Symmetric normal matrices and the Marcus-de Oliveira Determinantal Conjecture

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Abstract

We present notes on the Marcus-de Oliveira conjecture. The conjecture concerns the region in the complex plane covered by the determinants of the sums of two normal matrices with prescribed eigenvalues. Call this region Δ_U. Let Δ_O be the restriction of Δ_U to determinants of sums of *symmetric* normal matrices. In this paper, we conjecture that Δ_O has the same boundary as Δ_U. We prove the conjecture for the cases: 1) at least one of the two matrices has just one eigenvalue, 2) at least one of the two matrices has distinct eigenvalues. The implication of this theorem is that proving the Marcus-de Oliveira conjecture for symmetric normal matrices would prove it for the general case. This paper builds on work in [1].

Keywords:
determinantal conjecture; Marcus-de Oliveira; determinants; normal matrices; convex-hull

MSC:
15A15

1 Introduction

Marcus [4] and de Oliveira [2] made the following conjecture. Given two normal matrices A and B with prescribed eigenvalues $a_1, a_2 ... a_n$ and $b_1, b_2 ... b_n$ respectively, $det(A + B)$ lies within the region:

$co\{ \prod(a_i + b_{\sigma(i)}) \}$

where $\sigma \in S_n$. co denotes the convex hull of the $n!$ points in the complex plane.

As described in [1], the problem can be restated as follows:

Given two diagonal matrices, $A_0 = diag(a_1, a_2...a_n)$ and $B_0 = diag(b_1, b_2...b_n)$, let:

$\Delta_U = \{ det(A_0 + UB_0U^*) : U \in U(n) \}$, where $U(n)$ is the set of $n \times n$ unitary matrices. Then we can write the conjecture as:

(D.1)
Marcus-de Oliveira conjecture
$\Delta_U \in co\{ \prod(a_i + b_{\sigma(i)}) \}$.

We can write a similar conjecture regarding orthogonal matrices:

(D.2)
Restricted Marcus-de Oliveira conjecture
$\Delta_O \in co\{ \prod(a_i + b_{\sigma(i)}) \}$.

(E.1)
let $R(U) = det(A_0 + UB_0U^*)$.

Then the points forming the convex hull are at $R(P_0), R(P_1)...R(P_{n!-1})$, where the P's are the $n \times n$ permutation matrices. We will refer to these as corner points from now on.

Figure 1, (F.1) illustrates an example done with SageMath Cloud[3]. The black dots represent $R(U_i)$ where U_i's are random unitary matrices.

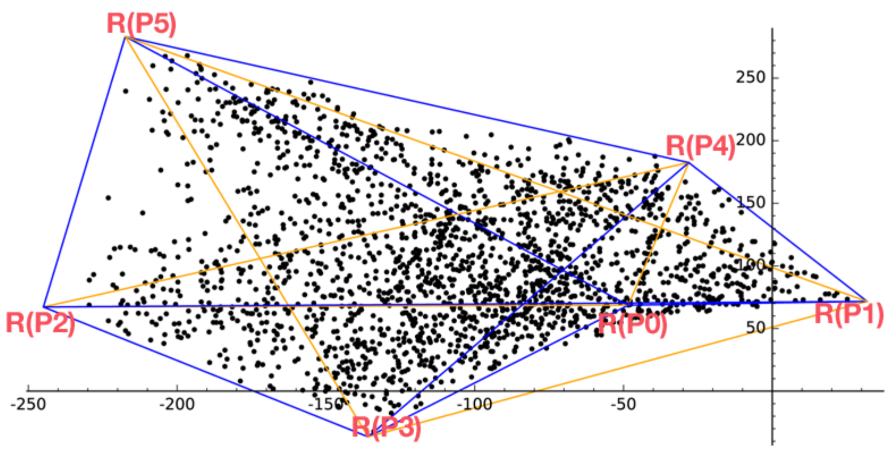
In this example:
$n = 3, A_0 = diag(5 + 7i, 2.2, 0.5), B_0 = diag(12i, 0.7 - i, 1.3i)$

$$P_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(F.1)

Figure 1:



Let

$\Delta_O = \{det(A_0 + UB_0U^*) : O \in O(n)\}$, where $O(n)$ is the set of $n \times n$ real orthogonal matrices.

As proven in [5], p.207, theorem 4.47, a matrix is normal and symmetric if and only if it is diagonalizable by a real orthogonal matrix.

Therefore Δ_O is the set of determinants of sums of normal symmetric matrices with prescribed eigenvalues.

2 Main Ideas

Conjecture:

(C.1)
Symmetric Sufficiency Conjecture (SSC)
Δ_O and Δ_U have the same boundary.

This conjecture is supported by experiments done in SageMath Cloud[3].

Suppose B is a complex number that lies on the boundary of Δ_U. We call B a boundary point of Δ_U.

Suppose U is a matrix such that $R(U) = B$, then U is a boundary matrix of Δ_U. A boundary point may have multiple boundary matrices.

We know that $\Delta_O \subseteq \Delta_U$.

If every boundary point on Δ_U has an orthogonal boundary matrix, this would prove (C.1). We will prove the conjecture true under restricted conditions.

Conjecture:

(C.2)
Restricted Symmetric Sufficiency Conjecture (RSSC)
Δ_O and Δ_U have the same boundary when:
1) At least A_0 or B_0 has only one eigenvalue.
2) At least A_0 or B_0 has distinct eigenvalues.

Under condition 1, we may freely choose B_0 to be the matrix with one eigenvalue. By (E.1), for any unitary matrix $U, R(U) = det(A_0 + UB_0U^*) = det(A_0 + B_0)$. So Δ_U consists of only one point which is itself the border of the region, and can be written as $R(O)$ for any $n \times n$ orthogonal matrix O. So this proves RSSC under condition 1.

We will refer to condition 2 as DEC (distinct eigenvalue condition) throughout the rest of the paper.

The rest of this paper concerns proving RSSC, (C.2), under DEC.

3 Slope of the tangent at the boundary

Suppose we have a smooth function of matrices $U(t)$ going through a boundary matrix of interest, U_0. We alter $U(t)$ while keeping $U(0) = U_0$.

Every unitary matrix can be written as an exponential of a skew-hermitian matrix. So we can write:

$U(t) = U_0 e^{S(t)}$, for a smooth function of skew hermitian matrices $S(t)$, with $S(0) = 0$. So by changing $S(t)$ to all the possible skew-hermitian matrix functions starting at 0, we get all possible $U(t)$ going through U_0.

For small Δt,

$U(t + \Delta t) = U_0 e^{S(t+\Delta t)}$

$U(t + \Delta t) = U_0 e^{S(t)+S'(t)\Delta t}$

$U(t + \Delta t) = U_0 e^{S(t)} e^{S'(t)\Delta t}$

(E.2)
$U(t + \Delta t) = U(t) e^{S'(t)\Delta t}$

(E.3)
let $B(t) = U(t) B_0 U^*(t)$

(E.4)
let $C(t) = A_0 + B(t)$

(E.5)
let $F(t) = B(t)C^{-1}(t) - C^{-1}(t)B(t)$

As shown in [1], $F(t)$ only goes to zero at the corner points.

(E.6)
let $R(t) = \det(A_0 + U(t)B_0U^*(t))$

Note that $S'(t)$ is also skew-hermitian.

As proved in [1] using (E.2),(E.3), (E.4), (E.5) and (E.6), when $F(t) \neq 0$:

$R(t + \Delta t) = R(t) + (\Delta t) det(C(t)) tr(S'(t)F(t)) + O((\Delta t)^2)$

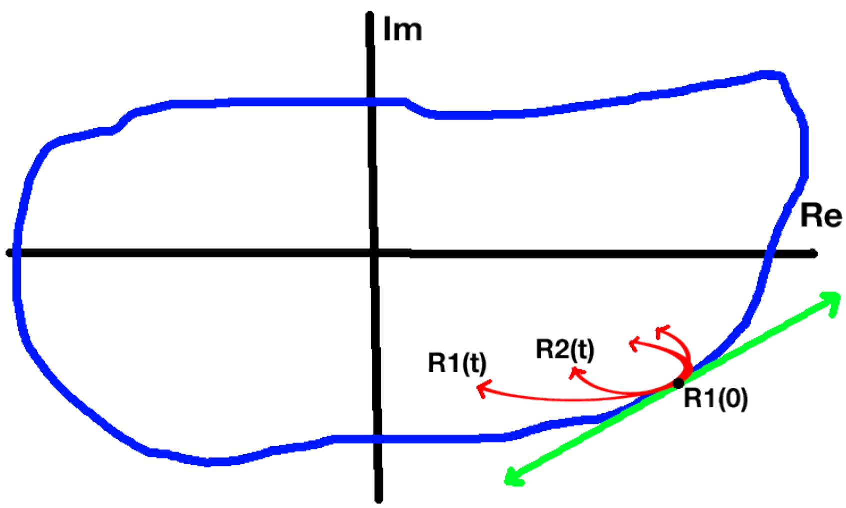
So

(E.7)
$R'(t) = det(C(t)) tr(S'(t)F(t))$

Since U_0 is a boundary matrix, the slope of the tangent to the curve $R(t)$ at $t = 0$ must remain the same regardless of our choice of $S(t)$. This is illustrated in (F.2) where the blue curve indicates the boundary of Δ_U. Curves going through the same boundary point have the same tangent line at the boundary point. This means that whenever $R'(0) \neq 0$, the phase of $R'(0)$ (phase of a complex number) must stay the same. Here we count phases π apart as the same since our concern is with the slope only. The phase of $det(C(0))$ is fixed since it only depends on U_0. So the phase of $tr(S'(0)F(0))$ should remain the same. Note that $S'(0)$ can turn out to be any skew-hermitian matrix. Proof: suppose we choose an arbitrary skew-hermitian matrix, multiply each element of the matrix by t, then we get a smooth function of skew-hermitian matrices $S(t)$ such that $S'(0)$ is the skew-hermitian matrix we initially chose.

(F.2)

Figure 2:



(E.8)
let $Z = S'(0)$.

We now focus solely on the values of the functions at $t = 0$ and discard the use of functions of t. We rewrite (E.3), (E.4), (E.5) as:

(E.9)
let $B = UB_0U^*$

We call this the B-matrix of a unitary matrix U. It is a normal matrix.

(E.10)
let $C = A_0 + B$

(E.11)
let $F = BC^{-1} - C^{-1}B$

We call this the F-matrix of a unitary matrix U.

Given a unitary matrix U, we calculate F according to (E.11). If there exists a constant real number p, $\frac{-\pi}{2} \leq p \leq \frac{\pi}{2}$, such that for any skew-hermitian matrix Z, whenever $tr(ZF) \neq 0$, then the phase of $tr(ZF) = p$, then we say that U satisfies the **Constant Phase Condition (CPC)**.

The paragraph between (E.7) and (E.8) proves the following theorem:

(T.1)
THEOREM 1
Every boundary matrix other than a corner satisfies CPC.

CPC is a necessary condition for a unitary matrix to be a boundary matrix. But it is not a sufficient condition. We may have two unitary matrices giving the same point. eg: $R(U_1) = R(U_2)$. U_1 may satisfy CPC, but U_2 may not. In this case, $R(U_1)$ is not a boundary point therefore U_1 is not a boundary matrix.

The rest of this paper concerns proving the following theorem:

(T.2)
THEOREM 2
Under DEC, given a unitary matrix U that satisfies CPC, there exists an orthogonal matrix O such that $R(O) = R(U)$.

(T.2) along with (T.1) proves that for every boundary point B of Δ_U, there exists an orthogonal matrix O such that $R(O) = B$, under DEC. Thereby proving (C.2).

We need some other tools first before proving (T.2).

4 F-matrix and phase-shift theorems

(T.3)
THEOREM 3
Given a unitary matrix U that satisfies CPC, its F-matrix is zero-diagonal.

Proof:

We require some simple matrix algebra theorems, which we won't prove:

1. If A and B commute, then A and B^{-1} commute.
2. If A commutes with B and A commutes with C, A commutes with B+C
3. Every matrix commutes with itself.
4. Every matrix commutes with the identity.
5. If D is a diagonal matrix, then DAD^{-1} has the same diagonal elements as A.

Now we can use (E.9),(E.10) and (E.11) to prove that F has zero diagonal.

$C = A_0 + B$
$C = A_0(1 + A_0^{-1}B)$
$C^{-1} = (1 + A_0^{-1}B)^{-1}A_0^{-1}$
$A_0^{-1}B$ commutes with $I + A_0^{-1}B$ (using 2,3,4)
$B^{-1}A_0$ commutes with $I + A_0^{-1}B$ (using 1)
$B^{-1}A_0(I + A_0^{-1}B) = (I + A_0^{-1}B)B^{-1}A_0$ (definition of commutativity)
$B^{-1}C = (I + A_0^{-1}B)B^{-1}A_0$ (simplification)
$C^{-1}B = A_0^{-1}B(I + A_0^{-1}B)^{-1}$ (inverses)
$C^{-1}B = A_0^{-1}BC^{-1}A_0$ (substitution)
$C^{-1}B$ and BC^{-1} have the same diagonal (using 5)

$F = BC^{-1} - C^{-1}B$ has zero-diagonal.

(T.4)

THEOREM 4

Given a unitary matrix U that satisfies CPC, with F-matrix F, $\|F_{mn}\| = \|F_{nm}\|$.

Proof:

We're given a matrix U that satisfies CPC, with F-matrix F.

The phase of all non-zero $tr(ZF)$ for all skew hermitian matrices Z, is the same.

Since F is zero-diagonal, the trace is unaffected by the diagonal elements of Z. We will look at the set of skew-hermitian matrices Z with zero diagonal.

This set forms a vector space over the reals. For example, for n=3, the following 6 matrices form a basis for the skew-hermitian matrices:

$$Z_{01} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Z_{02} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, Z_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Z_{01,i} = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Z_{02,i} = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, Z_{12,i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

Note the commas do not indicate tensors. They're just used here to indicate the real vs imaginary component.

Suppose $F_{mn} = F_{mn,r} + iF_{mn,i}$

$tr(Z_{mn}F) = F_{mn} - F_{nm}$

$tr(Z_{mn,i}F) = (F_{mn} + F_{nm})i$

Substitute in for F_{mn}

$tr(Z_{mn}F) = (F_{mn,r} - F_{nm,r}) + i(F_{mn,i} - F_{nm,i})$

$tr(Z_{mn,i}F) = (-F_{mn,i} - F_{nm,i}) + i(F_{mn,r} + F_{nm,r})$

The phase of these two is the same by CPC. So:

$(F_{mn,i} - F_{nm,i})(-F_{mn,i} - F_{nm,i}) = (F_{mn,r} + F_{nm,r})(F_{mn,r} - F_{nm,r})$

We can simplify this to get:

$F_{mn,r}^2 + F_{mn,i}^2 = F_{nm,r}^2 + F_{nm,i}^2$

$\|F_{mn}\| = \|F_{nm}\|$

(T.5)
THEOREM 5
Given a unitary matrix U, with B-matrix B and F-matrix F, and given a unitary diagonal matrix D, let $U_2 = D^*U$. then:
1) $R(U_2) = R(U)$
2) the F-matrix of U_2 is D^*FD.

Proof: (using (E.1))
$R(U_2) = det(A_0 + D^*UB_0U^*D)$
$R(U_2) = det(D)det(A_0 + D^*UB_0U^*D)det(D^*)$
$R(U_2) = det(DA_0D^* + UB_0U^*)$
$R(U_2) = det(A_0 + UB_0U^*)$
$R(U_2) = R(U)$

$B = UB_0U^*$

$B_2 = U_2B_0U_2^*$
$B_2 = D^*UB_0U^*D$
$B_2 = D^*BD$

$C = A_0 + B$
$C = A_0(I + A_0^{-1}B)$
$C^{-1} = (I + A_0^{-1}B)^{-1}A_0^{-1}$

$C_2^{-1} = (I + A_0^{-1}B_2)^{-1}A_0^{-1}$
$C_2^{-1} = (I + D^*A_0^{-1}BD)^{-1}A_0^{-1}$
$C_2^{-1} = (D^*(I + A_0^{-1}B)D)^{-1}A_0^{-1}$
$C_2^{-1} = D^*(I + A_0^{-1}B)^{-1}A_0^{-1}D$

$C_2^{-1} = D^*CD$

$F = BC^{-1} - C^{-1}B$
$F_2 = B_2C_2^{-1} - C_2^{-1}B_2$
$F_2 = D^*(BC^{-1} - C^{-1}B)D$
$F_2 = D^*FD$

(T.6)
THEOREM 6
For every $n \times n$ complex matrix C, and every real $n \times n$ skew-symmetric matrix P, there is a complex $n \times n$ matrix A such that $A = D^*CD$, $phase(A_{ij}) - phase(A_{ji}) = P_{ij}$ and $A_{ii} = C_{ii}$ where D is an $n \times n$ unitary diagonal phase matrix.

Proof:
The first step is to set the phases of the elements in the first row and column:
$i = 0, 0 \leq j \leq n-1$
$j = 0, 0 \leq i \leq n-1$

Let D_1 be a diagonal matrix of n variable phases (each element is of the form e^{ϕ_i}). We can expand $A = D_1^*CD_1$ and set $phase(A_{ij}) - phase(A_{ji}) = P_{ij}$ for all elements in the region above. This will give us $n-1$ linear equations with n variables. We have 1 free variable so we can always find a diagonal matrix D_1 to accomplish this.

We repeat the step above but for the region:
$i = 1, 1 \leq j \leq n-1$
$j = 1, 1 \leq i \leq n-1$

We leave the first row and column unchanged. Let D_2 be a diagonal matrix with 1 in the first spot, and variables in the other spots. Expanding $A = D_2^*D_1^*CD_1D_2$, we set $phase(A_{ij}) - phase(A_{ji}) = P_{ij}$ for the elements in the above region. This time we have $n-2$ equations with $n-1$ variables. Again we can find a diagonal matrix D_2.

Repeating this process we'll get a diagonal matrix

$D = D_1 D_2 D_3 D_4 ... D_{n-1}$ that sets the phases of all pairs of elements A_{ij} and A_{ji}.

This is a very useful theorem as it allows us to take any matrix, and construct a similar one, with the difference of phases of transpositional elements set to whatever we want.

(T.7)
THEOREM 7
If A_0 has distinct eigenvalues, then given a unitary matrix U, its F-matrix is skew-symmetric if and only if its B-matrix is symmetric.

$C = A_0 + B$

$C^{-1} = (A_0 + B)^{-1}$

$F = BC^{-1} - C^{-1}B$

$F = B(A_0 + B)^{-1} - (A_0 + B)^{-1}B$

$F = ((A_0 + B)B^{-1})^{-1} - (B^{-1}(A_0 + B))^{-1}$

$F = (A_0 B^{-1} + I)^{-1} - (B^{-1}A_0 + I)^{-1}$

$F_T = (B_T^{-1}A_0 + I)^{-1} - (A_0 B_T^{-1} + I)^{-1}$

If B is symmetric, it is clear that F is skew-symmetric. For the opposite direction:

$F_T = -F$

$(B_T^{-1}A_0 + I)^{-1} - (A_0 B_T^{-1} + I)^{-1} = (B^{-1}A_0 + I)^{-1} - (A_0 B^{-1} + I)^{-1}$

$A_0^{-1}(B_T^{-1} + A_0^{-1})^{-1} - (B_T^{-1} + A_0^{-1})^{-1}A_0^{-1} = A_0^{-1}(B^{-1} + A_0^{-1})^{-1} - (B^{-1} + A_0^{-1})^{-1}A_0^{-1}$

$(B_T^{-1} + A_0^{-1})^{-1}A_0 - A_0(B_T^{-1} + A_0^{-1})^{-1} = (B^{-1} + A_0^{-1})^{-1}A_0 - A_0(B^{-1} + A_0^{-1})^{-1}$

let $R = (B^{-1} + A_0^{-1})^{-1}$.

$A_0(R_T - R) = (R_T - R)A_0$

let $S = R_T - R$. S is skew-symmetric and A_0 commutes with S.

A diagonal matrix with distinct eigenvalues only commutes with another diagonal matrix. So S must be diagonal. The only diagonal skew-symmetric matrix is the zero-matrix.

So

$R_T - R = 0$
$R_T = R$
$(B_T^{-1} + A_0^{-1})^{-1} = (B^{-1} + A_0^{-1})^{-1}$
$(B_T^{-1} + A_0^{-1}) = (B^{-1} + A_0^{-1})$
$B_T^{-1} = B^{-1}$
$B_T = B$

We now have the tools necessary to prove theorem 2. (T.2)

5 Proof of Theorem 2

Under DEC, we are given a CPC unitary matrix U with F-matrix F. Without loss of generality, we may choose A_0 to be the matrix with distinct eigenvalues.

By (T.3), F is zero-diagonal.
By (T.4), $\|F_{mn}\| = \|F_{nm}\|$.

By (T.6), there exists a diagonal phase matrix D where
$F_2 = D^*FD$ has all pairs of transpositional elements with phases π apart. We just set all non-diagonal elements of P to π and we can solve for D. And since F_2 is zero-diagonal with transpositional elements of equal magnitude, it is skew-symmetric.

By (T.5), if we choose $U_2 = D^*U$, then the F-matrix of U_2 is F_2 and $R(U_2) = R(U)$.

So the F-matrix of U_2 is skew-symmetric and by (T.7), its B-matrix B_2 must be symmetric.

B_2 is a normal, symmetric matrix, so by [5], p.207, theorem 4.47, there exists an orthogonal matrix O, such that:

$B_2 = OB_0O^*$

But we also know that:

$B_2 = U_2B_0U_2^*$

So

$OB_0O^* = U_2B_0U_2^*$

Therefore by (E.1):

$R(O) = R(U_2) = R(U)$. This proves (T.2), thereby proving (C.2).

6 Conclusions - Implication of SSC

(T.8)
THEOREM 8
If SSC is true, then the restricted Marcus-de Oliveira conjecture implies the full Marcus-de Oliveira conjecture.

Proof:

The unitary group forms a closed set [6]. Therefore Δ_U is a closed set.

Suppose we are given that SSC, (C.1), is true. And suppose we know that (D.2) is true. This means the boundary of Δ_O is within the convex-hull. By SSC, we then know that the boundary of Δ_U is inside the convex-hull. Can we have a unitary matrix U such that $R(U)$ is outside the convex-hull? No, because that would mean we have points of Δ_U on both the inside and outside of the boundary of Δ_U which is impossible since Δ_U is a closed set. So Δ_U is within the convex hull proving (D.1).

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