The spatial elliptical movement in a non-inertial frame of reference

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In this paper the spatial two-body problem in a Newtonian non-inertial frame of reference is analyzed. The Lagrange's equations of elliptical movement are established and applied to estimate the advance of perihelion in this case. Finally the theoretical results are compared with the observational results.

Introduction

The theory of non-inertial frames of reference, [1], is a recently proposed Newtonian theory which is attempting to compete general relativity theory of gravitation. Moreover, it seems to capitalize some local academic recognition and it seems , unless for a few, a very reliable line of research, [2]. In two previous papers, [3,4], the author of the present one showed some results, with reference at planar movement, which contradict the statements of this proposed theory. This paper is inscribed in the same argumentation and try to develop a correct modality of thinking, or unless a point of view much closer to reality.

Lagrange's equations

In order to this purpose let we first consider the spatial movement. Schematically it can be represented as Fig.1 does.

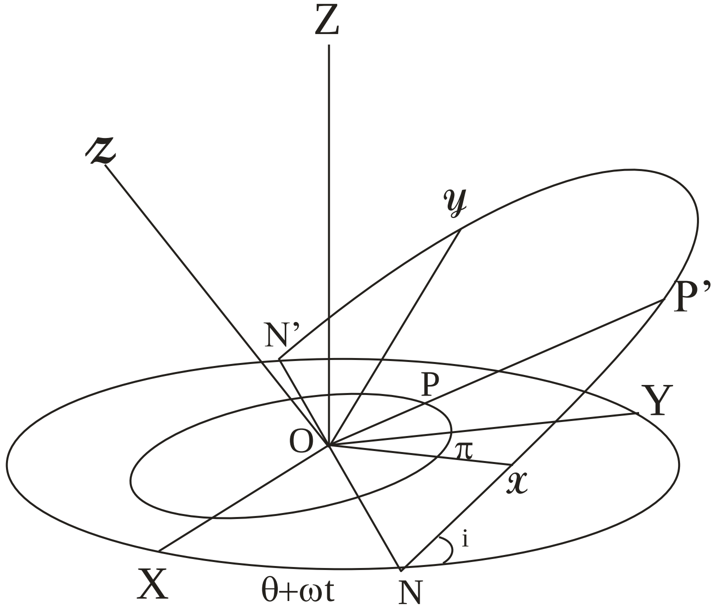

Fig.1

Where:

$XN + Nx = \overline{\omega}$ -perihelion longitude

$Nx = \overline{\omega} - \theta'$

$\theta' = \theta + \omega t = XN$ -node longitude

i - the slope

τ - perihelion passing time
are the movement parameters.
According to [5] the kinetic energy to mass ratio in polar coordinates is:

$$2T = r'^2 + r^2\alpha'^2 + r^2\cos^2\alpha\psi'^2$$

where into α is considered the influence of rotating non-inertial frame of reference too . By introducing the following canonical variables :

$$\frac{\partial T}{\partial r'} = r' = R \quad \frac{\partial T}{\partial \alpha'} = r^2\alpha' = A \quad \frac{\partial T}{\partial \psi'} = r^2\cos^2\alpha\psi' = \Psi$$

the Hamiltonian function will be:

$$T + U = \frac{R^2}{2} + \frac{A^2}{2r^2} + \frac{\Psi^2}{2r^2\cos^2\alpha} - \frac{\mu}{r}$$

With this function we can find then the Hamilton-Jacobi equation:

$$\frac{1}{2}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial S}{\partial \alpha}\right)^2 + \frac{1}{2r^2\cos^2\alpha}\left(\frac{\partial S}{\partial \psi}\right)^2 - \frac{\mu}{r} = h$$

where S is the solution.

The term $\frac{1}{2r^2\cos^2\alpha}\left(\frac{\partial S}{\partial \psi}\right)^2$ is depending on two variables. Thus to find an appropriate solution is difficult. Therefore we have to neglect it and search a solution as a sum between a term depending on ω and the other depending on function:

$$\left(\frac{\partial S_1}{\partial r}\right)^2 = -\frac{G^2}{r^2} + \frac{2\mu}{r} + 2h$$

The solution will be then:

$$S = G\alpha + \int \pm\sqrt{-\frac{G^2}{r^2} + \frac{2\mu}{r} + 2h}dr$$

with G a constant. Unknown function S must satisfy the relations:

$$\frac{\partial S}{\partial h} = t - \tau, \quad \frac{\partial S}{\partial G} = g \quad \frac{\partial S}{\partial \theta'} = -\Theta \tag{1}$$

which are of course much easier to solve. The last one is simpler and it can be written as:

$$G\frac{\partial \alpha}{\partial \theta'} = -\Theta$$

where the derivative is a function of slope angle, in an infinitesimal aproach:

$$d\alpha = -d\theta' \cos i$$

We have then:

$$\frac{\partial \alpha}{\partial \theta'} = -\cos i \ \text{ and } G\cos i = \Theta$$

Let now consider the equation $\frac{\partial S}{\partial G} = g$. By integration we obtain:

$$\alpha - g = \int \frac{\frac{dr}{r^2}}{\pm\sqrt{-\frac{1}{r^2} + \frac{2\mu}{G^2 r} + \frac{2h}{G^2}}} \tag{2}$$

By making the derivative of equation (2) we obtain:

$$\left(\frac{d\frac{1}{r}}{d\alpha}\right)^2 = -\frac{1}{r^2} + \frac{2\mu}{G^2 r} + \frac{2h}{G^2}$$

which represent the classical equation of planar elliptical movement. The study of second degree right side polynomial give the well known relations between roots and coefficients, from celestial mechanics:

$$\frac{\mu}{-h} = 2a \quad h = -\frac{\mu}{2a}$$

$$\frac{G^2}{-2h} = a^2\left(1-e^2\right) \quad G = \sqrt{\mu a\left(1-e^2\right)}$$

Integral (2) solution it results after the change of variable:

$$\frac{1}{r} = \frac{1+e\cos\varphi}{a\left(1-e^2\right)}$$

and it is:

$$\alpha - g = \varphi$$

By introducing this result in previous equation it follows immediately the parametric equation of ellipse:

$$\frac{1}{r} = \frac{1+e\cos(\alpha-g)}{a\left(1-e^2\right)}$$

Taking into account the origin conditions, in particular we may have:

$$g = \varpi - \theta',$$

so we have found another solution.

To solve the first equation (1):

$$\int_{a(1-e)}^{r} \frac{rdr}{\pm\sqrt{-G^2 + 2\mu r + 2hr^2}} = t - \tau$$

it is necessary to operate another change of variable:

$$r = a(1-e\cos u)$$

After we integrate the expression:

$$\int_0^u \pm\sqrt{\frac{a^3}{\mu}}(1-e\cos u)du = t - \tau$$

we find Kepler's equation:

$$u - e\sin u = n(t-\tau)$$

According to the previous calculations we have found the following constants:

$$\begin{cases} h = -\frac{\mu}{2a} & G = \sqrt{\mu a\left(1-e^2\right)} & \Theta = G\cos i \\ t - \tau & g = \varpi - \theta' & \theta' \end{cases}$$

which in fact are a complete set of canonical elements:

$$\alpha_1 = -h = \frac{\mu}{2a} \quad \alpha_2 = G = \sqrt{\mu a\left(1-e^2\right)} \quad \alpha_3 = \Theta = G\cos i$$

$\beta_1 = \tau \quad \beta_2 = g = \varpi - \theta^{'} \quad \beta_3 = \theta'$

and the solutions obtained through Lagrange's variation of constants method from a general canonical equations system:

$$\frac{d\alpha_i}{dt} = \frac{\partial F}{\partial \beta_i} \text{ and } \frac{d\beta_i}{dt} = \frac{\partial F}{\partial \alpha_i}, \text{ i=1,2,3} \tag{3}$$

The general function $F = F(\alpha_i, \beta_i, t)$ is the derivative of a potential function R.

Thus we can write the system (3), after we have been operated the required simplifications , as:

$$\frac{da}{dt} = -\frac{2}{n^2}\frac{\partial R}{\partial \tau} \tag{a}$$

$$\frac{de}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \varpi} - \frac{1-e^2}{n^2 a^2 e}\frac{\partial R}{\partial \tau} \tag{b}$$

$$\frac{di}{dt} = -\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \theta'} - \frac{\tan 1/2}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \overline{\varpi}} \tag{c}$$

$$\frac{d\theta'}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \tag{d}$$

$$\frac{d\varpi}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan 1/2}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} \tag{e}$$

$$\frac{d\tau}{dt} = \frac{2}{n^2 a}\frac{\partial R}{\partial a} + \frac{1-e^2}{n^2 a^2 e}\frac{\partial R}{\partial e} \tag{f}$$

where $n = \sqrt{\frac{\mu}{a^3}}$ is a notation.

The system of equations (a-f) represent the Lagrange's perturbations equations for a non-inertial frame of reference spatial two body problem. The equations (c) and (d) are the difference between inertial and non-inertial cases. As we can observe from above, the parameter $\theta' = \theta + \omega t$ contains the case of inertial movement as a particular case when $\omega = 0$, according to [5].

The advance of perihelion

Finally, using the above equations let we estimate now the advance of perihelion as an application. We have:

$$\theta' = \theta + \omega t \quad F = -\frac{\mu}{r^3} + \frac{C^*}{r^4} \quad R = \frac{\mu}{2r^2} - \frac{C^*}{4r^3}$$

where F and R are the gravitational force and de gravitational potential which appear in non-inertial frames of reference, as it result from [4].

From equation (e), taking into account (d), we obtain the intermediate result:

$$\frac{d\varpi}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan i/2}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} = \frac{\sqrt{1-e^2}}{na^2 e}F\frac{\partial r}{\partial e} + \frac{\tan i/2}{\sin i\left(\dot{\theta} + \omega\right)}$$

The derivative $\partial r / \partial e$ is established from expression $r = a(1 - e\cos u)$:

$$\frac{\partial r}{\partial e}=a\left(-\cos u+e\sin u\frac{\partial u}{\partial e}\right)$$

and from Kepler's equation we can write now:

$$u-e\sin u=u(t-\tau)\quad (u-e\cos u)\frac{\partial u}{\partial e}=\sin u$$

result which replaced in the previous expression it gives:

$$\frac{\partial r}{\partial e}=a\left(-\cos u+\frac{e\sin^2 u}{1-e\cos u}\right)=a\frac{-\cos u+e}{1-e\cos u}$$

Equation (e) become therefore:

$$\frac{\partial\varpi}{\partial t}=\frac{\sqrt{1-e^2}}{na^2e}\left[\frac{-\mu}{r^3}+\frac{C^*}{r^4}\right]a\frac{-\cos u+e}{1-e\cos u}+\frac{\tan i/2}{\sin i(\dot{\theta}+\omega)}$$

Replacing the variable u with t it follows:

$$\frac{\partial\varpi}{\partial u}=\frac{\sqrt{1-e^2}}{na^2e}\left[\frac{-\mu}{r^3}+\frac{C^*}{r^4}\right]a\frac{-\cos u+e}{1-e\cos u}\frac{1-e\cos u}{n}+\frac{\tan i/2}{\sin i(\dot{\theta}+\omega)}\frac{1-e\cos u}{n}$$

and:

$$\frac{\partial\varpi}{\partial u}=\frac{\sqrt{1-e^2}}{ae}\frac{\cos u-e}{(1-e\cos u)^3}+\frac{\sqrt{1-e^2}}{nae}\frac{\cos u-e}{n}\frac{C^*}{r^4}+\frac{\tan i/2}{\sin i(\dot{\theta}+\omega)}-\frac{1-e\cos u}{n}$$

Considering then:

$$\frac{r}{a}=1-e\cos u$$

$$\frac{C^*}{r^4}=\frac{C_1(C_1+2C_0)}{r^4}=\frac{\omega^2r^4+2\omega\dot{\theta}r^4}{r^4}$$

the advance of perihelion can be written as:

$$\delta\overline{\varpi}=I^1+I^2+I^3$$

where the three I are the three resulted integrals corresponding the right hand terms.The first one is the most complicated and it can be solved as it follows:

$$I^1=\int_0^{2\pi}\frac{\sqrt{1-e^2}}{ae}\frac{\cos u-e}{(1-e\cos u)^3}du=\int_0^{2\pi}\frac{\sqrt{1-e^2}}{ae}\frac{-(1-e\cos u)+1-e^2}{(1-e\cos u)^3}du$$

Or:

$$I^1=\frac{\sqrt{1-e^2}}{ae^2}\left[-I_2+(1-e^2)I_3\right]$$

The right hand integrals can be calculated from general formula:

$$I_n=\int_0^{2\pi}\frac{du}{(1-e\cos u)^n}=\int_{-\pi}^{\pi}\frac{du}{(1-e\cos u)^n}$$

making the changes of variable:

$$\tan\frac{u}{2}=z\quad u=2\arctan z\Rightarrow$$

$$z=\sqrt{\frac{1-e}{1+e}}\tan\frac{\varphi}{2}$$

Thus:

$$I_n = \frac{1}{\left(1-e^2\right)^{n-\frac{1}{2}}} \int_0^{2\pi} \left(1+e\cos\varphi\right)^{n-1} d\varphi$$

Consequently:

$$I_1 = \frac{2\pi}{\sqrt{1-e^2}} \quad I_2 = \frac{2\pi}{\left(1-e^2\right)^{3/2}} \quad I_3 = \frac{\left(2+e^2\right)\pi}{\left(1-e\right)^{5/2}}$$

And:

$$I^1 = \frac{\pi}{a\left(1-e^2\right)}$$

The other two integrals are solved more straightforward:

$$I^2 = \int_0^{2\pi} \frac{\sqrt{1-e^2}}{n^2 ae} \left(-\cos u + e\right)\left[\omega\left(\omega + 2\dot{\theta}\right)\right] du = \frac{a^2\sqrt{1-e^2}}{\mu} 2\pi\left[\omega\left(\omega+2\dot{\theta}\right)\right]$$

And:

$$I^3 = \int_0^{2\pi} \frac{\tan i/2}{\sin i\left(\dot{\theta}+\omega\right)} \frac{1-e\cos u}{n} du = 2\pi a\sqrt{\frac{a}{\mu}} \frac{\tan i/2}{\sin i\left(\dot{\theta}+\omega\right)}$$

Finally the expected result is:

$$\delta\varpi = \frac{\pi}{a\left(1-e^2\right)} + \frac{2\pi a^2\sqrt{1-e^2}}{\mu}\left[\omega\left(\omega+2\dot{\theta}\right)\right] + 2\pi a\sqrt{\frac{a}{\mu}} \frac{1}{\dot{\theta}+\omega} \frac{\tan i/2}{\sin i} \tag{4}$$

Comparison with observational data

Considering $\mu = KM$, with $K = 6.67\cdot 10^{-11}\ N\cdot m^2/kg^2$ and $M = 1.989\cdot 10^{30} kg$, the slow rotation of our galaxy $\omega = 0.0068$ arcsec/year and the following data from Astronomical Data Center :

planets	a($\cdot 10^9$ m)	e	Rev/cent	Slope (degrees)	Observed Advance (arcsec/ century)	GR Advance (arcsec/ century)
Mercury	57.9	0.206	414.9378	7.004	43.1+/-0.5	43.09
Venus	108.2	0.0070	162.6016	3.394	8.4+/-4.8	8.78
Earth	149.6	0.0170	100	0	5.0+/-1.2	3.89

For Mercury, taking into account (4), we find a very disturbing value for advance of perihelion. $1.95\cdot 10^{18}$ arcsec/century. This result is due to the middle right side term. By observing the fact that all data are the same order of magnitude we can conclude without doubt that something is wrong with this non-inertial frame of reference theoretical model.

Conclusions

The observational results don't fit at all theoretical results. These results corroborated with those regarding to planar movement, [3,4], give rise a great doubt concerning the correctness of non-inertial frames of reference theory. In all cases the theoretical results are too far from reality and therefore there is no question about the truthfulness of general relativity.

References

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