Scrutiny of Schwarzschild's Original Solution (1916)

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Abstract: In the present paper, the Schwarzschild's original solution (1916) is scrutinized and proven to be logically, mathematically and physically not only wrong but basically meaningless because the '*easy trick*' used by Schwarzschild violated the fundamental concepts of analytic geometry (rectangular coordinates), trigonometry (triangles) and dimensional analysis (consistency and homogeneity). It seems that Schwarzschild had systematically and deliberately violated these fundamental concepts in order to avoid/break an unavoidable/unbreakable impasse (the determinant $\neq 1$). Then he had mathematically cheated to have the determinant =1 in an *anti*-mathematical manner since he was not attached to his initial claim, *viz.*, – x_1, x_2, x_3 and x, y, z are rectangular coordinates with $r = \sqrt{x^2 + y^2 + z^2}$ – Thus, as scientists we should not forget *one* very important thing, namely, mathematics is not only an exact science, but *it* is the language of Science itself.

Keywords: Schwarzschild's original solution (1916), analytic geometry, trigonometry, dimensional analysis

1. Introduction

Historically, the Schwarzschild solution (also known as the Schwarzschild metric) is named in honor of Karl Schwarzschild (1873-1916), who found the so-called exact solution in 1915 and published it in 1916 [1], that is, a little more than a month after the publication of Einstein's general relativity theory [2]. The explicit expression of this solution is

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1} dR^2 - R^2\left(d\vartheta^2 + \sin^2 \vartheta d\phi^2\right), \quad R = \left(r^3 + \alpha^3\right)^{1/3}. \tag{i}$$

However, it is our duty to draw readers' attention to the fact that many research articles, textbooks, historians and specialists of general relativity theory (GRT) have incorrectly attributed the following solution/metric

$$ds^2 = \left(1 - \frac{r_S}{r}\right) c^2 dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2 \theta d\varphi^2\right), \quad r_S = 2GM/c^2, \tag{ii}$$

to Schwarzschild as *his* exact (vacuum) solution to the Einstein's field equations in empty space

$$R_{\mu\nu} = 0. \tag{iii}$$

The solution (ii) is supposed to be the correct description of the gravitational field outside a spherically symmetric mass. Also, the metric (ii) is usually acknowledged as the conceptual basis for the investigation of GR-effects and leading to the concept of black hole. According to Birkhoff's

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theorem, the metric (ii) is the most general spherically symmetric, vacuum solution of the Einstein's field equations (iii).

– Singularities and Black Holes

In the Schwarzschild's original solution (i) there is only one singularity at $r = 0$, however, the solution (ii), which was wrongly accredited to Schwarzschild, appears to have two singularities at $r = 0$ and at $r = r_S$ (the so-called Schwarzschild radius of the massive body, a scale factor which is related to its mass M by $r_S = 2GM/c^2$). In reality, the metric (ii) is Hilbert's solution [3] on which a more complete analysis of the singularity structure was given by Hilbert himself and he identified the two singularities. Although there was general consensus that the singularity at $r = 0$ was a 'real physical' singularity, the exact nature of the singularity at $r = r_S$ remained unclear [4]. Consequently, the concept of black hole was originated from these two singularities. –But why did historians and experts of GRT wrongly attribute the metric (ii) to Schwarzschild? Maybe because they did not read the Schwarzschild's original paper or maybe they ignored or neglected to do such a task in spite of the fact that the original paper has been translated from German to English.

– Profound difference between Mathematics and Physics

The existence of the two singularities in the metric (ii), which, as we know, supposed to be an exact (vacuum) solution to the Einstein's (gravitational) field equations (iii) shows us that the historians and experts of GRT –who erroneously credited the metric (ii) to Schwarzschild– are completely unable to distinguish Mathematics from Physics. Let us begin by recalling the profound difference between mathematics and physics. Such a recall is indispensable because in the framework of GRT there is no clear and explicit distinction between a *physical equation* (an equation written in a purely physical context) and a *mathematical equation* (an equation written in a purely mathematical context).

First, Mathematics is not Physics, and Physics is not Mathematics. The inhabitants of the mathematical world are purely abstract objects characterized by an absolute freedom. However, the inhabitants of the physical world are purely concrete objects – in the theoretical sense and/or in the experimental/observational sense – and are characterized by very relative and restricted freedom.

When applied outside its original context, mathematics should play the role of an accurate language and useful tool, and gradually should lose its abstraction. However, when we are dealing with physical equations, abstraction and freedom together lose their absolutism and become very relative, and thus restricted, because each parameter contained in the physical equation has a well-defined role, fixed by its own physical dimensions.

–Concept of infinity/singularity is absolutely irrelevant to Nature

One of most fundamental and profound distinction between a physical theory and a mathematical theory is relative to the concept of *infinity/singularity*. While in Mathematics we can associate and attribute, in perfectly logical and coherent way, the infinite value to a parameter, a dimension, or to a limit or boundary conditions, such associations are meaningless when related as results to a physical theory. And this is because in *Nature* nothing is infinite; all physical parameters of phenomena and material objects (time, space, dimension, mass, energy, temperature, pressure, volume, density, force, velocity...etc) are defined and characterized by finite values and only finite values like: minimum, average, maximum, critical and limit values. Nature cannot be described through infinite concepts and values as they are devoid of any meaning in the physical world. Nevertheless, the concept of infinity/singularity is suited only during mathematical treatment into the realm of the theories of natural sciences in order to obtain equations with finite parameters.

Indeed, any physical theory predicting, at some special upper limit conditions, singularities for any of its physical parameters is a theory based on fundamental flawed principles and concepts. But what Mathematics is to be used in particular study of Nature is in reality the critical question, which needs to be elucidated before embarking into any credible physical theory. Therefore, to use willy-nilly mathematical models for attempting to describe a particular phenomenon of Nature without physical justification for such an undertaking is an illogical act. So, we need constantly to be remained that all ways provided by Mathematics are abstract ways with no counterpart in the real physical world. The clever way therefore is to be able to find a foundation of Mathematics trough which we can communicate with the real physical world and show a convincing justification for its employment.

Hence, any claim such as "The solution/metric (ii) has two singularities at $r = 0$ and at $r = r_S$" becomes completely meaningless because singularities are absolutely irrelevant to *Nature*. Thus, the metric (ii) is in fact a pure mathematical solution without any physical connection.

2. Relationship between rectangular coordinates and spherical (polar) coordinates

It is judged imperative to start by recalling the well-known relationship between rectangular (Cartesian) coordinates (x, y, z) and spherical (polar) coordinates (r, θ, φ):

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases} , \qquad r = \sqrt{x^2 + y^2 + z^2} . \tag{iv}$$

By direct application of the dimensional analysis (DA) to (x, y, z) and (r, θ, φ), we get the following dimensional expressions:

$$[x] = [y] = [z] = \mathrm{L}, \tag{v}$$

$$[r] = \mathrm{L}, \tag{vi}$$

and

$$[\theta] = [\varphi] = 1. \tag{vii}$$

Relations (v) and (vi) are dimensionally identical, *i.e.*, the rectangular coordinates (x, y, z) and the radial distance $r = \sqrt{x^2 + y^2 + z^2}$ have the same dimensional quantity L (length).

3. Schwarzschild's Original Solution (1916)

Now, we are arriving at our main subject, that is, the scrutiny of Schwarzschild's Original Solution (1916) in order to show more conclusively that the '*easy trick*' used by Schwarzschild to derive his solution is not only wrong but more precisely *it* is logically, mathematically and physically meaningless because he systematically and deliberately violated the fundamental concepts of analytic geometry, trigonometry and dimensional analysis with the intention of avoiding/breaking an unavoidable/unbreakable impasse as we will see soon.

Historically, and as it was already mentioned, Schwarzschild found the so-called exact solution in 1915 and published it in 1916, that is, a little more than a month after the publication of Einstein's GRT [2]. It is supposed to be the exact solution of the Einstein's (gravitational) field equations in empty space (ii). Schwarzschild died shortly after his paper was published, as a result of a disease he contracted while serving in the German army during First World War. In order to make our scrutiny more comprehensible, we are obliged to rewrite the author's central claims, word by word. Thus, in his original article entitled "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie" or equivalently "On the gravitational field of a mass point according to Einstein's theory" Schwarzschild wrote (page 190):

'*Let* x_1, x_2, x_3 *stand for rectangular coordinates,* x_4 *for the time; furthermore, the mass at the origin shall not change with time, and the motion at infinity shall be rectilinear and uniform. Then, according to Mr. Einstein's list, loc. cit. p. 833, the following conditions must be fulfilled too*:
1. *All the components are independent of the time* x_4 .
2. *The equations* $g_{\rho 4} = g_{4\rho} = 0$ *hold exactly for* ρ =1, 2, 3.
3. *The solution is spatially symmetric with respect to the coordinate system in the sense that one finds again the same solution when* x_1, x_2, x_3 *are subjected to an orthogonal transformation* (*rotation*).
4. *The* $g_{\mu\nu}$ *vanish at infinity, with the exception of the following four limits different from zero*:

$$g_{44} = 1, \quad g_{11} = g_{22} = g_{33} = -1.$$

The problem is to find out a line element with coefficients such that the field equations, the equation of the determinant and these four requirements are satisfied.'

And in paragraph 3, he wrote:

'*If one calls t the time, x, y, z the rectangular coordinates, the most general line element that satisfies the condition* 1-3 *is clearly the following*:

$$ds^2 = Fdt^2 - G\left(dx^2 + dy^2 + dz^2\right) - H\left(xdx + ydy + zdz\right)^2$$

Where F, G, H are functions of $r = \sqrt{x^2 + y^2 + z^2}$.
The condition (4) *requires*: *for* $r = \infty$: $F = G = 1$, $H = 0$.
When one goes over to polar coordinates according to $x = r\sin\vartheta\cos\phi$, $y = r\sin\vartheta\sin\phi$, $z = r\cos\vartheta$, *the same line element reads*:

$$ds^2 = Fdt^2 - G\left(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\phi^2\right) - Hr^2 dr^2$$
$$= Fdt^2 - \left(G + Hr^2\right)dr^2 - Gr^2\left(d\vartheta^2 + \sin^2 \vartheta d\phi^2\right) \quad . \tag{6}$$

Now the volume element in polar coordinates is equal to $r^2 \sin\vartheta dr d\vartheta d\phi$*, the functional determinant* $r^2 \sin\vartheta$ *of the old with respect to the new coordinates is different from* 1; *then the field equations would not remain in unaltered form if one would calculate with these polar coordinates, and one would have to perform a cumbersome transformation.*'

It is clear from the above passage, it seems Schwarzschild situated in an impasse corresponding to the functional determinant different from 1, *i.e.*, the determinant $\neq 1$. But in order to avoid/break this unavoidable/unbreakable impasse, he should adapt and adopt the following strategy.

'*However there is an* <u>*easy trick*</u> *to circumvent this difficulty. One puts*:

$$x_1 = \frac{r^3}{3}, \quad x_2 = -\cos\vartheta, \quad x_3 = \phi. \tag{7}$$

Then we have for the volume element: $r^2 dr \sin\vartheta d\vartheta d\phi = dx_1 dx_2 dx_3$. *The new variables are then polar coordinates with the determinant* 1. *They have the evident advantages of polar coordinates for the treatment of the problem, and at the same time, when one includes also* $t = x_4$, *the field equations and the determinant equation remain in unaltered form.*

In the new polar coordinates the line element reads:

$$ds^2 = Fdx_4^2 - \left(\frac{G}{r^4} + \frac{H}{r^2}\right)dx_1^2 - Gr^2\left[\frac{dx_2^2}{1 - x_2^2} + dx_3^2\left(1 - x_2^2\right)\right] \quad , \tag{8}$$

... '

It is understandable that the '*easy trick*' (7) used by Schwarzschild to avoid/break this impasse violates the fundamental concepts of analytic geometry (rectangular coordinates), trigonometry (triangles) and dimensional analysis (consistency and homogeneity).Why? –Because from the beginning Schwarzschild considered x_1, x_2, x_3 and x, y, z as rectangular coordinates with $r = \sqrt{x^2 + y^2 + z^2}$, that's why the transformation (7) is mathematically meaningless, it is in contradiction with Schwarzschild's initial statement and with the relations (v), (vi) and (vii), respectively. To be precise, let us apply the DA to the transformation (7):

$$[x_1] = \mathrm{L} \neq \left[\tfrac{1}{3} r^3\right] = \mathrm{L}^3, \quad [x_2] = \mathrm{L} \neq [-\cos\vartheta] = 1, \quad [x_3] = \mathrm{L} \neq [\phi] = 1. \tag{viii}$$

Therefore, x_1, x_2, x_3 have not the same dimensional quantity L (length) to be identified as rectangular coordinates or even spherical (polar) coordinates. Thus contrary to Schwarzschild's initial claim, the expression $r^2 dr \sin\vartheta d\vartheta d\phi = dx_1 dx_2 dx_3$ has nothing to do with the volume element since x_1, x_2, x_3 in (7) are mathematically meaningless, *i.e.*, they are not rectangular coordinates. Consequently, the line element (8) is mathematically and physically meaningless, and the Schwarzschild's '*easy trick*' (7) is in fact –a mathematical cheat– because he systematically and deliberately violated the fundamental concepts of analytic geometry, trigonometry and dimensional analysis with the aim of avoiding/breaking an unavoidable/unbreakable impasse as we have seen. In view of the fact that the relations (7) are the cornerstone of Schwarzschild's derivation of the so-called exact solution (i), for this reason, this solution and Birkhoff's theorem are logically, mathematically and physically meaningless. All that shows us, more conclusively, that Schwarzschild had mathematically cheated to have the determinant=1 in an *anti*-mathematical manner since he was not attached to his initial claim, *viz*., $-x_1, x_2, x_3$ and x, y, z are rectangular coordinates with $r=\sqrt{x^2+y^2+z^2}$ –. Furthermore, in spite of its mathematical meaningless, transformation (7), may be *illegitimately* used to show that its properties are identical to those of

$$x = r\sin\vartheta\cos\phi\,,\ \ y = r\sin\vartheta\sin\phi,\ \ z = r\cos\vartheta. \qquad \text{(ix)}$$

Hence, after differentiation and some rearrangement, we get:

(a) – The Jacobean of transformation (7) is $J = r^2\sin\vartheta$,
(b) – The volume element is $dx_1dx_2dx_3 = r^2dr\sin\vartheta d\vartheta d\phi$.
(a') – The Jacobean of transformation (ix) is $J = r^2\sin\vartheta$,
(b') – The volume element is $dxdydz = r^2dr\sin\vartheta d\vartheta d\phi$.

Thus, contrary to Schwarzschild's false claim, *i.e*., the determinant of transformation (7) is not equal to 1. Schwarzschild employed the expression "*polar coordinates with determinant* 1" just as a '*word-game*' to justify the use of (7) with the purpose of transforming (6) into (8). Finally, it is obvious that Schwarzschild's procedure is not only incorrect but *anti*-mathematics.

4. Conclusion

The so-called Schwarzschild exact solution (1916) to the Einstein's (gravitational) field equations in empty space is scrutinized and proven to be not only wrong but logically, mathematically and physically meaningless because the '*easy trick*' used by Schwarzshild to avoid/break an avoidable/unbreakable impasse *was* and *is* completely in violation of the fundamental concepts of analytic geometry, trigonometry and dimensional analysis. Thus as scientists we should not forget *one* very important thing, namely, mathematics is not only an exact science, but *it* is the language of Science itself.

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