')\right]\cdot\bar{\bar{G}}^{JE}_\Omega(\bar{r}',\bar{r})dS'\\ &-\oiint_{\partial\Omega}\left[\bar{E}_\Omega(\bar{r}')\times\hat{n}_{\to\Omega}\right]\cdot\bar{\bar{G}}^{JH}_\Omega(\bar{r}',\bar{r})dS'\end{aligned}\qquad\text{(C-4.1)}$$

here $\partial\Omega$ is the boundary of Ω. In the procedure to derive (C-4.1), the conclusion that there doesn't exist surface magnetized magnetic current [13] has been utilized, and the sufficient differentiability for material parameters are assumed. If $\bar{P}=\bar{H}(\bar{r})$ and $\bar{\bar{Q}}=\bar{\bar{G}}^{MH}(\bar{r},\bar{r}')$ are inserted into (C-3), the following integral expression for the magnetic field $\bar{H}$ at any position $\bar{r}$ in Ω can be obtained.

$$\begin{aligned}\bar{H}_\Omega(\bar{r}) =& -\iiint_\Omega\bar{J}(\bar{r}')\cdot\bar{\bar{G}}^{ME}_\Omega(\bar{r}',\bar{r})d\Omega'\\ &+\iiint_\Omega\bar{M}(\bar{r}')\cdot\bar{\bar{G}}^{MH}_\Omega(\bar{r}',\bar{r})d\Omega'\\ &-\oiint_{\partial\Omega}\left[\hat{n}_{\to\Omega}\times\bar{H}_\Omega(\bar{r}')\right]\cdot\bar{\bar{G}}^{ME}_\Omega(\bar{r}',\bar{r})dS'\\ &+\oiint_{\partial\Omega}\left[\bar{E}_\Omega(\bar{r}')\times\hat{n}_{\to\Omega}\right]\cdot\bar{\bar{G}}^{MH}_\Omega(\bar{r}',\bar{r})dS'\end{aligned}\qquad\text{(C-4.2)}$$

In the procedure to derive (C-4.2), the conclusion that there doesn't exist surface ohmic and polarized electric currents [13] has been utilized. In (C-3) and (C-4), the $\hat{n}_{\to\Omega}$ is the unity normal vector at boundary $\partial\Omega$, and it points to the interior of domain Ω. The subscripts "Ω" in (C-4) represent that the relevant expressions for fields and Green's functions are only valid for the domain Ω.

When the sources $\{\bar{J}^{inc}(\bar{r}),\bar{M}^{inc}(\bar{r})\}$, which lead to the field $\bar{F}^{inc}$, don't distribute in scatterer, the total field $\bar{F}^{tot}$ in V satisfies the following Maxwell's equations [13].

$$\begin{aligned}\nabla\times\bar{H}^{tot}_-(\bar{r}) &= j\omega\varepsilon_c\bar{E}^{tot}_-(\bar{r})\\ \nabla\times\bar{E}^{tot}_-(\bar{r}) &= -j\omega\mu\bar{H}^{tot}_-(\bar{r})\end{aligned}\quad,\quad(\bar{r}\in V)\qquad\text{(C-5)}$$

or equivalently written as the following form

$$\begin{aligned}\nabla\times\bar{H}^{tot}_-(\bar{r}) &= \bar{J}^{vop}(\bar{r})+j\omega\varepsilon_0\bar{E}^{tot}_-(\bar{r})\\ \nabla\times\bar{E}^{tot}_-(\bar{r}) &= -\bar{M}^{vm}(\bar{r})-j\omega\mu_0\bar{H}^{tot}_-(\bar{r})\end{aligned}\quad,\quad(\bar{r}\in V)\qquad\text{(C-5')}$$

The total field $\bar{F}^{tot}$ in $\mathbb{R}^3\setminus V$ satisfies the following Maxwell's equations [13].

$$\begin{aligned}\nabla\times\bar{H}^{tot}_+(\bar{r}) &= \bar{J}^{inc}(\bar{r})+j\omega\varepsilon_0\bar{E}^{tot}_+(\bar{r})\\ \nabla\times\bar{E}^{tot}_+(\bar{r}) &= -\bar{M}^{inc}(\bar{r})-j\omega\mu_0\bar{H}^{tot}_+(\bar{r})\end{aligned}\quad,\quad\left(\bar{r}\in\mathbb{R}^3\setminus V\right)\qquad\text{(C-6)}$$

The above subscripts "+" and "−" represent that the relevant fields respectively correspond to the external and internal domains of scatterer.

By inserting the (C-5) into (C-4), and restricting the region Ω to V, the following integral expressions for $\bar{F}^{tot}_-$ can be derived.

$$\begin{aligned}\bar{E}^{tot}_-(\bar{r}) =& +\oiint_{\partial V}\left[\hat{n}_-\times\bar{H}^{tot}_-(\bar{r}')\right]\cdot\bar{\bar{G}}^{JE}_-(\bar{r}',\bar{r})dS'\\ &-\oiint_{\partial V}\left[\bar{E}^{tot}_-(\bar{r}')\times\hat{n}_-\right]\cdot\bar{\bar{G}}^{JH}_-(\bar{r}',\bar{r})dS'\end{aligned}\qquad\text{(C-7.1)}$$

$$\begin{aligned}\bar{H}^{tot}_-(\bar{r}) =& -\oiint_{\partial V}\left[\hat{n}_-\times\bar{H}^{tot}_-(\bar{r}')\right]\cdot\bar{\bar{G}}^{ME}_-(\bar{r}',\bar{r})dS'\\ &+\oiint_{\partial V}\left[\bar{E}^{tot}_-(\bar{r}')\times\hat{n}_-\right]\cdot\bar{\bar{G}}^{MH}_-(\bar{r}',\bar{r})dS'\end{aligned}\qquad\text{(C-7.2)}$$

for any $\bar{r}\in V$. The $\hat{n}_-$ is the unity normal vector on ∂V, and points to the interior of V. The subscript "−" represents that the relevant Green's functions correspond to the internal domain of scatterer.

Inserting the (C-5') into (C-4), and restricting the Ω to V, the following relations are obtained.

$$\begin{aligned}\bar{E}_-^{tot}(\bar{r}) &= +\iiint_V \bar{J}^{vop}(\bar{r}')\cdot\bar{\bar{G}}_0^{JE}(\bar{r}',\bar{r})d\Omega' \\ &\quad -\iiint_V \bar{M}^{vm}(\bar{r}')\cdot\bar{\bar{G}}_0^{JH}(\bar{r}',\bar{r})d\Omega' \\ &\quad +\oiint_{\partial V}\left[\hat{n}_-\times\bar{H}_-^{tot}(\bar{r}')\right]\cdot\bar{\bar{G}}_0^{JE}(\bar{r}',\bar{r})dS' \\ &\quad -\oiint_{\partial V}\left[\bar{E}_-^{tot}(\bar{r}')\times\hat{n}_-\right]\cdot\bar{\bar{G}}_0^{JH}(\bar{r}',\bar{r})dS'\end{aligned} \tag{C-7.1'}$$

$$\begin{aligned}\bar{H}_-^{tot}(\bar{r}) &= -\iiint_V \bar{J}^{vop}(\bar{r}')\cdot\bar{\bar{G}}_0^{ME}(\bar{r}',\bar{r})d\Omega' \\ &\quad +\iiint_V \bar{M}^{vm}(\bar{r}')\cdot\bar{\bar{G}}_0^{MH}(\bar{r}',\bar{r})d\Omega' \\ &\quad -\oiint_{\partial V}\left[\hat{n}_-\times\bar{H}_-^{tot}(\bar{r}')\right]\cdot\bar{\bar{G}}_0^{ME}(\bar{r}',\bar{r})dS' \\ &\quad +\oiint_{\partial V}\left[\bar{E}_-^{tot}(\bar{r}')\times\hat{n}_-\right]\cdot\bar{\bar{G}}_0^{MH}(\bar{r}',\bar{r})dS'\end{aligned} \tag{C-7.2'}$$

for any $\bar{r}\in V$.

Inserting the (C-6) into (C-4), and restricting the Ω to $\mathbb{R}^3\setminus V$, and employing the Sommerfeld's radiation conditions for the fields and various Green's functions [19], the following integral expressions for $\bar{F}_+^{tot}$ are derived.

$$\begin{aligned}\bar{E}_+^{tot}(\bar{r}) &= +\iiint_{\mathbb{R}^3\setminus V} \bar{J}^{inc}(\bar{r}')\cdot\bar{\bar{G}}_0^{JE}(\bar{r}',\bar{r})d\Omega' \\ &\quad -\iiint_{\mathbb{R}^3\setminus V} \bar{M}^{inc}(\bar{r}')\cdot\bar{\bar{G}}_0^{JH}(\bar{r}',\bar{r})d\Omega' \\ &\quad +\oiint_{\partial V}\left[\hat{n}_+\times\bar{H}_+^{tot}(\bar{r}')\right]\cdot\bar{\bar{G}}_0^{JE}(\bar{r}',\bar{r})dS' \\ &\quad -\oiint_{\partial V}\left[\bar{E}_+^{tot}(\bar{r}')\times\hat{n}_+\right]\cdot\bar{\bar{G}}_0^{JH}(\bar{r}',\bar{r})dS'\end{aligned} \tag{C-8.1}$$

$$\begin{aligned}\bar{H}_+^{tot}(\bar{r}) &= -\iiint_{\mathbb{R}^3\setminus V} \bar{J}^{inc}(\bar{r}')\cdot\bar{\bar{G}}_0^{ME}(\bar{r}',\bar{r})d\Omega' \\ &\quad +\iiint_{\mathbb{R}^3\setminus V} \bar{M}^{inc}(\bar{r}')\cdot\bar{\bar{G}}_0^{MH}(\bar{r}',\bar{r})d\Omega' \\ &\quad -\oiint_{\partial V}\left[\hat{n}_+\times\bar{H}_+^{tot}(\bar{r}')\right]\cdot\bar{\bar{G}}_0^{ME}(\bar{r}',\bar{r})dS' \\ &\quad +\oiint_{\partial V}\left[\bar{E}_+^{tot}(\bar{r}')\times\hat{n}_+\right]\cdot\bar{\bar{G}}_0^{MH}(\bar{r}',\bar{r})dS'\end{aligned} \tag{C-8.2}$$

for any $\bar{r}\in\mathbb{R}^3\setminus V$. The $\hat{n}_+$ is the unity normal vector on ∂V, and points to the exterior of V. The subscripts "0" in Green's functions represent that these functions are the vacuum versions.

The Maxwell's equations for incident field $\bar{F}^{inc}$ can be written as the following (C-9) for any $\bar{r}\in\mathbb{R}^3$.

$$\begin{aligned}\nabla\times\bar{H}^{inc}(\bar{r}) &= \bar{J}^{inc}(\bar{r})+j\omega\varepsilon_0\bar{E}^{inc}(\bar{r}) \\ \nabla\times\bar{E}^{inc}(\bar{r}) &= -\bar{M}^{inc}(\bar{r})-j\omega\mu_0\bar{H}^{inc}(\bar{r})\end{aligned}\quad,\quad\left(\bar{r}\in\mathbb{R}^3\right) \tag{C-9}$$

Inserting the (C-9) into (C-4), and letting the Ω be whole space $\mathbb{R}^3$, and employing the Sommerfeld's radiation condition for the fields and various Green's functions [19], the following integral expressions for $\bar{F}^{inc}$ are obtained.

$$\begin{aligned}\bar{E}^{inc}(\bar{r}) &= +\iiint_{\mathbb{R}^3\setminus V} \bar{J}^{inc}(\bar{r}')\cdot\bar{\bar{G}}_0^{JE}(\bar{r}',\bar{r})d\Omega' \\ &\quad -\iiint_{\mathbb{R}^3\setminus V} \bar{M}^{inc}(\bar{r}')\cdot\bar{\bar{G}}_0^{JH}(\bar{r}',\bar{r})d\Omega'\end{aligned} \tag{C-10.1}$$

$$\begin{aligned}\bar{H}^{inc}(\bar{r}) &= -\iiint_{\mathbb{R}^3\setminus V} \bar{J}^{inc}(\bar{r}')\cdot\bar{\bar{G}}_0^{ME}(\bar{r}',\bar{r})d\Omega' \\ &\quad +\iiint_{\mathbb{R}^3\setminus V} \bar{M}^{inc}(\bar{r}')\cdot\bar{\bar{G}}_0^{MH}(\bar{r}',\bar{r})d\Omega'\end{aligned} \tag{C-10.2}$$

for any $\bar{r}\in\mathbb{R}^3$.

The scattering fields satisfy the following Maxwell's equations for any $\bar{r}\in\mathbb{R}^3$ [14]-[15].

$$\begin{aligned}\nabla\times\bar{H}^{sca}(\bar{r}) &= \bar{J}^{vop}(\bar{r})+j\omega\varepsilon_0\bar{E}^{sca}(\bar{r}) \\ \nabla\times\bar{E}^{sca}(\bar{r}) &= -\bar{M}^{vm}(\bar{r})-j\omega\mu_0\bar{H}^{sca}(\bar{r})\end{aligned}\quad,\quad\left(\bar{r}\in\mathbb{R}^3\right) \tag{C-11}$$

By repeating a similar procedure to derive (C-10), the following integral expressions for $\bar{F}^{sca}$ can be derived from (C-11).

$$\begin{aligned}\bar{E}^{sca}(\bar{r}) &= +\iiint_V \bar{J}^{vop}(\bar{r}')\cdot\bar{\bar{G}}_0^{JE}(\bar{r}',\bar{r})d\Omega' \\ &\quad -\iiint_V \bar{M}^{vm}(\bar{r}')\cdot\bar{\bar{G}}_0^{JH}(\bar{r}',\bar{r})d\Omega'\end{aligned} \tag{C-12.1}$$

$$\begin{aligned}\bar{H}^{sca}(\bar{r}) &= -\iiint_V \bar{J}^{vop}(\bar{r}')\cdot\bar{\bar{G}}_0^{ME}(\bar{r}',\bar{r})d\Omega' \\ &\quad +\iiint_V \bar{M}^{vm}(\bar{r}')\cdot\bar{\bar{G}}_0^{MH}(\bar{r}',\bar{r})d\Omega'\end{aligned} \tag{C-12.2}$$

for any $\bar{r}\in\mathbb{R}^3$.

Comparing of the (C-8) with (C-10) and considering of that $\bar{F}^{tot}=\bar{F}^{inc}+\bar{F}^{sca}$, the following integral expressions for $\bar{F}_+^{sca}$ are obtained.

$$\begin{aligned}\bar{E}_+^{sca}(\bar{r}) &= +\oiint_{\partial V}\left[\hat{n}_+\times\bar{H}_+^{tot}(\bar{r}')\right]\cdot\bar{\bar{G}}_0^{JE}(\bar{r}',\bar{r})dS' \\ &\quad -\oiint_{\partial V}\left[\bar{E}_+^{tot}(\bar{r}')\times\hat{n}_+\right]\cdot\bar{\bar{G}}_0^{JH}(\bar{r}',\bar{r})dS'\end{aligned} \tag{C-13.1}$$

$$\begin{aligned}\bar{H}_+^{sca}(\bar{r}) &= -\oiint_{\partial V}\left[\hat{n}_+\times\bar{H}_+^{tot}(\bar{r}')\right]\cdot\bar{\bar{G}}_0^{ME}(\bar{r}',\bar{r})dS' \\ &\quad +\oiint_{\partial V}\left[\bar{E}_+^{tot}(\bar{r}')\times\hat{n}_+\right]\cdot\bar{\bar{G}}_0^{MH}(\bar{r}',\bar{r})dS'\end{aligned} \tag{C-13.2}$$

for any $\bar{r}\in\mathbb{R}^3\setminus V$.

Comparing of the (C-7') with (C-12), the following integral expressions for $\bar{F}_-^{inc}$ are obtained.

$$\begin{aligned}\bar{E}_-^{inc}(\bar{r}) &= +\oiint_{\partial V}\left[\hat{n}_-\times\bar{H}_-^{tot}(\bar{r}')\right]\cdot\bar{\bar{G}}_0^{JE}(\bar{r}',\bar{r})dS' \\ &\quad -\oiint_{\partial V}\left[\bar{E}_-^{tot}(\bar{r}')\times\hat{n}_-\right]\cdot\bar{\bar{G}}_0^{JH}(\bar{r}',\bar{r})dS'\end{aligned} \tag{C-14.1}$$

$$\begin{aligned}\bar{H}_-^{inc}(\bar{r}) &= -\oiint_{\partial V}\left[\hat{n}_-\times\bar{H}_-^{tot}(\bar{r}')\right]\cdot\bar{\bar{G}}_0^{ME}(\bar{r}',\bar{r})dS' \\ &\quad +\oiint_{\partial V}\left[\bar{E}_-^{tot}(\bar{r}')\times\hat{n}_-\right]\cdot\bar{\bar{G}}_0^{MH}(\bar{r}',\bar{r})dS'\end{aligned} \tag{C-14.2}$$

for any $\bar{r}\in V$.

Comparing the (C-7) with (C-14), the following integral expressions for $\bar{F}_-^{sca}$ are derived.

$$\begin{aligned}\bar{E}_-^{sca}(\bar{r}) &= +\oiint_{\partial V}\left[\hat{n}_-\times\bar{H}_-^{tot}(\bar{r}')\right]\cdot\Delta\bar{\bar{G}}_-^{JE}(\bar{r}',\bar{r})dS' \\ &\quad -\oiint_{\partial V}\left[\bar{E}_-^{tot}(\bar{r}')\times\hat{n}_-\right]\cdot\Delta\bar{\bar{G}}_-^{JH}(\bar{r}',\bar{r})dS'\end{aligned} \tag{C-15.1}$$

$$\begin{aligned}\bar{H}_-^{sca}(\bar{r}) &= -\oiint_{\partial V}\left[\hat{n}_-\times\bar{H}_-^{tot}(\bar{r}')\right]\cdot\Delta\bar{\bar{G}}_-^{ME}(\bar{r}',\bar{r})dS' \\ &\quad +\oiint_{\partial V}\left[\bar{E}_-^{tot}(\bar{r}')\times\hat{n}_-\right]\cdot\Delta\bar{\bar{G}}_-^{MH}(\bar{r}',\bar{r})dS'\end{aligned} \tag{C-15.2}$$

for any $\bar{r}\in V$, here

$$\Delta\bar{\bar{G}}_-^{JE}(\bar{r}',\bar{r}) = \bar{\bar{G}}_-^{JE}(\bar{r}',\bar{r}) - \bar{\bar{G}}_0^{JE}(\bar{r}',\bar{r}) \tag{C-16.1}$$

$$\Delta\bar{\bar{G}}_{-}^{JH}(\bar{r}',\bar{r}) = \bar{\bar{G}}_{-}^{JH}(\bar{r}',\bar{r}) - \bar{\bar{G}}_{0}^{JH}(\bar{r}',\bar{r}) \quad \text{(C-16.2)}$$
$$\Delta\bar{\bar{G}}_{-}^{ME}(\bar{r}',\bar{r}) = \bar{\bar{G}}_{-}^{ME}(\bar{r}',\bar{r}) - \bar{\bar{G}}_{0}^{ME}(\bar{r}',\bar{r}) \quad \text{(C-16.3)}$$
$$\Delta\bar{\bar{G}}_{-}^{MH}(\bar{r}',\bar{r}) = \bar{\bar{G}}_{-}^{MH}(\bar{r}',\bar{r}) - \bar{\bar{G}}_{0}^{MH}(\bar{r}',\bar{r}) \quad \text{(C-16.4)}$$

In the (C-16), $\bar{r}',\bar{r} \in V$.

If the surface equivalent sources $\{\bar{J}_{\pm}^{SE},\bar{M}_{\pm}^{SE}\}$ are defined as follows

$$\bar{J}_{\pm}^{SE}(\bar{r}) = \left[\hat{n}_{\pm}(\bar{r})\times\bar{H}_{\pm}^{tot}(\bar{r}_{\pm})\right]_{\bar{r}_{\pm}\to\bar{r}} \quad , \quad (\bar{r}\in\partial V) \quad \text{(C-17.1)}$$
$$\bar{M}_{\pm}^{SE}(\bar{r}) = \left[\bar{E}_{\pm}^{tot}(\bar{r}_{\pm})\times\hat{n}_{\pm}(\bar{r})\right]_{\bar{r}_{\pm}\to\bar{r}} \quad , \quad (\bar{r}\in\partial V) \quad \text{(C-17.2)}$$

and considering of that $\left[\bar{F}_{+}^{tot}(\bar{r}_{+})\right]_{\bar{r}_{+}\to\bar{r}\in\partial V}^{\tan} = \left[\bar{F}_{-}^{tot}(\bar{r}_{-})\right]_{\bar{r}_{-}\to\bar{r}\in\partial V}^{\tan}$, and $\hat{n}_{+}(\bar{r}) = -\hat{n}_{-}(\bar{r})$ on whole ∂V, the following relations exist

$$\bar{J}_{-}^{SE}(\bar{r}) = \bar{J}^{SE}(\bar{r}) = -\bar{J}_{+}^{SE}(\bar{r}) \quad , \quad (\bar{r}\in\partial V) \quad \text{(C-18.1)}$$
$$\bar{M}_{-}^{SE}(\bar{r}) = \bar{M}^{SE}(\bar{r}) = -\bar{M}_{+}^{SE}(\bar{r}) \quad , \quad (\bar{r}\in\partial V) \quad \text{(C-18.2)}$$

and they are illustrated in Fig. 2. In (C-17), $\bar{r}_{+}\in\mathbb{R}^3\setminus V$, and $\bar{r}_{-}\in V$.

Inserting the (C-17) and (C-18) into the (C-13) and (C-15), the $\bar{F}^{sca}$ can be written as the following operator form.

$$\bar{F}^{sca}(\bar{r}) = \bar{F}^{sca}(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) = \begin{cases}\bar{F}_{-}^{sca}(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) & , \quad (\bar{r}\in V)\\ \bar{F}_{+}^{sca}(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) & , \quad (\bar{r}\in\mathbb{R}^3\setminus V)\end{cases} \quad \text{(C-19)}$$

Inserting the (C-17) and (C-18) into the (C-7), the $\bar{F}_{-}^{tot}$ can be written as the following operator form.

$$\bar{F}_{-}^{tot}(\bar{r}) = \bar{F}_{-}^{tot}(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) \quad , \quad (\bar{r}\in V) \quad \text{(C-20)}$$

Inserting the (C-17) and (C-18) into the (C-14), the $\bar{F}_{-}^{inc}$ can be written as the following operator form.

$$\bar{F}_{-}^{inc}(\bar{r}) = \bar{F}_{-}^{inc}(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) \quad , \quad (\bar{r}\in V) \quad \text{(C-21)}$$

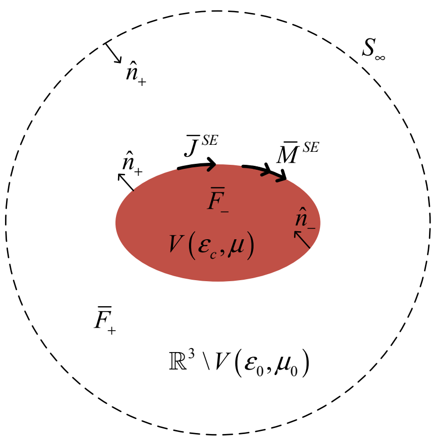

Fig. 2. The surface equivalent sources on the boundary of material scatterer.

Based on the (C-20) and (A-2), the various scattering currents on scatterer can be written as the following operator forms for any $\bar{r}\in V$.

$$\bar{J}^{vo}(\bar{r}) = \sigma\bar{E}_{-}^{tot}(\bar{r}) = \bar{J}^{vo}(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) \quad \text{(C-22.1)}$$
$$\bar{J}^{vp}(\bar{r}) = j\omega\Delta\varepsilon\bar{E}_{-}^{tot}(\bar{r}) = \bar{J}^{vp}(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) \quad \text{(C-22.2)}$$
$$\bar{J}^{vop}(\bar{r}) = j\omega\Delta\varepsilon_c\bar{E}_{-}^{tot}(\bar{r}) = \bar{J}^{vop}(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) \quad \text{(C-22.3)}$$
$$\bar{M}^{vm}(\bar{r}) = j\omega\Delta\mu\bar{H}_{-}^{tot}(\bar{r}) = \bar{M}^{vm}(\bar{J}^{SE},\bar{M}^{SE};\bar{r}) \quad \text{(C-22.4)}$$

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REFERENCES

[1] R. J. Garbacz, "Modal expansions for resonance scattering phenomena," *Proc. IEEE*, vol. 53, no. 8, pp. 856–864, August 1965.

[2] A. Zettl, *Sturm-Liouville Theory*. American Mathematical Society, 2005.

[3] R. F. Harrington and J. R. Mautz, "Theory of characteristic modes for conducting bodies," *IEEE Trans. Antennas Propag.*, vol. AP-19, no. 5, pp. 622-628, May 1971.

[4] R. F. Harrington, J. R. Mautz, and Y. Chang, "Characteristic modes for dielectric and magnetic bodies," *IEEE Trans. Antennas Propag.*, vol. AP-20, no. 2, pp. 194-198, March 1972.

[5] Y. Chang and R. Harrington, "A surface formulation for characteristic modes of material bodies," *IEEE Trans. Antennas Propag.*, vol. 25, no. 6, pp. 789–795, November 1977.

[6] A. H. Nalbantoglu, "New computation method for characteristic modes," *Electron. Lett.*, vol. 18, no. 23, pp. 994-996, November 1982.

[7] Q. I. Dai, Q. S. Liu, H. U. I. Gan, and W. C. Chew, "Combined Field Integral Equation-Based Theory of Characteristic Mode," *IEEE Trans. Antennas Propag.*, vol. 63, no. 9, pp. 3973-3981, September 2015.

[8] Y. K. Chen and C.-F. Wang, *Characteristic Modes: Theory and Applications in Antenna Engineering*. Hoboken: Wiley, 2015.

[9] Y. K. Chen, "Alternative surface integral equation-based characteristic mode analysis of dielectric resonator antennas," *IET Microw. Antennas Propag.*, vol. 10, no. 2, pp. 193-201, January 2016.

[10] R. Z. Lian, "Electromagnetic-power-based characteristic mode theory for perfect electric conductors," arXiv: 1610.05099.

[11] R. Z. Lian, "Electromagnetic-power-based characteristic mode theory for material bodies," viXra citation number: 1610.0340.

[12] J. V. Bladel, *Singular Electromagnetic Fields and Sources*. New York: IEEE Press, 1991.

[13] M. I. Sancer, K. Sertel, J. L. Volakis, and P. V. Alstine, "On volume integral equations," *IEEE Trans. Antennas Propag.*, vol. 54, no. 5, pp. 1488-1495, May 2006.

[14] A. F. Peterson, S. L. Ray, and R. Mittra, *Computational Methods for Electromagnetics*. New York: IEEE Press, 1998.

[15] J. V. Volakis and K. Sertel, *Integral Equation Methods for Electromagnetics*. SciTech, 2012.

[16] G. A. E. Vandenbosch, "Reactive energies, impedance, Q factor of radiating structures," *IEEE Trans. Antennas Propag.*, vol. 58, no. 4, pp. 1112–1127, April 2010.

[17] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*. Addison-Wesley Publishing Company, 1964.

[18] G. Y. Wen, "Stored energies and radiation Q," *IEEE Trans. Antennas Propag.*, vol. 63, no. 2, pp. 636-645, February 2015.

[19] C. T. Tai, *Dyadic Green Functions in Electromagnetic Theory, 2nd*. New York: IEEE Press, 1994.
[20] R. A. Horn and C. R. Johnson, *Matrix Analysis, 2nd*. New York: Cambridge University Press, 2013.
[21] K. Rektorys, *Variational Methods in Mathematics, Science and Engineering, 2nd.* Dordrecht: D. Reidel Publishing Company, 1980.
[22] J. M. Jin, *The Finite Element Method in Electromagnetics, 2nd*. New York: Wiley, 2002.