A Newtonian equivalent for the cosmological constant

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We deduce from Newtonian mechanics the cosmological constant, following some older ideas. An equivalent to this constant in classical mechanics it was found. Therefore in our development, the cosmological constant appears in a natural way into Friedmann cosmological model. But the theoretical context in which it appears tells almost nothing about the nature of expanding universe force.

Introduction

The Newtonian derivation of the Friedmann equations by Milne [1] and McCrea [2] came after Friedmann [3] and Lamaitre [4,5] demonstration. They, [3, 4, 5] had used general relativity, for an unbounded homogeneous universe, to show the validity of the Friedmann equations. Milne and McCrea, using a Newtonian formalism, had obtained the same results like relativistic cosmology theory. Nevertheless this Newtonian cosmology can not explain all the observational data since it does not contain a theory of light propagation. Even so, this approach is quite legitimate, since the structural similarity of general relativity and Newtonian celestial mechanics were pointed out by Cartan [6,7]. Following his ideas [8] showed a correct derivation of the Friedmann equations from Newtonian theory.

The development of Milne [1] leads to some problems:1) the force on any particle of an infinite homogeneous distribution is undetermined [9]; 2) the finiteness of force requires either the mass distribution to be finite, and thus homogeneous, or inhomogeneous and infinite, for cosmological distances; 3) the recession speed of the mass distribution particles is close to the speed of light, which presume a relativistic treatment of the expanding sphere. These problems were partially solved in [10], considering that the expanding sphere describe a small region compared with the size of the observable universe. Since all regions of an uniform and isotropic universe expend the same way, the study of a small region may give information about the whole universe, in which case the Newtonian treatment is correct [11].

However, this solution solved some issues and leads to other problems regarding the center of a local isotropic and homogeneous expansion that is a point impossible to establish. Carter [12] and McCrea [13] indicated that each point can be chosen as center with local isotropy and homogeneity surrounding, but there is a zone at the boundary of the expanding sphere where these properties are broken. But the same problem occurs in general relativity too.

By taking into account all above considerations we can conclude that the Newtonian cosmological models derived from Friedmann's equations are perfectly valid for an homogeneous universe and for small scale of length.

We presently develop a follow-up of Milne ideas by introducing in a natural manner the cosmological constant Λ. Empirically introduced by Einstein in 1917 [14], the cosmological constant, after the discovery of expanding universe [15], was repelled by its author. However, in Friedmann model, Λ appears from general relativity equations of field deduced by Lemaitre in 1927, [3,4,16]. Milne, [1,2], demonstrated his Newtonian derivation, in which the cosmological constant appears by postulation.

The main aim of this paper is to show that Λ can be introduced directly, without postulation.

Theoretical treatment

If two spheres attract each other with a force proportional with $\frac{1}{r^2}$, as demonstrated by Newton's theorem, then we can replace the two spheres with points that have the mass of the associated spheres. In this case one can study [17], the general form of the gravitational potential and the limits of this approximation.

Considering a spherical surface with radius α, density σ, situated at distance r from a center O and an arbitrary exterior point P, one observes that the gravitational potential in P is equivalent to one generated by the mass $m_{(\alpha)}$ placed in O. We can write this as a function of the gravitational potential due to a central mass at distance r, $\Phi_{(r)}$:

$$m_{(\alpha)}\Phi_{(r)} + 2\pi\sigma\alpha\gamma_{(\alpha)} = \frac{2\pi\sigma\alpha}{r}\int_{r-\alpha}^{r+\alpha}\beta\Phi_{(\beta)}d\beta \tag{1}$$

where $\gamma_{(\alpha)}$ is a constant which can be added to the potential without altering its associated law force.

Equation (1) has two classes of solutions. The first one, [18], is function of the Yukawa type potentials:

$$\Phi_{(r)} = \frac{A_1 e^{\xi\cdot r} + A_2 e^{-\xi\cdot r}}{r} + A_3 \tag{2}$$

with the equivalent mass:

$$m_{(\alpha)} = 4\pi\sigma\alpha\,\frac{sh(\xi\cdot\alpha)}{\xi} \tag{3}$$

and A_1, A_2, A_3 being arbitrary real constants, ξ is a constant $\in R$ or C and

$\gamma_{(\alpha)} = 2A_3\alpha$

When $A_3 = 0$ and $\xi = 0$ we obtain the mass of sphere in the Newtonian particular case:

$m_{(\alpha)} = 4\pi\sigma\alpha^2$

The second class of solutions of equation (1) contains the algebraic potentials [19]:

$$\Phi_{(r)} = \frac{B_1}{r} + B_2 r^2 + B_3 \tag{4}$$

with the same equivalent mass:

$m_{(\alpha)} = 4\pi\sigma\alpha^2$,

B_1, B_2, B_3 arbitrary real constants and

$\gamma_{(\alpha)} = 2B_3\alpha + 2B_2\alpha^2$

The Newtonian potential $\Phi \approx r^{-1}$ is obtained by taking into account another property of inverse square radius force. For the potential Φ the interior of a spherical surface must be an equipotent region. In general $\Phi_{(r)}$ will have this property if:

$$\gamma_{(\alpha)} \cdot r = \int_{\alpha-r}^{\alpha+r} \beta \cdot \Phi_{(\beta)} d\beta \quad \text{, for } r < \alpha \tag{5}$$

Equation (5) has the unique solution:

$$\Phi_{(r)} = \frac{B}{r} + C \tag{6}$$

where the constant C can be zero without it's associated force law being altered.

The potential (6) has been used by Milne to derive the first Friedmann equation from the conservation of the total energy:

$$(\frac{dr/dt}{r})^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{r^2} + \frac{\Lambda c^2}{3} \tag{7}$$

where the cosmological constant has been introduced by postulation. Nevertheless if we observe that equation (4) includes the term $B_2 r^2$ which is the Newtonian equivalent of a cosmological constant, one may reconsider Milne's derivation.

First thing we must do is to presume valid the potential (4). In other words to consider as valid the hypothesis that the interior region of the supposed spherical surface will not to be an equipotent region. Consequently we have:

$$\Phi_{(r)} = \frac{B_1}{r} + B_2 r^2 \tag{8}$$

which is the potential (4) with $B_3 = 0$, an operation which simplifies (4) without the associated force law being altered.

Then we have:

$B_1 = -GM$, where G is the Newton's gravitational constant and M is the entire mass within the sphere,

$M = \frac{4\pi}{3}\rho \cdot r^3$, and ρ is the mass density. The constant B_2 is presumed positive, it correspond to a repulsive force.

The gravitational energy of a particle of mass m on motion within the potential (8) is:

$$E_g = -\frac{GMm}{r} + B_2 r^2 m$$

The kinetic energy of the same particle can be written as:

$$E_k = \frac{1}{2} m (dr/dt)^2$$

The conservation of the total energy leads, after an elementary calculus, to equation:

$$(\frac{dr/dt}{r})^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{r^2} - 2B_2 \tag{9}$$

with k a dimensionless constant given by:

$$k = \frac{2E}{mc^2}$$

The total force on a particle with mass m is:

$$F = m(-\frac{4\pi G}{3}\rho + 2B_2)r \tag{10}$$

If $F = 0$, leading to $dr/dt = 0$ and $\rho = const.$, we obtain:

$$2B_2 = \frac{4\pi G}{3}\rho \tag{11}$$

Using the same procedure like in the case of equation (7) we find the same result for the quantity $-\frac{\Lambda c^2}{3}$.This indicates that there is a physical equivalence between the two quantities, observed directly from similarities between equations (7) and (9). The only significant difference between them is the fact that one is deduced naturally from equation (8), the other is postulated.

In conclusion if we neglect equation (7) and set:

$$-2B_2 = \frac{\Lambda c^2}{3}$$

we have been deduced the cosmological constant from Newtonian theory.

Discussions

The derivation of cosmological constant from Newton's theory doesn't solve the problem of its physical nature. We can't conceive a Newtonian cosmology pure and simple based on this theory. To establish a concordance with observational data we must exceed the Newtonian theory and make the assumptions which lead us to Λ's physical nature. Thus the constant B_2 will result from other theories, as until now. Some early generalizations of general relativity theory found that the cosmological constant arose naturally from the mathematics, [20, 21]. But these ideas don't have a natural support and were soon abandoned. Inevitable we must consider B_2 as an intrinsic energy density of the vacuum in which case the form (11) it will be conserved. A positive cosmological constant is generated by a negative vacuum pressure. The sign minus between the constant B and the cosmological constant legitimate this idea. A positive repelling force corresponds only to a negative pressure. Hence the cosmological constant problem is occurring, which is known like the most difficult situation of fine-tuning in physics. There is no coherent procedure to derive the cosmological constant from particle physics, and also the modern field theories are pessimistic regarding this matter.

Even so the constant Λ seams to be the best solution for any complete cosmological model, like a cyclic one. Following [22] we observe that (6) is, for $l = 0$, the depending part on r solution of the Laplace's equation, [23]:

$$R_{l(r)} = A_l r^l + B_l r^{-l-1} \quad , \ l \in [0, \infty) \tag{12}$$

but the potential (4) it is not.

As illustrated in ref. [22] , the potential (4) could have a distribution with absolute minima or maxima. It means that a Newtonian universe without a cosmological constant is instable and generates paradoxes. Under these circumstances we have only an expanding universe. By including the cosmological constant makes sense to a cyclic cosmological model. For this reason we need Λ, otherwise we don't have alternative to an eternal expanding universe.

A coherent theory could be build only if we accept the modified potential (4) instead the common potential (6). On the other hand a Newtonian celestial mechanics governed by (4) it doesn't make sense at small distances. But the opposite situation of having two Newtonian mechanics, one for small distances, (eq. 6) and another for cosmological distances, (eq.4), introduce too much ambiguity into the theory. If we imagine the situation of two existing force laws, only one operable, this is completely unscientific. There is only one Newtonian mechanics, governed by only one potential, no matter what distances are claimed.

Comparing the potentials expressed by equation (4) and (6), one may observe that the former has a simpler representation but there is a reconciliatory way, regarding to all aspects, for potential (4): to make it looks like (6). In other words, one could formulate a new potential with a modified gravitational constant. The cumulated effects of the attracting and repelling potentials within relation (4) can be written as an effect of a single sensibly smaller gravitational constant. Further work will consider this problem.

Conclusions

In this paper we obtain the complete Newtonian derivation of the first Friedmann's equation. The cosmological constant was introduced in a natural way, as it resulted from (4), by calculus not by postulation. It is discussed then the importance of Λ in context of a cyclic cosmological model, in which is indispensable. As another consequence it appears to be the modified gravitational constant which holds for great values of the universe mass, at any scale we want.

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