Proof of Beal’s conjecture

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Abstract

Equation constituting the Beal conjecture is rearranged and squared, then rearranged again and raised to power 4. The result, standing as an equivalent having the same property, is emerging as a singular primitive Pythagorean equation with no solution. So, the conjecture is proved. General line of proving the Pythagorean equation is observed as a moving spirit.

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Intoduction

Originally, the Beal conjecture [1] states that if

$$A^x + B^y = C^z \tag{1.1}$$

where *A*, *B*, *C*, *x*, *y* and *z* are positive integers and *x*, *y* and *z* are all greater than 2, then *A*, *B* and *C* must have a common prime factor;

Identically [1], [2], the conjecture states that there are no solutions to the equation

$$a^x + b^y = c^z \tag{1.2}$$

in positive integers *a*, *b*, *c*, *x*, *y*, *z* with all of *x*, *y*, *z* being greater than 2 and with *a*, *b*, and *c* being pairwise coprime.

The point can be seen as a generalization of Pythagorean issue ($x = y = z = 2$) embracing the Fermat last theorem as well.

Unmodern mathematical tools are used to prove the theorem (hitherto conjecture). General line of proving the Pythagorean triple [3] is observed as a moving spirit. Generally, original (1.1) and alike (1.2) statements concerning the Bealian form refer to non-primitive and primitive triple, respectively. So, trivial or non-trivial solutions are to be under consideration. More, there is an unexplored potential in consecutive steps known from ancient times, exploided barely now.

The alternative statements concerning the Bealian form (1.2), point to two sub-forms indicated in the proof below. The known identities, like $7^3 + 13^2 = 2^9$ and $3^5 + 11^4 = 122^2$ and else $17^7 + 76271^3 = 21063928^2$ are in the bottom of a forethought about.

Proving the conjecture, we refere to divisibiity and parity of integers [3, chap. 2]. In particular, there is

Lemma 1. If gcd(*u*,*v*)=1, then gcd(*u*+*v*,*u*-*v*)=1 or 2; definitely it is 1 provided that *u* and *v* are of different parity and it is 2 otherwise.

This is almost to the letter citation from elementary theory providing also a formal proof [3, pp. 69;353]. □

The lemma admits conversion:

Lemma 2. If gcd(*u*+*v*,*u*-*v*)=1, then gcd(*u*,*v*)=1 or 2, and it is 1 when *u*+*v* and *u*-*v* are of different parity or it is 2 otherwise.

For most elementary demonstration in the subject suppose *u* and *v* are integers. Then gcd(*u*,*v*) divides *u* and *v*, and hence divides *u*+*v* and *u*-*v*, and thence divides gcd(*u*+*v*,*u*-*v*). So, if gcd(*u*+*v*,*u*-*v*)=1 or 2, then gcd(*u*,*v*)=1 or 2, and vice versa. □

The Lemmas leads to conclusions [3, p.353].

Corollary 1. If gcd(u,v)=1 and also gcd(u+v,u-v)=1, then u and v are of opposite parity and more there is an alternative for (u+v)/2 and (u-v)/2 to be either of different parity or odd, one and the other.

Corollary 2. If u and v are odd and gcd(u,v)=1, then gcd(u+v,u-v)=2 and more (u+v)/2 and (u-v)/2 are of different parity.

The proof is arranged like the pages of elementary theory [3, pp.76-77] concerning the Pythagorean triples, preserving moving spirit above all.

The Beal theorem

The 3-tuple (a,b,c) of positive integers with the property that a,b and c have no positive common divisor other than unity will be referred to as primitive.

In the Pythagorean spirit, the theorem is slightly restated.

The Beal theorem. If (a,b,c) is a primitive 3-tuple, where x, y and z are positive integers greater than 2, then there are no solutions in positive integers a, b, c, x, y, z to the equation (1.2).

Proof. If (a,b,c) is a primitive 3-tuple, then a, b and c are coprime in pairs and so are their powers. If a and b are even then c is even, a contradiction. If a and b are odd then c is even. If c is odd then one of a and b is even and the other is odd. If c is even then a and b are both either odd or even. Hence, there are at most two potentially possible sub-forms of the Bealian (1.2).

The firth case is: a and b must be of different parity and c odd. Without loss of generality, let a be even and b be odd.

The second case is: a and b must be odd and c even. Without loss of generality, let b>a..

The rearranged sub-forms are

$$a_e^{\,x} = c_o^{\,z} - b_o^{\,y} \qquad (2.1),$$

or

$$c_e^{\,z} = a_o^{\,x} + b_o^{\,y} \qquad (2.2)$$

where the subscripts e or o refer to parity of a positive integer involved. For instance, when a refers to a positive integer then a_e and a_o refer to even and odd parity of a, respectively.

Supposition. In addition, for a proof by contradiction suppose the equation (1.2) is satisfied.

The rearranged equivalent equations (2) are squared

$$a_e^{2x} = (c_o^{\;z} - b_o^{\;y})^2 \text{, or } c_e^{2z} = (a_o^{\;x} + b_o^{\;y})^2$$

then rearranged again and raised to power *k*,

$$(a_o^x c_o^z)^k = [(b_o^{2y} + c_o^{2z} - a_e^{2x})/2]^k \quad (3.1)$$

or

$$(a_o^x b_o^y)^k = [(a_o^{2x} + b_o^{2y} - c_e^{2z})/2]^k \quad (3.2)$$

where

$$a_o^x c_o^z = w \text{ or } a_o^x b_o^y = w, \text{ respectively}$$

and

$$w = a^x c^z \text{ where } a^x \text{ and } c^z \text{ are odd} \quad (4.1)$$

or

$$w = a^x b^y \text{ where } a^x \text{ and } b^y \text{ are odd} \quad (4.2).$$

Since c-a and c+a must be even, or b-a and b+a must be even, respectively, let

$$(c^{kz} + a^{kx}) = 2s \text{ and } (c^{kz} - a^{kx}) = 2t \quad (5.1)$$

or

$$(b^{ky} + a^{kx}) = 2s \text{ and } (b^{ky} - a^{kx}) = 2t \quad (5.2)$$

then

$$w^k = s^2 - t^2 \quad (6)$$

where s and t are relatively prime numbers, one of which is even and the other odd (by the Corollaries 1 and 2 above).

Specifying the power k as k=4 yields

$$t^2 + (w^2)^2 = s^2 \quad (7)$$

to be considered as a singular primitive Pythagorean triple. For, s has to be odd and t even, while w is odd as a product of odd integers. It follows [3] that t=2uv, w^2=u^2-v^2, s=u^2+v^2 where u and v are relatively prime numbers of opposite parity. Hence, u-v and u+v must be of different parity, in the virtue of Corollary 1 above. Since w^2=(u-v)(u+v) and gcd(u-v, u+v)=1, u-v and u+v must be perfect squares, say u-v=f^2 and u+v=g^2, where one of f and g is even and the other is odd. It folows that eqution (1.2) is equivalent to

$$f^4 + g^4 = 2(u^2 + v^2) \quad (8),$$

when supposed to have the solution.

Left-hand side of the equation (8) is odd as a sum of integers of different parity, whereas the right-hand side is visibly even, a contradiction terminating the proof, for fallacy then lack of solutions. □

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