**THE BORSUK-ULAM THEOREM: AN UNIVERSAL PRINCIPLE FOR PHYSICAL SYSTEMS**

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The Borsuk-Ulam Theorem (BUT) states that a single point, if embedded in one spatial dimension higher, gives rise to two antipodal points with matching descriptions and similar features. Novel BUT variants allow the assessment of countless physical systems, from entropies to quantum entanglement. We argue that BUT, cast in a quantitative fashion which has the potential of being operationalized, is a universal principle underlying a number of natural phenomena.

Topology, which assesses the properties that are preserved through deformations, stretchings and twistings of objects (Manetti; Krantz), is a underrated methodological approach with countless possible applications. In particular, we show that novel incarnations of the "classical" Borsuk-Ulam theorem (BUT) lead to a better comprehension and assessment of several physical phenomena. BUT and its variants provide indeed a topological methodology for the evaluation of the most general features of systems activity, cast in an empirical fashion that has the potential to be operationalized.

**The "standard" version of the Borsuk-Ulam theorem (BUT)**. The notation $S^n$ denotes an n-sphere, which is a generalization of the circle (Weeks). A *n*-sphere is a n-dimensional structure of constant, positive curvature, embedded in a *n*+1 space (Marsaglia; Henderson). For example, a 1-sphere ($S^1$) is the one-dimensional circumference surrounding a 2-dimensional disk, while a 2-sphere ($S^2$) is the 2-dimensional surface of a 3-dimensional ball (a beach ball is a good example). A 3-sphere is a 3-dimensional manifold which is enclosed in a Euclidean 4-dimensional space called a 4-ball. A 3-sphere is thus the surface of a 4-dimensional ball, while a 4-dimensional ball is the interior of a 3-sphere, in the same way as a bottle of water is made of a glass surface and a liquid content. The Borsuk-Ulam Theorem (Borsuk 1933; Dodson) states that, if a sphere $S^n$ is mapped continuously into a *n*-dimensional Euclidean space $R^n$, there is at least one pair of antipodal points on $S^n$ which map onto the same point of $R^n$ (Beyer) (**Figure A**). Points on $S^n$ are *antipodal*, provided they are diametrically opposite. Examples of antipodal points are the opposite points along the circumference of a circle, or the poles of a sphere (Matousek). The **SUPPLEMENTARY INFORMATION 1** provides a mathematical treatment for technical readers.

**Matching signals (Signal-BUT).** The concept of antipodal points can be generalized to countless types of systems' signals (Borsuk 1958-59; Borsuk 1969). They can be used not just for the description of simple topological points, but also of more complicated structures, such as shapes of space (spatial patterns), of shapes of time (temporal patterns), vectors or tensors, functions, signals, thermodynamical parameters, movements, trajectories and general symmetries (Peters 2016) (**SUPPLEMENTARY INFORMATION 2)**. If we simply evaluate systems activity instead of "signals", BUT leads naturally to the possibility of a region-based, not simply point-based, geometry, with many applications. A region can have indeed features such as area, diameter, average signal value, and so on. We are thus allowed to describe systems features as antipodal points on a n-sphere. If we map the two points on a n-1 –sphere, we obtain a single point. This means that signal shapes can be compared (Weeks; Peters 2016): the two antipodal points standing for systems features are assessed at one level of observation, while the single point at a lower level (Tozzi 2016a).

**BUT for non-antipodal points (Re-BUT).** The BUT can be generalized not just for the evaluation of antipodal, but also of non-antipodal points on an *n*-sphere (**Figure B**). We can consider regions on an *n*-sphere that are either adjacent or far apart (Tozzi 2016a). And ReBUT applies, provided there are a pair of regions on *n*-sphere with the same feature value. Therefore, the two points (or regions) do not need necessarily to be antipodal, in order to be described together (Peters 2016). This makes it possible to evaluate matching signals, even if they are not "opposite", but "near" each other: the antipodal points restriction from the "standard" BUT is no longer needed.

**Generalization of BUT to antipodal points occurring on hyperbolic manifolds (Hyper-BUT).** The original formulation of BUT describes antipodal points on spatial manifolds in every dimension, provided the n-sphere is a convex structure with positive curvature (i.e, a ball). However, many natural phenomena occur on manifolds endowed with other types of geometry. BUT can be generalized also to symmetries occurring either on flat manifolds, or on Riemannian hyperbolic n-manifolds of constant sectional curvature -1 and concave shape (i.e, a saddle) (Mitroi-Symeonidis). In other words, whether the systems function displays a concave, convex or flat structure, it does not matter: we may always find the points with matching description predicted by BUT. For further details, see Tozzi (2016a).

**Changes in the n value of $S^n$ spheres ($S^n$-BUT)**. Although BUT was originally described just in case of n being a natural number which expresses a structure embedded in a spatial dimension, nevertheless the value of n can stand for other types of numbers. The n value of $S^n$ can be also cast as an integer, a rational or an irrational number (Tozzi 2016). The n value could express completely different parameters: for example, we might regard functions or shapes as embedded in a sphere in which n does not stand for a spatial dimension, but for the time or a fractal dimension. This makes it possible to use the *n* parameter as a versatile tool for the description of systems features.

**Systems' symmetry breaking (Sym-BUT)**. Symmetries are widespread invariances underlining countless physical systems (Weyl). A symmetry break occurs when the symmetry is present at one level of observation, but "hidden" at another level (Roldàn). BUT tells us that symmetries can be found when evaluating the system in a proper dimension, while they disappear (are hidden or broken) when we evaluate the same system in just one dimension lower. The symmetries are widespread at every level of organization and may be regarded as the most general feature of systems, perhaps more general than free-energy and entropy constraints too (Tozzi 2016). Thus, giving insights into symmetries provides a very general approach to every kind of systems function.

**BUT without euclidean spaces (No-R-BUT).** A $S^n$ manifold can also not map to a $R^{n-1}$ Euclidean space, but straight to a $S^{n-1}$ manifold. In other words, in this BUT formulation the Euclidean space is not mentioned. In many applications (for example, in fractal systems), we do not need the Euclidean manifold (the ball) at all: by an intrinsic, "internal" point of view, a manifold may exist in - and on – itself, and does not need to lie in any dimensional space (Weeks). Therefore, we do not need a $S^n$ manifold curving into a dimensional space $R^n$: we may think that the manifold just does exist by itself. Without the BUT limitation of the Euclidean space, we are allowed to modify the $S^n$ exponent such that it can be not just a natural number, but also other kinds of numbers, as already described above for the $S^n$-BUT. Another important consequence is that a n-sphere may map on itself: the projection of two antipodal points to a single point into a dimension lower can be internal to the same n-sphere.

**Energy-BUT.** There exists a physical link between the abstract concept of BUT and the energetic features of the system formed by two spheres $S^n$ and $S^{n-1}$. We start from a n-sphere $S^n$ equipped with two antipodal points, standing for a symmetry according to Sym-BUT. When these opposite points map to a *n*-Euclidean manifold (where $S^{n-1}$ lies), a symmetry break/dimensionality reduction occurs, and a single point is achieved (Tozzi and Peters 2016). However, it is widely recognized that a decrease in symmetry goes together with a decrease in entropy and free-energy (in a closed system). It means that the single mapping function on $S^{n-1}$ displays energy parameters lower than the two corresponding antipodal functions on $S^n$. Thus, a decrease of dimension in a system gives rise to a decrease of energy and energy requirements. In such a way, BUT and its variants become physical quantities, because we achieve a system in which the energetic changes do not depend anymore on thermodynamic parameters, rather on affine connections and homotopies.

BUT's four versatile ingredients (summarized in the **Table**) can be modified in different guises, in order to achieve a wide range of uses. To make an example, when we just consider quantum systems just one dimension higher, we are able to assess entanglement from an unusual perspective. A quantum entanglement's composite system does not display separable states and a single constituent cannot be fully described without considering the other states. If we introduce quantum entanglement on a hypersphere -, derived from signals originating on the surface of an ordinary 3D sphere - a separable state can be achieved for each of the entangled particles, just by embedding them in a higher dimensional space. When the particles are entangled at the 3D level and un-entangled at the 4D hypersphere level, we accomplish a composite system in which each local constituent is equipped with a pure state. Other examples of BUT applications are provided in **Figures C-E**. For additional applications in nonlinear chaotic dynamics (evaluation of $S^n$ spheres equipped with a n = Feigenbaum constant), see Tozzi (2016a and 2016b).
A shift in conceptualizations is evident in the BUT approach: the opportunity to treat physical systems as topological structures gives us the invaluable chance to evaluate them through correspondences from topological spaces to algebraic groups (Matoušek; Yang; Dol'nikov). Embracing systems in the framework of algebraic topology (Willard; Dodson) means that transformations (the antipodal points) can be described as paths or trajectories on "abstract" structures: the BUT perspective allows a system's property located in the real space (the physical melieu's geometric space) to be translated to an abstract space (called topological configuration space manifolds) and *vice-versa*, enabling us to achieve maps from one level to another. It makes it possible for us to study systems interactions in terms of affine connections and "proximity" among signals, in order to explain, for example, how network communities integrate or segregate information (**SUPPLEMENTARY INFORMATION 2)**.

| INGREDIENTS | GENERAL FEATURES | SPECIFIC FEATURES |
| --- | --- | --- |
| **1) CONTINUOUS FUNCTION** | LANDSCAPE | Observation |
| **2) TWO ANTIPODAL POINTS (BUT) or REGIONS (ReBUT)** | SPATIAL PATTERNS | Points |
| | | Regions |
| | | Diameters |
| | | Areas |
| | | Shapes |
| | | Concave manifolds |
| | TEMPORAL PATTERNS | Movements |
| | | Trajectories |
| | FUNCTIONS | Symmetries |
| | | Proximities |
| | | Affine connections |
| | | Vectors |
| | | Regions |
| | | Homologies |
| | SIGNALS | i.e., average signal value |
| **3) N-VALUES** | NUMBER | Natural |
| | | Fractional |
| | | Irrational |
| | DIMENSION | Spatial |
| | | Temporal |
| | | Thermodynamical parameters |
| | | Index dimension |
| | | Absence of Euclidean space |
| **4) MAPPING FROM A HIGHER TO A LOWER DIMENSION (and vice versa)** | FIXED | Description |
| | IN MOTION | Real-Time Description |
| | SYMMETRIES | Increased in a n-dimension; decreased in a n-1 dimension |

**Table.** Different features of each one of the four ingredients of BUT and its variants**.**

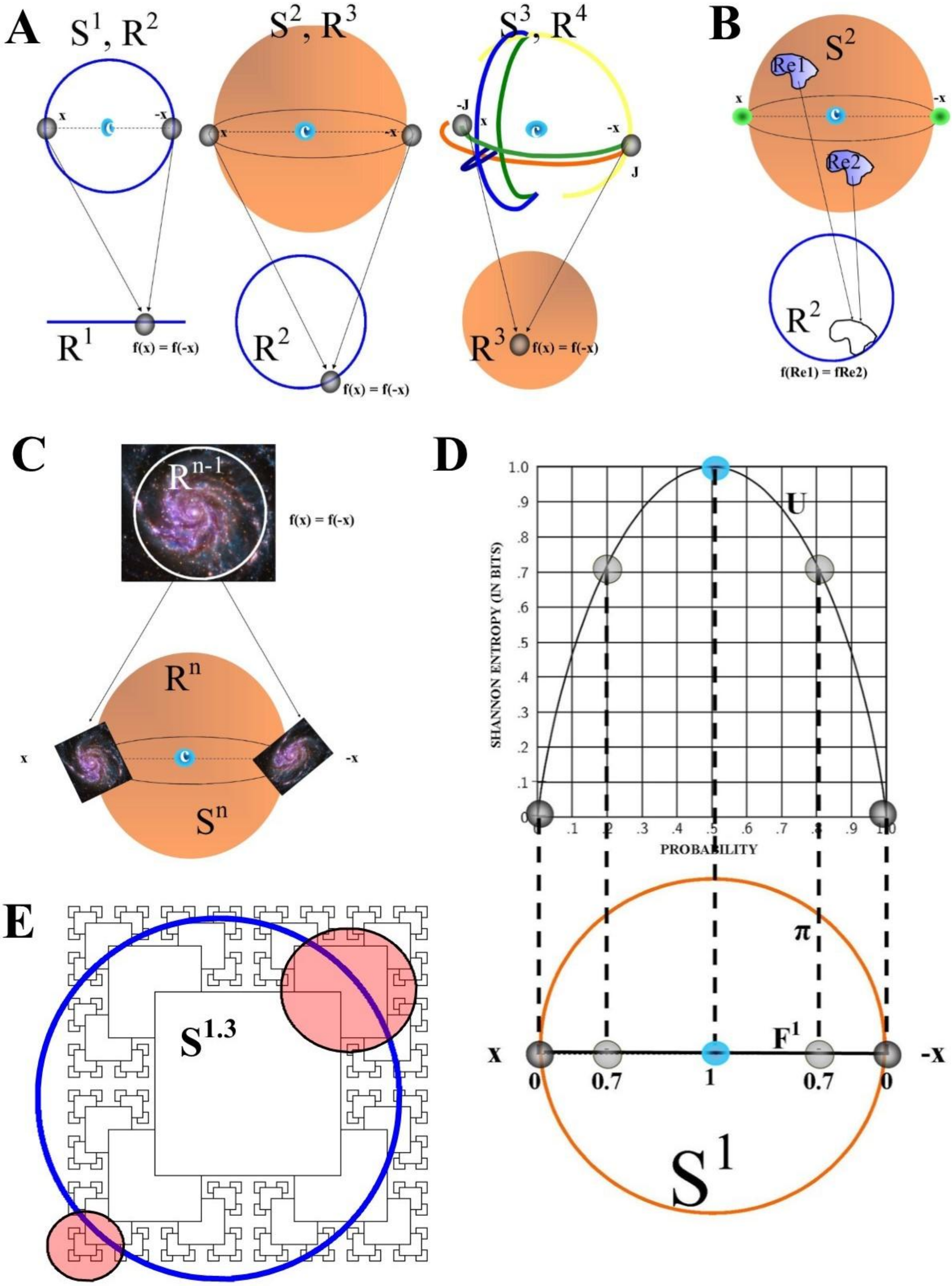


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**Figure. Many phenomena can be described by BUT and its variants. A:** the "standard" BUT for different values of $S^n$ (a circle, a sphere and a hypersphere). **B:** A simplified sketch of Re-BUT. **C**: ReBUT allows the evaluation of gravitational lenses. **D**: Shannon (ergodic) entropy values projected to a $S^1$ sphere: i.e, when entropy = 0.7, two antipodal points are displayed on a $S^1$'s diameter. Also "non-ergodic" entropies (i.e. not following the Shannon's curve) could be evaluated on other diameters of $S^1$: the use of BUT variants wipes away this long-standing limit of

Shannon entropy. **E**: on a self-similar structure, two antipodal points (corresponding to the distinctive scale-free's higher and lower magnifications) are embedded in a n-sphere equipped with n = rational number (the fractal dimension: in this example, 1.3).

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**SUPPLEMENTARY INFORMATION 1**

**The "standard" version of the Borsuk-Ulam theoerem (BUT)**. The Borsuk-Ulam Theorem (BUT) is a remarkable finding by K. Borsuk (Borsuk 1933) about Euclidean n-spheres and antipodal points. It states that (Dodson 1997):

*Every continuous map* $f : S^n \rightarrow R^n$ *must identify a pair of antipodal points (on* $S^n$*).* (**Figure 1**).

An n-sphere is formed by points which are constant distance from the origin in ($n$+1)-dimensions (Marsaglia). For example, a 3-sphere (also called *glome* or *hypersphere*) of radius $r$ (where r may be any positive real number) is defined as the set of points in 4D Euclidean space at distance $r$ from some fixed center point **c** (which may be any point in the 4D space) (Henderson). From a geometer's perspective, we have the following n-spheres, starting with the perimeter of a circle ($S^1$) and advancing to $S^3$, which is the smallest hypersphere (**Figure 1**), embedded in a 4-ball:

1-sphere $S^1 : x_1^2 + x_2^2$ (circle perimeter),

2-sphere $S^2 : x_1^2 + x_2^2 + x_3^2$ (surface of the common sphere, i.e., a beach ball),

3-sphere $S^3 : x_1^2 + x_2^2 + x_3^{2+} x_4^2$(hypersphere surface), ...,

n-sphere $S^n : x_1^2 + x_2^2 + x_3^{2+} ... + x_n^2$

**SUPPLEMENTARY INFORMATION 2**

**Re-BUT variant for non-antipodal signals.** Let a pair of systems modules (for sake of simplicity, we will talk about a pair of cortical or subcortical neural modules), represented by $A^n$, $B^n$, be n-spheres that interact and then separate. This is an example of the situation described by Schrödinger (1935). Let p1,−p1 be a pair of antipodal points on $A^n$ and let q1,−q1 be a pair of antipodal points on $B^n$. Also let f, g be continuous functions and, using BUT, assume that:

$f : A^n \rightarrow R^n$, such that f(p1) = f(−p1), and

$g : B^n \rightarrow R^n$, such that g(q1) = g(−q1).

From BUT, the signal value f(p1) from $A^n$ can be used to make observations about $B^n$, provided f(p1) = g(−q1). This situation is analogous to the one described by Schrödinger in his paper on separated systems (Schrödinger 1935). That is, we look for an affine connexion (affinity) between modules represented by their feature vectors and by a sameness described by continuous functions on the modules. Using BUT, if antipodal points on $A^n$, $B^n$ emit a common a signal value, then an affine connexion between the physical entities is established. However, this situation is less than satisfactory, since it is not apparent how to make the connexion between the $n$-spheres $A^n$, $B^n$ based on the fact that both n-spheres have pairs of antipodal points that map to the same vector in $R^n$.

Schrödinger was dissatisfied with attempting to find a connexion between vectors using a principle from Euclidean geometry, namely, two vectors are equal when their components are equal. He thought that it is more satisfactory to establish a connexion between physical entities using the notion of continuous transfer (of energy) between points on a curve finitely apart from each other (Schrödinger 1944). But this alternative is still not satisfactory, because curvature is described using rather complicated tensors such as the Riemann-Christoffel curvature tensor, which has 96 components, and the Weyl's metrical tensor, which has 40 components. An alternative to Schrödinger's approaches to establishing affinities between physical entities (neural modules) is obtainable by introducing a region-based form of BUT (denoted by ReBUT). To use our knowledge of one n-sphere to gain knowledge about another d-sphere with d ≤ n, we consider separated regions instead of antipodal points on an n-sphere.

The terminology we use is explained, next. **Region**. A "region" (denoted by *Re*) on an $n$-sphere (i.e, a neural module) is a set containing one or more points. A singleton region is the same as point. Each region is a set of modules. Each module has only feature, namely, its location identified by a vector. By contrast, such a region Re (into a module) not only has location represented by its centroid (center of mass) but also region features such as diameter, area, entropy, shape, porous (shape with holes, *i.e*., some of the points in Re have gaps between them), average signal value and so on.

**Family of regions $2^{Sn}$**. The collection (family) of all subsets on the n-sphere $S^n$ is denoted by $2^{Sn}$. **Regions Re1, Re2 $\in 2^{Sn}$**. Regions Re1, Re2 are members in $2^{Sn}$. **Feature value probe $\varphi$ : Re → R**. A feature value probe is a real-valued function that extracts a feature value from a region. **Region feature k-vector**. Let $\varphi_1, \varphi_2, \ldots, \varphi_k$ be $k$ region feature value probes. A feature value vector for a region on an n-sphere has the following form:

$(\varphi_1(Re), \varphi_2(Re), \ldots, \varphi_k(Re))$, called region feature vector.

**Theorem 1. ReBUT**. Assume that the regions on an n-sphere have n features. If $f : 2^{Sn} \rightarrow R^n$ is a continuous function, then f(Re1) = f(Re2)$R^n$ for at least one pair of regions Re1,Re2 $\in 2^{Sn}$ (Peters 2016). (**Figure B**).

**Proof**. The function f(Re1) is a feature vector in $R^n$ for region Re1 $\in 2^{Sn}$. Given region Re1 $\in 2^{Sn}$ on an n sphere, we know, from the symmetry of the latter, that there are many possible regions Re2 with feature vectors that match the feature vector of Re1. Hence, f(Re1) = f(Re2) for some Re2 $\in 2^{Sn}$.

For example, let each each region on a 2-sphere (a brain module) have two features, namely, area and diameter. Since there are many regions on a 2-sphere with the same area and diameter, from ReBUT, we know that f(Re1) = f(Re2)$R^2$ for at least one pair of regions Re1,Re2 $\in S^2$. This means that the antipodal points restriction of the "classical" BUT is no longer needed, when we consider regions on an n-sphere. In fact, we can now consider neural regions that are either adjacent or separated (far apart). And ReBUT applies, provided the feature vector for a region on an n-sphere (a nervous module) has n feature values.

**Observations Via ReBUT for symmetrical physical entities (two separated cortical modules)**. In touch with Schrödinger's Programme of Observations, we want to use our knowledge on physical entity A (a cortical module) to gain knowledge about another physical entity B (another cortical module) that is separated from A. We do this using ReBUT. Let $S_1^n$, $S_2^n$ be a pair of n-spheres that represent a pair of separated cortical modules. The regions on the surface of an n-sphere are comparable to mailboxes, each one containing information about some part of a module. All that we know about an n-sphere is summarized by the collection of feature vectors that describe
the regions on an n-sphere.

Apply ReBUT to the pair of n-spheres $S_1^n$, $S_2^n$. Let $f : S_1^n \to \mathbb{R}^n$ be a continuous function on $S_1^n$and let $g : S_2^n \to \mathbb{R}^n$ be a continuous function on $S_2^n$. Let $Re_1^1$, $Re_1^2$ be a pair of regions on $S_1^n$ and let $Re_2^1$, $Re_2^2$ be a pair of regions on $S_2^n$ that satisfy ReBUT. That is, from ReBUT, we obtain f($Re_1^1$) = f($Re_1^2$) and f($Re_2^1$) = f($Re_2^2$) on the separated n-spheres. Then the pair of n-spheres$S_1^n$, $S_2^n$ resemble each other (similarity between the physical entities is established, provided f($Re_1^1$) = f($Re_2^1$)).

**Observations Via ReBUT for asymmetrical physical entities (a cortical and a subcortical module)**. Let $S^n$, $S^d$, d ≤ n be n-sphere and d-sphere, respectively, that represent a pair of physical entities (in this case, a cortical and subcortical nervous module). Assume that each region on $S^n$ is described by a feature vector with n-components and assume that each region on $S^d$ is described by a feature vector with d-components with n ≠ d. All that we know about the n-sphere (the cortical module) is summarized by the collection of feature vectors (each with n-components) that describe the regions on an n-sphere. By contrast, all that we know about the d-sphere (the subcortical module) is summarized by the collection of feature vectors (each with d-components) that describe the regions on an d-sphere.
Let Re1,Re2 on $S^n$ that satisfy ReBUT and let Re3,Re4 on $S^d$ that satisfy ReBUT. This means that f(Re1) = f(Re2) on $R^n$ and g(Re3) = g(Re4) on $R^d$. Hence, the feature vectors f(Re1), g(Re3) are not comparable, since each has a different number of components. Even so, we can compare f(Re1), g(Re3) componentwise. Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be n region feature value probes. A feature value vector for a region on the n-sphere $S^n$ has the following form:
$\Phi$(Re1) = ($\varphi_1$(Re1), $\varphi_2$ (Re1), . . . , $\varphi_n$ (Re1)), i.e, a $S^n$ region Re1 feature vector.
Further, let $\varphi_1, \varphi_2, \ldots, \varphi_d$ be d region feature value probes, n ≠d. A feature value vector for a region on the n-sphere $S^n$ has the following form:
$\Phi$(Re3) = ($\varphi_1$(Re3), $\varphi_2$ (Re3), . . . , $\varphi_d$ (Re3)), i.e a $S^d$ region Re3 feature vector.
The continuation of the Schrödinger's Programme of Observations via ReBUT for asymmetrical physical entities (i.e., cortical and subcortical modules represented by hyperspheres with different dimensions), is made possible by the fact that *d* region features are the same for both the modules. The pair of hyperspheres' feature vectors f(Re1), g(Re3) are descriptions of similar regions, provided $S^n$, $S^d$ each contains at least one region Re1 on $S^n$ and at least one region on $S^d$ with feature vectors $\Phi$ (Re1), $\Phi$ (Re3) with up to n−d matching components. In other words, separated neural modules, represented by hyperspheres with different dimensions, have feature vectors that are componentwise comparable. The separated modules are similar, provided there is a feature vector f(Re1) for a region Re1 on a hypersphere $S^n$ which has *d* components with values that match the values of the corresponding components in the feature vector f(Re1) for a region Re3 on a hypersphere $S^d$. For example, we may evaluate the case of similarity between nural regions on hyperspheres with different dimensions (a brain surface and a thalamus surface). Let $\varphi_1$, $\varphi_2$, $\varphi_3$ be three region feature value probes for region features diameter, area, shape, respectively, on a 3-sphere (a cortical module). And let $\varphi_1$, $\varphi_2$, be two region feature value probes for region features diameter, area, respectively, on a 2-sphere (for example, a thalamic module). Let Re1,Re2 be regions on a hypersphere $S^2$ that satisfy ReBUT and let Re3,Re4 be regions on a hypersphere $S^1$ that satisfy ReBUT. If $\Phi$(R1) = ($\varphi_1$(R3), $\varphi_1$ (R2) = $\varphi_1$ (R4) (matching components), then the hypersphere $S^2$ is similar to $S^1$. This tells us that there are similarities between the separated nervous structures represented by the hyperspheres $S^2$ and $S^1$.

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