A THEOREM FROM TOPOLOGY UNVEILS THE MYSTERY OF FRACTALS AND POWER LAWS

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The (spatial) fractals and (temporal) power laws are ubiquitously displayed by large classes of biological systems. Nevertheless, they are controversial phenomena with still unexplained genesis. From the far-flung branch of topology, a helpful concept comes into play, namely the Borsuk-Ulam theorem, shedding new light on the scale-free origin's long-standing enigma. The theorem states that a single point, if embedded in just one spatial dimension higher, gives rise to two antipodal points that have matching descriptions and similar features. Here we demonstrate that, when we introduce into a system the proper fractal extra-dimension instead of a spatial one, we are able to achieve two antipodal self-similar shapes, corresponding to the distinctive scale-free's higher and lower magnifications. By showing that the elusive phenomena of fractals and power laws can be explained and analyzed in a topological framework, we make clear why the Borsuk-Ulam theorem is the most general principle underlying their pervasive occurrence in nature.

Scale-free dynamics – also called $1/f^{\alpha}$ behavior, pink noise, power law, self-similarity, fractal-like distribution (Newman; Slomczynski) - are an intrinsic feature of a large class of natural models, from earthquakes to brain activity (Milstein; Linkenkaer-Hansen). In particular, in many dynamical systems, the frequency spectrum displays a scale-invariant behaviour $S(f)= 1/f^{\alpha}$, where S(f) is the power spectrum and f is the frequency. Further, α stands for an exponent, the so-called "dimension" of the fractal, which equals the negative slope of the line in a log power versus log frequency plot (Pritchard, Van de Ville) (**Figure 1A**). Such a general scheme stands both for (spatial) fractals and (temporal) power laws. For example, when a one-dimensional curve is self-similar, its fractal dimension α (also called D) is a continuous quantity ranging from 0 to infinity, characterized by the expression:

$$D = \frac{-\log N}{\log r(N)}.$$

This tells us that the fractal segment can be exactly decomposed into N overlapping segments (N stands in this case for any positive integer) (Mandelbrot). The fractal dimension gives rise to a dimension greater than the "classical" one-dimension which is generally attributed to the "normal" curves embedded in a standard Euclidean space. More importantly, the fractal dimension encourages us to have another look at the the mechanism that gives rise to the ubiquitous power law distributions, both in physical and biological systems.

Several possibilities have been suggested to explain the origin of power law distributions: from combination of exponentials, to successive fractionation, from inverses of quantities to the Yule process, to multiplicative noise (Newman; Beggs & Timme). It has also been proposed that power laws are involved in random walks, phase transitions and self-organized criticality, leading to physical/biological systems of increasing complexity (Bak). Recent papers start to uncover connections between the exponent of a fractal scaling in escape paths from energy basins and the activation free energy (Perkins). The ongoing fluctuations with complex scale-free properties have been absorbed into a free energy scheme (Friston), suggesting that the critical slowing implicit in power law scaling might be mandated by any system that minimizes its energetic expenditure. It has also been demonstrated that a pink noise can be achieved by adding together a random sine wave and a proper one (Tozzi), so that a fractal system might be produced by simply choosing the appropriate oscillation to include. In such a multifaceted framework, a simple theorem from the far-flung branch of the algebraic topology, namely the Borsuk-Ulam theorem (BUT) (Borsuk 1933; Manetti) is able to elucidate the general mechanism underlying different models of scale-free behavior.

MATERIALS AND METHODS

***n*-spheres.** The notation S^n denotes an n-sphere, which is a generalization of the circle (Weeks). An *n*-sphere, also called S^n, is a n-dimensional structure embedded in a n+1 euclidean space *(*Moura;Weeks*)*. For example, a 1-sphere (S^1) is the one-dimensional circumference surrounding a 2-dimensional disk, while a 2-sphere (S^2) is the 2-dimensional surface of a 3-dimensional space (a beach ball's surface is a good illustration) (Marsaglia). A *n*-sphere is a *n*-dimensional structure embedded in a n+1 space. An *n*-sphere is formed by points which are constant distance from the origin in (n+1)-dimensions (Marsaglia). For example, a 2-sphere (also called *glome* or *hypersphere*) of radius *r* (where r may be any positive real number) is defined as the set of points in a 3D Euclidean space at distance *r* from some fixed center point **c** (which may be any point in the 3D space) (Moura). A2-sphere is a simply connected 2-dimensional manifold of constant, positive curvature, which is enclosed in an Euclidean 3-dimensional space called a 3-ball. A 2-sphere is thus the surface or boundary of a 3-dimensional ball, while a 3-dimensional ball is the interior of a 2-sphere. From a geometer's perspective, we have different n-spheres, starting with the perimeter of a circle (S^1) and advancing to S^3, which is the smallest hypersphere, embedded in a 4-ball (**Figure 1**).

The Borsuk-Ulam theorem: Definition. The Borsuk-Ulam Theorem (BUT) is a finding by K. Borsuk (Borsuk 1933)about Euclidean n-spheres and antipodal points. It states that (Dodson):
Every continuous map $f : S^n \rightarrow R^n$ must identify a pair of antipodal points on S^n map to same point in R^n.
This means that diametrically opposite points (antipodes) on S^n are mapped to a single point in the n-dimensional Euclidean spaceR^n. Points on S^n are *antipodal*, provided they are diametrically opposite (Krantz). Examples of antipodal points are the endpoints of a line segment, or opposite points along the circumference of a circle, or poles of a sphere. A point embedded in an R^n manifold is projected to a pair of diametrically opposite points on a S^{n+1}-sphere, and vice versa. In effect, the Borsuk-Ulam theoremtells us that we can always find antipodal points on S^n (which is embedded in R^{n+1}) project to the same point on R^n (which contains S^{n-1}) (see **Figure 1B**). In other words, if a sphere is mapped continuously into a plane set, there is at least one pair of antipodal points having the same image; that is, they are mapped in the same point of the plane (Beyer; Borsuk 1958-1959). Put simply, two opposite points on a sphere, when projected on a circumference, give rise to a single point with a description that matches the description of both anipodes. This means that the projection from a higher dimension (equipped with two antipodal points) to a lower one gives rise to a single point. Conversely, we have pullback in which a point on S^{n-1} (embedded in R^n) projects back to two antipodal points on R^{n+1} (which contains S^n). As a result, this also means that the projection from a lower dimension (equipped with just one point) to a higher dimensional space, leads us back to information-carriers in the form of antipodal points that explain the origin of the single signal value predicted by the Borsuk-Ulam theorem. Note that, in the classical formulation of BUT, the function*f* needs to be continuous and *n* must be a natural number (although we will see that this is not always the case). For other definitions of BUT and its many proofs, see (Matoušek).

Description of a signal through BUT. In terms of activity, a feature vector $x \in R^n$ models the description of a signal. To elucidate the picture in the application of the BUT in signal analysis, we view the surface of a manifold as a *n*-sphere and the feature space for signals as finite Euclidean topological spaces (**Figure 1B**). The BUT states that for the description $f(-x)$ for a signal x, we expect to find an antipodal feature vector $f(-x)$ describing a signal on the opposite (antipodal) side of the manifold S^n. The pair of antipodal signals have matching descriptions on S^n.
Let *X*denote a nonempty set of points on the manifold's surface. A topological structure on *X*(called a topological space) is a structure given by a set of subsets τ of *X*, equipped with the following properties:
(Str.1) Every union of sets in τ is a set in τ .
(Str.2) Every finite intersection of sets in τ is a set in τ .
The pair (X, τ) is a topological space. Usually, *X*by itself is called a topological space, provided it has a topology τ on it. Let *X*,*Y* be topological spaces. Recall that a function or map $f : X \rightarrow Y$ on a set *X* to a set Y is a subset $X \times Y$ so that for each $x \in X$there is a unique $y \in Y$such that $(x,y) \in f$ (usually written $y = f(x)$). The mapping *f* is defined by a rule that tells us how to find $f(x)$ (Willard).
Shapes and homotopies. A mapping $f : X \rightarrow Y$ is continuous, provided, when $A \subset Y$ is open, that the inverse $f^{-1}(A) \subset X$ is also open (Krantz). In such a view of continuous mappings from the signal topological space X on the manifold's surface to the signal feature space R^n, we consider not just one signal feature vector $x \in R^n$, but also mappings from *X* to a set of signal feature vectors $f(X)$. This expanded view of signals is noteworthy, since every connected set of feature vectors $f(X)$ has a shape. It means that signal shapes can be compared.

A consideration of $f(X)$ (set of signal descriptions for a region X) instead of $f(x)$ (description of a single signal x) leads to a region-based view of signals. This region-based manifold's view arises naturally in terms of a comparison of shapes produced by different mappings from X (object space) to the feature space R^n. Continuous mappings from object spaces to feature spaces lead into homotopy theory and the study of shapes.

Let $f, g : X \to Y$ be continuous mappings from X to Y. The continuous map $H : X \times [0,1] \to Y$ is defined by:

$H(x,0) = f(x),\ H(x,1) = g(x),$ for every $x \in X$.

The mapping H is a *homotopy*, provided there is a continuous transformation (called a deformation) from f to g. The continuous maps f, g are called homotopic maps, provided $f(X)$ continuously deforms into $g(X)$ (denoted by $f(X) \to g(X)$). The sets of points $f(X)$,$g(X)$ are called shapes (Manetti; Cohen).

For the mapping $H : X \times [0,1] \to R^n$, where $H(X,0)$ and $H(X,1)$ are *homotopic*, provided $f(X)$ and $g(X)$ and have the same shape. That is, $f(X)$ and $g(X)$ are homotopic, if:

$\| f(X) - g(X) \| < \| f(X) \|$, for all $x \in X$.

Borsuk first associated the geometric notion of shape and homotopies. The early work on n-spheres and antipodal points led to the study of retraction and homotopic mappings (Borsuk 1958-59, Borsuk 1969, Borsuk 1980) and into the geometry of shapes and shapes of space (Collins). A pair of connected planar subsets in Euclidean space R^2 have equivalent shapes, provided the planer sets have the same number of holes. In terms of signals, it means that the connected graph for *f (X)* with, for example, an *e* shape, can be deformed into the 9 shape. This suggests another useful application of Borsuk's view of the transformation of a shape into another, in terms of signal analysis (Schleicher; Su). Sets of signals not only will have similar descriptions, but also dynamic character and the deformation of one signal shape into another occurs when they are descriptively near (Peters). The nice thing about antipodal points is that the concept can be generalized to countless types of systems' signals (Peters 2014). It is indeed worth of note that the two antipodal points can be used not just for the description of simple topological points, but also of more complicated structures and systems, such asshapes of space (spatial patterns), shapes of time (temporal patterns), movements and trajectories (Peters 2015; Collins). The two antipodal points on a S^n-sphere are characterized by the same function, have matching descriptions and display similar features (Borsuk 1969; Borsuk 1980). The collections of signals can be viewed as surface shapes, where one shape maps to another antipodal one (Cohen). It means that different phenomena can be studied in terms of opposite points, if we only consider them embedded in just one dimension higher than the usual one. If we simply evaluate biological scale-free dynamics instead of "signals", BUT leads naturally to the possibility of a region-based, not a simple point-based, geometry of fractals. When we apply BUT and its extensions to self-similar systems, we achieve interesting results. If we regard scale-free dynamics as embedded in a dimension different from the usual one, we are allowed to study $1/f^\alpha$ behaviorin terms of opposite points on a n-sphere.

RESULTS

We proceeded as follows. At first we enclosed fractal and power law structures, equipped with antipodal self-similar points, into the abstract spaces of *n*-spheres. In such a way, the $1/f^\alpha$ behavior – both in its spatial and temporal variants – can be evaluated in guise of projections on S^n. **Figure 2A** illustrates an example of "spatial" fractals in the framework of BUT. The two antipodal points *A* and *B* stand for two self-similar fractal structures, assessed at different level of observation: the point A displays the scale-free system's macrostates, while the point B displays its microstates. Although BUT has been originally described as valid just in case of n being a natural number, in this context the value of *n* in S^n is a fractional number: 1.3. In such a framework, the n exponent is not a natural number, as it occurs in the "classical" BUT: it is instead the scale-free dynamics' fractal dimension - which is a rational number – which stands for the n exponent. Are we allowed to modify the BUT's the exponent on an *n*-sphere, changing a natural number into a rational one, to describe *n* in a BUT fractal system equipped with two antipodal points? We here demonstrate that the answer is positive, by taking into account a Borsuk-Ulam theorem on *d*-spheres with Hausdorff dimension *d*, which is a fraction between 0 and 1.

We used the following terminology:

1) **Metric space**: Let X be a metric space with the metric $\mu_d(X)$ defined on it. This means that $\mu_d(X) \geq 0$ and μ_d has the usual symmetry and triangle inequality properties for all subsets of X.
2) **Hausdorff measure**: Let d be either 0 or a positive real number in R_0^+. The Hausdorff measure $\mu_d(X)$ equals a real number for each number d in $X = R^d$.
3) **Hausdorff dimension (informal)**: The threshold value of d denoted by $\dim_H(X)$ is the Hausdorff dimension of X, provided $\mu_d(X) = 0$, if $d > \dim_H(X)$, and $\mu_d(X) = \infty$, if $d < \dim_H(X)$.

Hausdorff Dimension- To arrive at the Hausdorff (fractional) dimension of a subset X in a metric space, we need to consider the Hausdorff measure of X.

Definition 1. Hausdorff measure. Let X be a subset of a metric space M and let d any real number in R_0^+ $\varepsilon \in R_0^+$(a real number that is either positive or zero) a nonempty subset of X, U_i, $i \in \{1, . . . ,n\}$ is a cover of X, i.e., X is a subset of

$X \subseteq C_i$ for all i (Schleicher 2007). Here n is any positive integer. Also, let diam(U_i) <ε be the diameter of the cover U_i. The d-dimensional Hausdorff measure $\mu_d(X)$ is defined by:

$$\mu_d(X) = \lim_{\varepsilon \to 0} \left[\inf_{U_i \supseteq X} \sum_{i=1}^{n} n\Big(diam\big(U_i\big)\Big)^d \right].$$

The basic idea is to cover X with sets U_i with small diameters and estimate the d-measure of X as the sum of the $(\text{diam}(U_i))^d$, i.e., the sum of the U_i diameters raised to the power d.

Lemma 1. Schleicher Lemma. Let d be any real number in R_0^+. For every bounded set X in a metric space, there is a unique value of $d := \dim_H(X)$ in $R_0^+ \cup \{\infty\}$ such that:

$\mu_d^I(X) = 0$, if $d^I > d$.

$\mu_d^I(X) = \infty$, if $d^I < d$.

Definition 2. Hausdorff dimension. The value of $d = \dim_H(X)$ in R_0^+ called the Hausdorff dimension of X. With $d = \dim_H(X)$, the Hausdorff measure $\mu_d(X)$ may be zero, positive or infinite.

Lemma 2. Schleicher Boundedness Lemma. Let d be any real number in R_0^+ and let Y be a metric space. If $X \subset Y$, then:

$\dim_H(X) \leq \dim_H(Y)$.

Proof. Immediate from the definition of the Hausdorff dimension of a nonempty set.

Assume that X is a nonempty subset (inner sphere) of an n-sphere and having the same center as S^n with the Hausdorff measure $\mu_d(X)$ defined on it and assume that $\mu_d(X)$ satisfies the Schleicher Lemma 1 conditions. The inner sphere S^d of an n-sphere S^n can be any sub-sphere in S^n, including S^n itself. Then, the inner sphere S^d has dimension $d = \dim_H(X)$, $d \leq n$. In addition, assume that R^d is a d-dimensional space which is a subset of the n-dimensional Euclidean space R^n, $d<n$. This gives us new form of the Borsuk-Ulam Theorem (Borsuk 1933).

Theorem 1. Hausdorff-Borsuk-Ulam Theorem. Let S^d with Hausdoff dimension d be an inner sphere of an n-sphere and let $f : S^d \longrightarrow R^d$ be a continuous map. There exists a pair of antipodal points on S^d that are mapped to the same point in R^d.

Proof. A direct proof of this theorem is symmetric with the proof of the Borsuk-Ulam Theorem is given by Su (1997), since we assume that S^d is an inner sphere of S^n symmetric about the center of S^n and, from the Schleicher Boundedness Lemma 2, $\dim_H(S^d) \leq \dim_H(S^n)$.

In mathematical words, we showed there that the Borsuk-Ulam theorem can be used for the description of antipodal points on d-spheres equipped with Hausdorff dimension d. Our results allow us to use the n parameter as a versatile tool not just for the description of topological manifolds, but also of biological and physical systems. It is thus suitable to make use of a rational numbers, instead of integer ones, as n exponents in S^n. The same mechanism described in Figure 2A for "spatial" fractals is valid for "temporal" power laws' plots, when they are equipped with a slope corresponding both to the α exponent and the n-dimension. **Figure 2B** displays an example of the "temporal" variant of scale-free behavior, in the framework of the BUT: if we take into account a system with, i.e., a fractal dimension D = 2.1, we may regard the spatial $1/f^{\alpha}$ structure (equipped with antipodal points A and B) as embedded in a sphere $S^{2.1}$.

CONCLUSION

Our study elucidates how (spatial) fractals and (temporal) power laws are produced, both in physical and biological systems. The Borsuk-Ulam theorem displays two very useful general featureswhich help us to explain a wide-range of phenomena, including fractals. First, when a single point is embedded in just one dimension higher, it gives rise to two antipodal points: it means that, by adding just a further dimension (in our case, the fractal one) to a biological or physical system, we are allowed to study it in terms of antipodal points. Second, if we evaluate scale-free dynamics instead of "signals", a collections of fractal signals could be viewed as surface shapes (or signals) where one shape maps to another antipodal one. Although BUT has been originally described as valid just in case of n being a natural number, recent studies investigated the theorem in the framework of more general conditions. In particular, it has been shown that this extension holds for the case of rationally independent numbers (Kim). We demonstrated here that BUT holds also for n-sphere with a fractional n, and thus for scale-free systems. What does a topologic reformulation brings on the table, in the evaluation of $1/f^{\alpha}$ behaviors? The opportunity to treat fractals as topological structures gives us the unvaluable chance to describe them through the powerful analytical tools of homology theory and functional analysis (Matoušek; Yang; Dol'nikov). The BUT perspective allows a symmetry property located in the real space (the environment) to be translated to an abstract space and *vice-versa*, enabling us to achieve a map from one dynamical system to another. Embracing self-similarityin the framework of BUT means that scale-free' symmetry transformations (the antipodal points) can be described as paths or trajectories on "abstract" structures (called topological configuration spacemanifolds). It takes us into the powerful realm of algebraic topology, where the abstract metric space (a projection of the physical and biological melieu's real geometric space) is able to elucidate countless relationships of large scale structures, through correspondences from topological spaces to algebraic groups (Willard; Dodson). In conclusion, we provided a general topological mechanism which explains the elusive phenomenon of fractals and power laws, casted in a physical/biological fashion which has the potential of being operazionalized.

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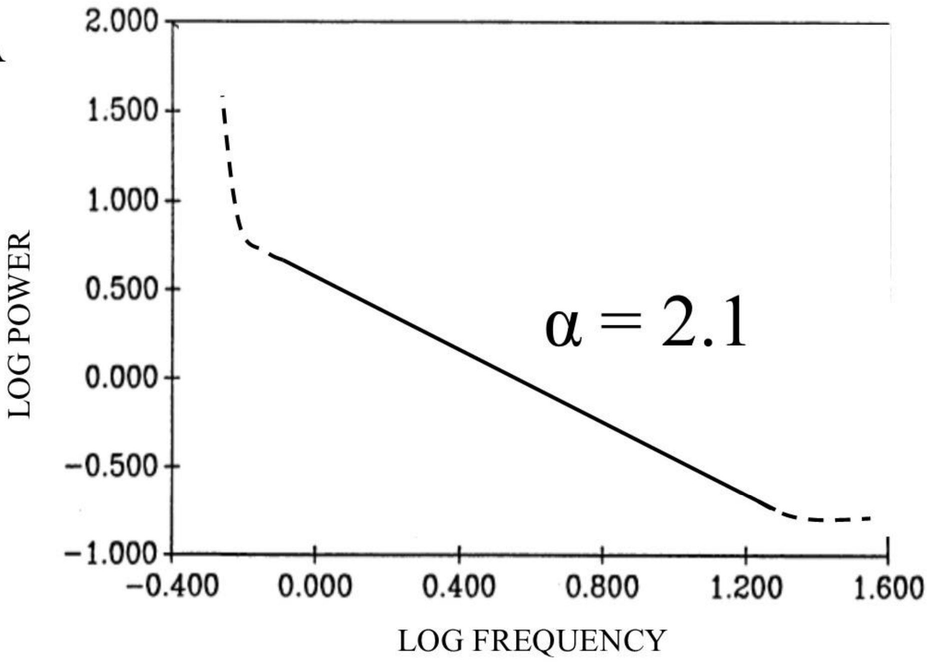
A



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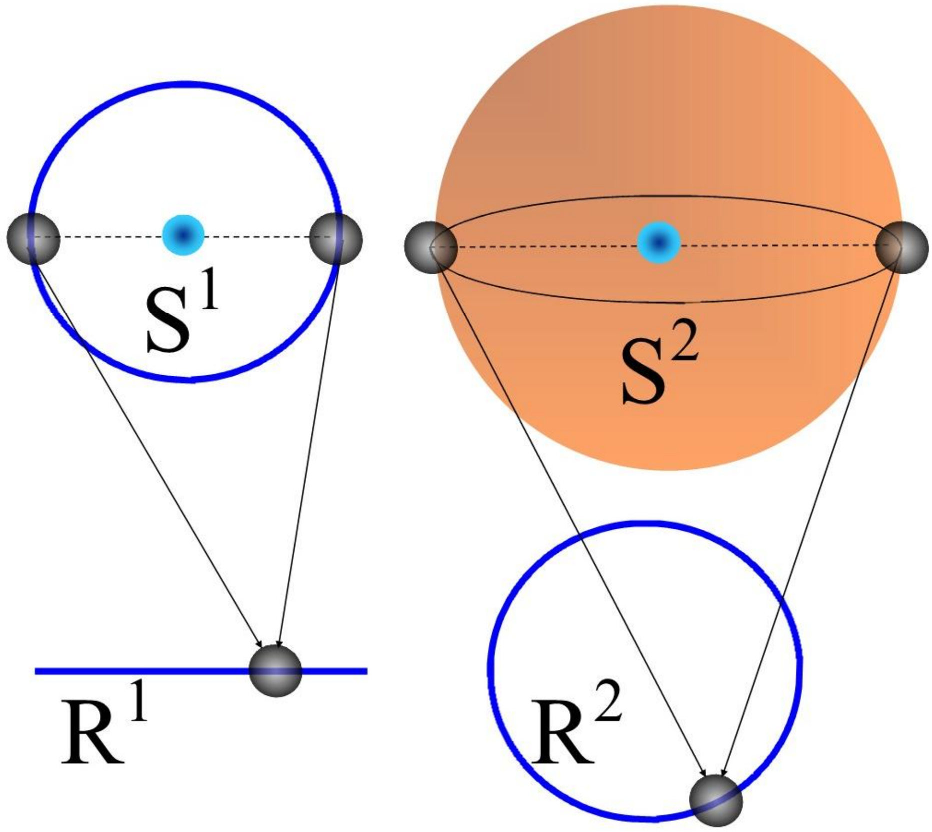

Figure 1A. Example of temporal power laws. The plot displays log power versus log frequency of an electric wave, along with the regression line (modified from Pritchard 1992). The regression line's slope (in this case, $\alpha = 2.1$) is the linear alignment of the data points reflecting the dominant power law. Note that at the slope's tails the α exponent is lost (dotted lines on the right and left of the main slope).

Figure 1B. The Borsuk-Ulam theorem for different values of S^n. Each S^n is embedded in an Euclidean space R^{n+1}. Two antipodal points (black circles) in S^n project to a single point in R^n, and vice versa. Note that the two antipodal points do not stand just for topological points, but they can also stand either for feature vectors, or shapes, or signals, or changes in their dynamic character.

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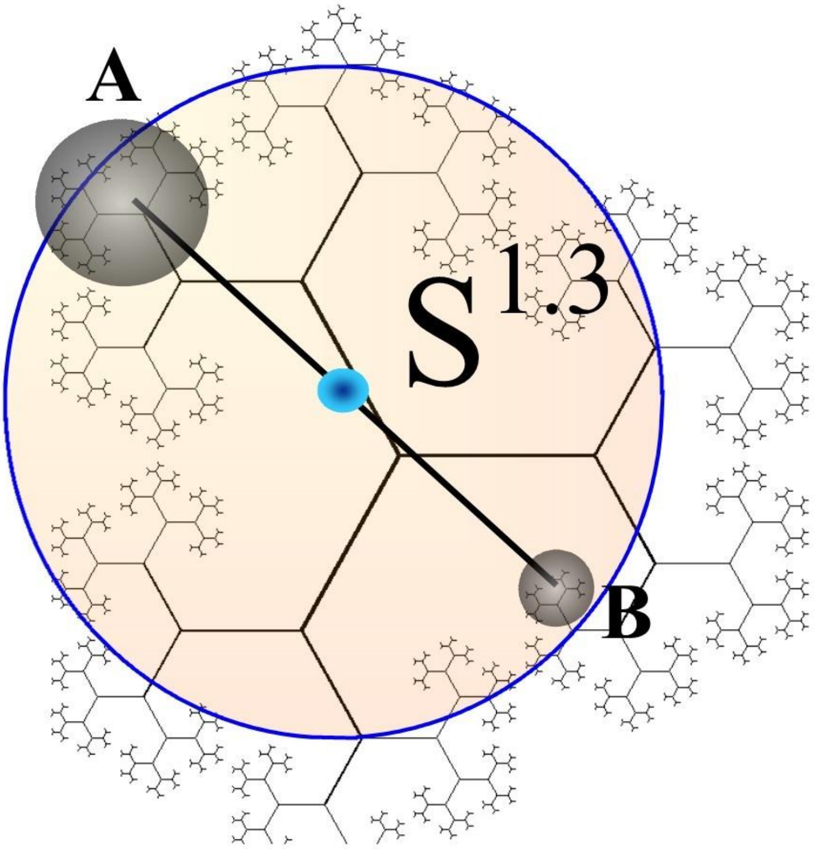

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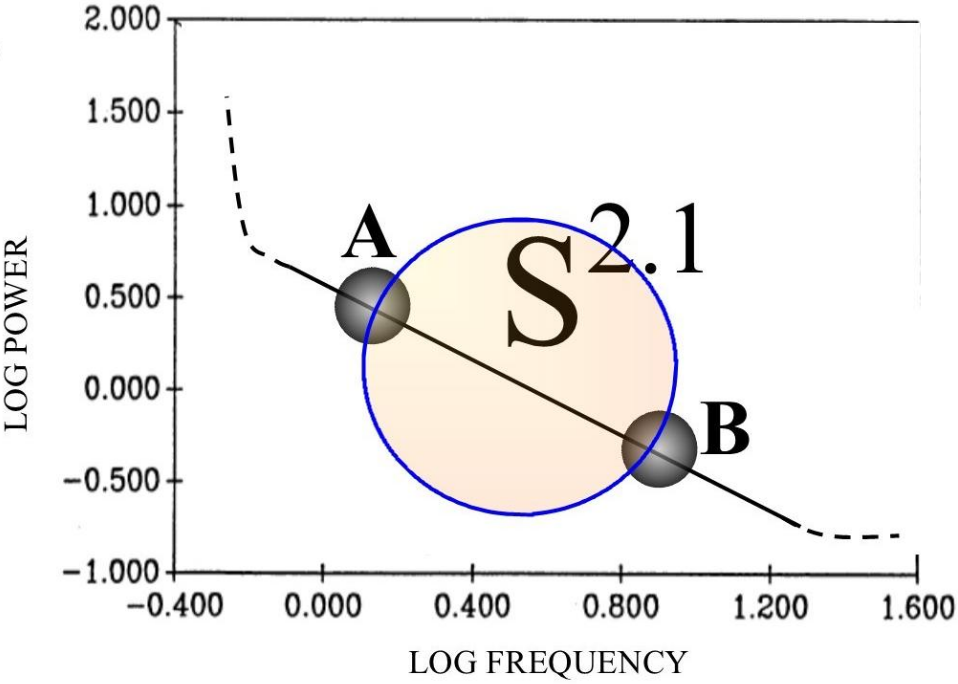

Figure 2. Antipodal points on self-similar structures.
Figure 2A. Spatial fractals embedded in a n-sphere equipped with n corresponding to a rational number. The black circles A and B depict fractals at lower and higher magnification, respectively.
Figure 2B. The figure, modified from our Figure 1A, illustrates an example of temporal power laws embedded in a n-sphere. The n-sphere, in this case, is equipped with a value of n corresponding to the fractal dimension α. Note that the Borsuk-Ulam theorem with its antipodal points is not valid at the slope's tails, where the α exponent is lost (dotted lines).