Four Dimensional Quantum Hall Effect for Dyons

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Abstract

Starting with division algebra based on quaternion, we have constructed the generalization of quantum Hall effect from two dimension to four dimension. We have constructed the required Hamiltonian operator and thus obtained its eigen values and eigen functions for four dimensional quantum Hall effect for dyons. The degeneracy of the four dimensional quantum Hall system has been discussed in terms of two integers ($P\,and\,Q$) related together where as the integer Q plays the role of Landau level index and accordingly the lowest Landau level has been obtained for four dimensional quantum Hall effect associated with magnetic monopole(or dyons). It is shown that there exists the integer as well the fractional quantum Hall effect and so, the four dimensional quantum Hall system provides a macroscopic number of degenerate states and at appropriate integer or fractional filling factions this system forms an incompressible quantum liquid.

Key Words: Quaternion, dyons, Hamiltonian operator, Landau level etc

1 Introduction

The similarity between particle physics and condensed matter physics has played a very important role in the physics [1]. The famous particle model (Standard Model), the idea of spontaneously broken symmetry and the Higgs mechanism were first originated from the BCS theory of superconductivity [2]. The discovery of quantum Hall effect(QHE) [2] was a remarkable achievement in condensed matter physics. Almost all condensed matter systems are well described by a non-relativistic Hamiltonian of the electron and nuclei [3]. But some theories like superconductivity, super-fluidity, antiferromagnetism, quantum Hall effect(integer and fractional) and the magnetism are not fully described [4] by this model. These system are best described by effective quantum field theories. In quantum Hall effect (i.e. integer and fractional) [2], the electrons are trapped in a thin layer at the interface between two semiconductors or between a semiconductor and an insulator [5]. Both the integer and the fractional quantum Hall effect [2, 5, 6] were discovered in the very special context of semiconductor hetro-structures subjected to very large magnetic fields at very low temperatures ($T = 2K$ or even less). The fundamental properties of quantum Hall effect are a consequences of the fact that the energy spectrum of electron system used for the experiments is a discrete energy spectrum in terms of three-dimensional Coulomb repulsion [2, 5, 6]. One dimensional systems theory developed by Bethe's,[7] defines exact ground state wave function of the corresponding Hamiltonian. On the other hand, the two dimensional theory [8] of quantum Hall effect describes an incompressible quantum fluid with fractional charged [5]. This incompressible liquid theory has also been described by a Chern-Simons-Landau-Ginzbure (CSLG)[9], while spherical geometry for fractional quantum Hall effect was first introduced by Haldane [10] in order to construct the hierarchy of fractional quantum Hall effect. Haldane [10] approach has been taken as very convenient to study the fractional quantum Hall effect, where the magnetic field is produced by Dirac magnetic monopole [11] and the flux is quantized in terms Dirac quantization condition[11], followed by the monopole vector potential in terms of spherical coordinates[12]-[15]. Further more, this theory has been extended by Fano[16], where the eigenfunctions are described in terms of monopole harmonics [17].

Now it is recognized that there are many connections between string theory and the above mentioned theory of quantum Hall effect [18]. The first of them constructed by Bernevig et.al. [19] reproducing the quantum Hall effect on a sphere, followed by a series of papers[20]-[23]. Another line of progress was started by the proposal of Susskind [24] that the granular structure of the quantum Hall fluid can be captured by making the ordinary Chern-Simons description of non-commutative geometry. This model can also be obtained from the Lagrangian of a charged particle moving in magnetic field on replacing its coordinates matrices, based on the matrix theory for D_0-brane. Recently an interesting extension of quantum Hall effect by Zhang and Hu [14] in four dimensions. Zhang [14] constructed a generalization of quantum Hall effect, where particles move in four dimensional space under a SU(2) gauge field. This system has a macroscopic number of degenerate single particle state. It should be noted that in the two dimensional quantum Hall effect, the current is always perpendicular to the applied magnetic field, while in four dimensional case there are three extra independent directions for this current. It means that there exists no unique direction for the current in quantum Hall system. Zhang and Hu [14] found a remarkable theoretical construction of a new physical phenomenon a four dimensional quantum Hall effect [9, 13, 14, 15, 16] on an S^4. The common feature in these models is to generalize the Landau problem on different higher dimensional manifolds. This is because the Landau problem is the cornerstone of quantum Hall effect(QHE).

Renewed interest [2, 9, 14, 26, 27, 25] shows that possibility of the existence of magnetic monopoles could be batter understood in condensed matter physics, where the magnetic materials [25]-[27] contain the magnetic field associated with magnetic monopoles. So, keeping in view the recent interests on monopole(dyons) and their possible role to produce strong magnetic field responsible for quantum Hall effect, in this paper we have made an attempt to investigate the role of non abelian dyons in order to explain the quantum Hall effect in four dimensions. Here two dimensional quantum Hall effect has been generalized to four dimensional case in terms of quaternions and non abelian gauge theory of dyons.

2 Quaternions

Quaternions (or division algebra) mean a set of four and introduce new methods in physics and mathematics. Quaternion represents the natural extension of complex numbers and form an algebra under addition and multiplication. They were first described by Irish mathematician Sir William Rowan Hamilton [28] and applied [29] to mechanics in three-dimensional space. Quaternions have the same properties as complex numbers but differ in the way that commutative law is not valid which gave the possibility of developing the fundamental laws in physics [30]-[32]. A striking feature of quaternions is that the product of two quaternions is non commutative, meaning that the product of two quaternions depends on which factor is to the left of the multiplication sign and which factor is to the right. The algebra of quaternion $\mathbb{H}$ is a four - dimensional algebra over the field of real numbers $\mathbb{R}$ and a quaternion ϕ is expressed in terms of its four base elements [28, 29]-as

$$\phi=\phi_{\mu}e_{\mu}=\phi_{0}+e_{1}\phi_{1}+e_{2}\phi_{2}+e_{3}\phi_{3},\quad(\mu=0,1,2,3) \tag{1}$$

where ϕ_0, ϕ_1, ϕ_2, ϕ_3 are the real quarterate of a quaternion and e_0, e_1, e_2, e_3 are known as quaternion unit (basis elements). A quaternion is also expressed as the combination of scalar and vector parts i.e.

$$\phi=\left(\phi_{0},\overrightarrow{\phi}\right);\tag{2}$$

here $\overrightarrow{\phi}=e_{1}\phi_{1}+e_{2}\phi_{2}+e_{3}\phi_{3}$ is vector part and ϕ_{0} is scalar part. The quaternion units $e_{A},(\forall\,A=0,1,2,3)$ satisfy the following relations

$$\begin{aligned}e_{0}e_{A} &= e_{A}e_{0}=e_{A};\\ e_{A}e_{B} &= -\delta_{AB}e_{0}+f_{ABC}e_{C}.\,(\forall\,A,B,C=1,2,3)\end{aligned}\tag{3}$$

Where δ_{AB} is the delta symbol and f_{ABC} is the Levi Civita three index symbol having value ($f_{ABC}=+1$) for cyclic permutation, ($f_{ABC}=-1$) for anti cyclic permutation and ($f_{ABC}=0$) for any two repeated indices. As such we may write the following relations among quaternion basis elements

$$\begin{aligned}\left[e_{A},\;e_{B}\right] &= 2\,f_{ABC}\,e_{C};\\ \left\{ e_{A},\;e_{B}\right\} &= -2\,\delta_{AB}e_{0};\\ e_{A}(\,e_{B}\,e_{C}) &= (e_{A}\,e_{B}\,)\,e_{C}.\end{aligned}\tag{4}$$

The brackets [,] and { , } are used respectively for commutation and the anti commutation relations while δ_{AB} is the usual Kronecker Dirac - Delta symbol. $\mathbb{H}$ is an associative but non commutative algebra. Alternatively, a quaternion is defined as a two dimensional algebra over the field of complex numbers $\mathbb{C}$ as

$$\phi=(\phi_{0}+e_{1}\phi_{1})+e_{2}\,(\phi_{2}-e_{1}\phi_{3})\tag{5}$$

The quaternion conjugate $\overline{\phi}$ is defined as

$$\overline{\phi}=\phi_{\mu}\overline{e_{\mu}}=\phi_{0}-e_{1}\phi_{1}-e_{2}\phi_{2}-e_{3}\phi_{3}\tag{6}$$

In practice ϕ is often represented as a 2×2 matrix where $e_{0}=I$, $e_{j}=-i\sigma_{j}$ ($j=1,\;2,\;3$)and σ_{j} are the usual Pauli spin matrices. Hence a quaternion can be decomposed in terms of its scalar

$(Sc(x))$ and vector $(Vec(x))$ parts as

$$Sc(\phi) = \frac{1}{2}(\phi + \overline{\phi});$$
$$Vec(x) = \frac{1}{2}(\phi - \overline{\phi}). \tag{7}$$

The norm of a quaternion is expressed as

$$N(\phi) = \phi\overline{\phi} = \overline{\phi}\phi = \mid \phi \mid^2 = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2. \tag{8}$$

Since there exists the norm of a quaternion, we have a division i.e. every ϕ has an inverse of a quaternion and is described as

$$\phi^{-1} = \frac{\overline{\phi}}{\mid \phi \mid}. \tag{9}$$

Rather the quaternion conjugation satisfies the following property

$$\overline{\phi_1 \phi_2} = \overline{\phi_1}\,\overline{\phi_2}. \tag{10}$$

The norm of the quaternion is positive definite and obey the composition law

$$N(\phi_1\phi_2) = N(\phi_1)\,N(\phi_2). \tag{11}$$

The sum and product of two quaternions are described as

$$(\alpha_0, \overrightarrow{\alpha}) + \left(\beta_0, \overrightarrow{\beta}\right) = \left(\alpha_0 + \beta_0, \overrightarrow{\alpha} + \overrightarrow{\beta}\right),$$
$$(\alpha_0, \overrightarrow{\alpha}) . \left(\beta_0, \overrightarrow{\beta}\right) = \left(\alpha_0\beta_0 - \overrightarrow{\alpha} . \overrightarrow{\beta}, \alpha_0 \overrightarrow{\beta} + \beta_0 . \overrightarrow{\alpha}\right). \tag{12}$$

Quaternion elements are non-Abelian in nature and thus represent a non commutative division ring.

3 Field Associated with Dyons

The generalized duality invariant Dirac Maxwell's equation in presence of electric and magnetic[33, 34] charges are expressed as

$$\vec{\nabla} \cdot \vec{E} = \rho_e \quad ; \qquad \vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{E}}{\partial t},$$
$$\vec{\nabla} \cdot \vec{H} = \rho_m \quad ; \qquad \nabla \times \vec{E} = -\vec{k} - \frac{\partial \vec{H}}{\partial t}, \tag{13}$$

where ρ_e is the charge source density due to electric charge , ρ_m is the charge source density due to magnetic charge (monopole), $\vec{j}$ is the current source density due to electric charge (e) and $\vec{k}$ is the current source density due to magnetic charge(g). This hypothesis of existence of magnetic charge(monopole) provides an explanation for the quantization of electric charge, Dirac [11, 35], gave an interesting result was that the product of electric charge (e) with magnetic monopole charge (g) must be quantized.

$$eg = I. \tag{14}$$

where I is an integer which could assume the values $1, 2, 3$........

In spite of many good point, Dirac's monopole theory encounters the difficulty of string. The vector potential can not be defined uniquely and definitely along this string. This condition was referred as Dirac's veto [36]. It is unnatural and undesirable condition because string are unphysical object. The name dyon was coined by Schwinger [36] for the particles carrying simultaneously the existence of electric and magnetic charges. Dyons are not strictly static, although they are stationary in certain gauges, and they have non - zero kinetic energy. A dyon with a zero electric charge is usually referred to as a magnetic monopole. Schwinger extended the Dirac quantization condition (14) to the dyon. So an alternative approach which is free from Dirac string involve a second potential in addition to the electric four potential. The electric and magnetic fields of dyons satisfying the generalized Dirac Maxwell's equations are now expressed in terms of components of two four potentials in a symmetrical manner i.e.

$$\vec{E} = -\vec{\nabla}\phi_e - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \times \vec{B},$$
$$\vec{H} = -\vec{\nabla}\phi_g - \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{A}, \tag{15}$$

Where $\{A^\mu\} = \left\{\phi_e, \vec{A}\right\}$ and $\{B^\mu\} = \left\{\phi_g, \vec{B}\right\}$ are the component of two four potential associated respectively

The complex vector electrodynamic field $\vec{\psi} = \vec{E} - i\vec{H}$ reduces the four GDM [33] equations to two differential equations as

$$\overrightarrow{\nabla}.\overrightarrow{\psi} = \rho;$$
$$\overrightarrow{\nabla}\times\overrightarrow{\psi} = -i\overrightarrow{j} - i\frac{\partial\overrightarrow{\psi}}{\partial t}. \tag{16}$$

Consequently, the Lorentz force equation of motion may be written in following form as

$$m\frac{d^2x}{dt^2} = \left(eF_{\mu\nu} + g\widetilde{F}_{\mu\nu}\right)u^{\nu}; \tag{17}$$

which may further be reduced to

$$m\frac{dv}{dt} = e\left(\overrightarrow{E} + \overrightarrow{u}\times\overrightarrow{H}\right) + g\left(\overrightarrow{H} - \overrightarrow{u}\times\overrightarrow{E}\right); \tag{18}$$

where m is the mass of the particle, e is the electric charge, $\{u^{\nu}\}$ is four-velocity of particle, space-time four vector is defined as $\{x^{\mu}\}= \{t, \overrightarrow{x}\}$ and g is magnetic charge. Electric and magnetic four-current are related as $j^{\mu} = eu^{\mu}$ and $k^{\mu} = gu^{\mu}$. As such the duality invariance is an intrinsic property of Maxwell's Lorentz theory of electrodynamics in presence of monopole(ie. for dyons).

let us introduce the generalized charge for dyon as $q = e - ig$, so that the Generalized four potential $V^{\mu} = \left(\phi, \vec{V}\right)$ associated with dyons is defined as

$$V^{\mu} = A^{\mu} - iB^{\mu}; \tag{19}$$

So the duality transformations for $\{A^{\mu}\}$ and $\{B^{\mu}\}$ are described as

$$\begin{aligned} A_{\mu} =& \quad A_{\mu}cos\theta + B_{\mu}sin\theta; \\ B_{\mu} =& \quad A_{\mu}sin\theta - B_{\mu}cos\theta; \end{aligned} \tag{20}$$

Hence, the covariant tensorial form of generalized Dirac-Maxwell's equations of dyons may be written as,

$$\begin{aligned} \partial_{\nu}F^{\mu\nu} =& \quad j^{\mu}; \\ \partial_{\nu}\widetilde{F_{\mu\nu}} =& \quad k^{\mu}; \end{aligned} \tag{21}$$

These equation are invariant under the duality transformations

$$\begin{aligned} \left(F, \widetilde{F}\right) =& \quad \left(Fcos\theta + \widetilde{F}sin\theta;\; Fsin\theta - \widetilde{F}cos\theta\right); \\ (j_{\mu}, k_{\mu}) =& \quad (j_{\mu}cos\theta + k_{\mu}sin\theta;\; j_{\mu}sin\theta - k_{\mu}cos\theta). \end{aligned} \tag{22}$$

where

$$\frac{g}{e}=\frac{B_\mu}{A_\mu}=\frac{k_\mu}{j_\mu}=\frac{\widetilde{F}}{F}=-tan\theta. \tag{23}$$

is described as constancy condition. The generalized charge may also be written as

$$q=\mid q\mid e^{-i\theta}. \tag{24}$$

In addition to the dual symmetry,the equation of motion (17) and the GDM field equation (21) leads to the following symmetries;

(a) invariance under a pure rotation in charge space or its combination with a transformation containing simultaneously space and time reflection(strong symmetry);

(b) a weak symmetry under charge reflection combined with space reflection or time reflection (not both);

(c) a weak symmetry under PT (combined operation of parity and time reversal) and strong symmetry under CPT (combined operation of charge conjugation, parity and time reversal).

using equation (23), the Interaction of i^{th}dyon in the field of j^{th} dyon may be written as follows

$$I_{ij}=\frac{A_\mu^j}{e_j}q_j^*q_iu_\mu^i,$$

where A_μ^j is the electric four potential describing the field of j^{th} dyon e_j is its electric charge and u_μ^i is the four-velocity of i^{th} dyon in the field of j^{th} this equation shows that

(a) interaction between two dyons is zero, when their generalized charges are orthogonal in their combined charge space.

(b) interaction depends on electric coupling parameter

$$\alpha_{ij}=e_ie_j+g_ig_j. \tag{25}$$

under the constancy condition $\frac{e_i}{g_i}=\frac{e_j}{g_j}$=constant.

(c) interaction depends on the magnetic coupling parameter(i.e. chirality)

$$\mu_{ij}=e_ig_j-g_ie_j; \tag{26}$$

under the condition$\frac{e_i}{g_j}=-\frac{e_j}{g_j}$ The coupling between two generalized charges q_i and $q_j$is described as

$$q_i^* \cdot q_j = (e_i e_j + g_i g_j) - i\ (e_i g_j - g_i e_j) = \alpha_{ij} - i\,\mu_{ij}; \tag{27}$$

where the real part α_{ij} is called the electric coupling parameter (the Coulomb like term) responsible for the existence of either electric charge or magnetic monopole while the imaginary part μ_{ij} is the magnetic coupling parameter and plays an important role for the existence of magnetic charge. Both of these parameters are invariant under the duality transformations. The parameter μ_{ij} has also been named as Chirality quantization parameter for dyons and leads the following charge quantization condition i.e.

$$\mu_{ij} = \pm I\ (I \in \mathbb{Z}); \tag{28}$$

where the half integral quantization is forbidden by chiral invariance and locality in commutator of the electric and magnetic vector potentials.

4 Generalized Angular Momentum in Presence of Non-Abelian Dyons

We have described the dyons and their interactions only in abelian context [37]. Here we have made an attempt to extend this theory to higher dimensional space leading to the generalization of Maxwell's fields to Yangs-Mills fields. In this case, the physical space is associated with the n- dimensional internal charge space known as isospin space. Any gauge transformation in this space must give rise to a local phase transformation of the type,

$$\Psi(x) \rightarrow \Psi^{'}(x) = exe\left(-i\chi_j(x)\,T^j\right)\Psi(x) \tag{29}$$

where$\chi_j(x)$is the parameter for gauge change and T^j are $n \times n$ matrices leading to the generators of the gauge transformation. For $SU(2)$, the two dimensional space in terms two dimensional Pauli-spin matrices,T^j $(\forall\, \mathrm{j} = 1, 2, 3)$ are Pauli matrices. These matrices satisfy the following commutation rules ,

$$\left[T^j, T^k\right] = i\varepsilon^{jkl}T^l \quad (\forall\ \mathrm{j,k,l}{=}1, 2, 3,) \tag{30}$$

In this case of dyon the complex charge space of dyons opens into a higher dimensional internal space. Now it is widely recognized that magnetic monopoles are better understood in non-

Abelian gauge theories. Thus for $SU(2)$ case there corresponds the three gauge fields $V_{\mu}^{c}(x)$, where $V_{\mu}(x)$ is the generalized potential of dyon. Using equation (19 &23) when the value of $\theta = 0$, we can easily see that the form of generalized potential V_{μ} is same as A_{μ}, So the generalized potential$V_{\mu}\,(= A_{\mu})$, generalized Yang-Mill's tensor $G_{\mu\nu}$ and the covariant derivative D^{μ} In terms of matrices may be written as follows[38];

$$\begin{aligned} A_{\mu} =& A_{\mu}^{c} T^{c} \\ G_{\mu\nu} =& G_{\mu\nu}^{c} T^{c} \\ D^{\mu} =& \partial^{\mu} + i\kappa A^{c\mu} T^{c} \end{aligned} \tag{31}$$

where κ is an arbitrary parameter, So the $G_{\mu\nu}$may now be written as

$$G_{\mu\nu} = V_{\mu,\nu} - V_{\nu,\mu} + i\kappa\left[V^{\mu}, V^{\nu}\right] \tag{32}$$

In previous paper,[40] we have already discussed the gauge invariant angular momentum operator of dyons. It has already been argued that this angular momentum is not rotationally symmetric. Hence the commutation relations do not show the proper commutation algebra. So, we have modified the gauge invariant and rotationally symmetric angular momentum operator for dyons by including the extra term $\left(\mu_{ij}\frac{\vec{r}}{r}\right)$ named as residual angular momentum term. The commutation relation demand an additional term $\frac{\mu_{ij}^{2}}{2mr^{2}}$ in the Hamiltonian, which possesses the higher symmetry similar to that of pure Coulomb Hamiltonian [40, 41]. So our generalized angular momentum, which is gauge invariant and rotational symmetric, depends on the magnetic coupling parameter (μ_{ij}). Without this extra term the angular momentum operator does not satisfy $SO(3)$ commutation relation. In the higher dimensional case(i.e. non-abelian) the modified angular momenta for quantum Hall system may now be written as i.e.

$$\begin{aligned} \Lambda_{ab} =& x_{a}\left(\partial_{b} - iT^{c}A_{b}^{c}\right) - x_{b}\left(\partial_{a} - iTA_{a}^{c}\right) + r^{2}\mathcal{F}_{ab}^{c}T^{c} \quad (\forall a, b = 1,\ 2,\ 3,\ 4,\ 5) \\ =& x_{a}\left(\partial_{b} + \grave{a}_{b}\right) - x_{b}\left(\partial_{a} + \grave{a}_{a}\right) + r^{2}\mathcal{F}_{ab}^{c}T^{c} \end{aligned} \tag{33}$$

where $\grave{a}_{a} = -iT^{c}A_{a}^{c}$, $\grave{a}_{b} = -iT^{c}A_{b}^{c}$ gauge potential and $\mathcal{F}_{ab}^{c}$ is the non abelian field strength due to the existence of non abelian dyons . Equation (33) may also be written as

$$\Lambda_{ab} = x_a (D_b) - x_b (D_a) + r^2 \mathcal{F}_{ab}^c T^c = J_{ab} + r^2 \mathcal{F}_{ab}^c T^c \tag{34}$$

where $x_a$denote coordinates on S^4 where a, $b = 1, 2, 3, 4, 5$ and $x_a^2 = 1$, $c = 1, 2, 3$ denote the three $SU(2)$ spin components and also denoted the space index on the three dimensional orbital space. Let us define the Dirac matrices for S^4 as

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma_\mu} \\ \sigma_\mu & 0 \end{pmatrix}, \ \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{35}$$

$$\gamma^{ab} = -\frac{1}{2} \left[\gamma^a, \gamma^b\right] \tag{36}$$

$$\gamma^{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \bar{\sigma_{\mu\nu}} \end{pmatrix}, \ \gamma^{\mu 5} = i \begin{pmatrix} 0 & \bar{\sigma_\mu} \\ \sigma_\mu & 0 \end{pmatrix} \tag{37}$$

As such equation (34) represented the conserved angular momentum on $SO(5)$. Where we have introduced the coupling term $\grave{a}_a$ which is responsible for gauge field strength. Here the partial derivative(∂_a) has been replaced by covariant derivative (D_b) for the development of gauge formulation. Here T^c are the usual Pauli spin matrices for $SU(2)$ group associated with isospin in isotropic spin space which is non abelian in nature. In equation (34) the third term $r^2 \mathcal{F}_{ab}^c T^c$ (non-abelian) is analogues to the rotational symmetric term $\mu_{ij} \frac{\vec{r}}{r}$ like the angular momentum [40] for abelian gauge theory. So this term $r^2 \mathcal{F}_{ab}^c T^c$ may now be identified as the gauge invariant as well as rotational symmetric angular momentum terms. The $SO(5)$ group is the group of orthogonal transformation in the five dimensional real vector space. It has 10 symmetry generators. A four dimensional sphere can be embedded in the five dimensional Euclidean space through the relation $x_a^2 = 1$. Since a orthogonal transformation preserves the length of a vector, the $SO(5)$ group is the symmetry group of the four dimensional sphere. Like wise, the $SO(4)$ group is the group of orthogonal transformation in the four dimensional real vector space. It has 6 symmetry generators and its Lie algebra is isomorphic to the direct sum of two $SU(2)$ Lie algebra. Further more, the $SU(2)$ group is the group of unitary transformations in the two dimensional complex vector space. It has 3 symmetry generators and its Lie algebra is isomorphic to the $SO(3)$ group, which is the group of orthogonal transformations in the three dimensional real vector space.

Here $SU(2)$ group determines the coupling between the particles and the background monopoles potential.

As such, the Hamiltonian for generalized theory of four dimensional quantum Hall effect on S^4 $(SO(5))$ may now be written as

$$\mathbb{H} = \frac{1}{2Mr^2} \sum_{a<b} \mid \Lambda_{ab} \mid^2 \tag{38}$$

which is further reduced to the following expression after making certain restrictions i.e.

$$\mathbb{H} = \frac{1}{2Mr^2} \sum_{a<b} \left(\mathit{\Lambda}_{ab}^2 - R^4 \mathcal{F}_{ab}^2 \right) = \frac{1}{2Mr^2} \sum_{a<b} \left(\mathit{\Lambda}_{ab}^2 - 2I\,(I+1) \right) \tag{39}$$

The angular momentum equation (33 and 34) thus satisfy the $SO(5)$ following well known commutation relation [17] of angular momentum operator i.e.

$$[\mathit{\Lambda}_{ab},\, \mathit{\Lambda}_{cd}] = i \left(\delta_{ac} \mathit{\Lambda}_{bd} + \delta_{bd} \mathit{\Lambda}_{ac} - \delta_{bc} \mathit{\Lambda}_{ad} - \delta_{ad} \mathit{\Lambda}_{bc} \right) \tag{40}$$

5 Energy eigen values for four dimensional quantum Hall effect

In general, $SO(5)$ irreducible representation is labeled by two integers $(P,\, Q)$, with $P \geqslant Q \geqslant 0$. For such representation, the Casimir operator may be written as are given by

$$C\,(P,\, Q) = \sum_{a<b} \Lambda_{ab}^2 = \frac{P^2}{2} + \frac{Q^2}{2} + 2P + Q \tag{41}$$

while the dimensionality formula is deduced as

$$d\,(P, Q) = (1+Q)\,(1+P-Q) \left(1 + \frac{P}{2} \right) \left(1 + \frac{P+Q}{3} \right) \tag{42}$$

For a given I , the two integer is related as $P = Q + 2I$. So the energy eigen value of the Hamiltonian (39) for a given I, may be then expressed as

$$E_n\,(P = Q + 2I,\, Q) = \frac{1}{2Mr^2} \left(C\,(P = Q + 2I,\, Q) - 2I\,(I+1) \right) \tag{43}$$

which may further be reduced as

$$E_n = \frac{1}{2Mr^2} \left(\frac{P^2}{2} + \frac{Q^2}{2} + 2P + Q - 2I\,(I+1) \right) = \frac{1}{2Mr^2} \left(Q^2 + Q\,(2I+3) + 2I \right) \tag{44}$$

while the degeneracy $d\left(P,Q\right)$ is obtained by the irreducible representation[13, 14, 15, 39] of the $SO\left(5\right)$. Here Q plays the role of landau level index. If we put $Q=0$, we get the lowest landau level energy,

$$E=\frac{1}{2Mr^{2}}\left(2I\right)=\frac{P}{2Mr^{2}} \tag{45}$$

So the ground state, which is the lowest $SO(5)$ level for a given I , is obtained by putting $Q=0$ and consequently the lowest state has $d\left(P,\,0\right)=\frac{1}{3!}\left(P+1\right)\left(P+2\right)\left(P+3\right)$ fold degeneracy. Therefore, the dimensionality of $SU(2)$ representation is associated with the magnetic flux in dyons. Accordingly the states with $Q>0$ are separated from the ground state by a finite energy gap.

6 LLL Eigenstates for four Dimensional quantum Hall effect

We may now easily construct the second Hopf $(S^{7}-S^{4})$extending the one imaginary unit of complex algebra to three imaginary units $(e_{1,}e_{2,}e_{3})$ of quaternion algebra for which the Pauli spin matrices are connected with quaternion element $e_{0}=I,\;e_{i}=-i\sigma_{i}$, Likewise, the second Hopf map is realized as

$$x_{a}=\psi^{\dagger}\left(\gamma_{a}\right)\psi \tag{46}$$

where $a=1,2,...5$ and ψ is the four component quaternionic spinor defined as

$$\psi=\left(\begin{array}{c}\psi_{1}\\ \psi_{2}\\ \psi_{3}\\ \psi_{4}\end{array}\right) \tag{47}$$

satisfying the normalization condition $\psi^{\dagger}\psi=1$. It should be noted that quaternionic group is isomorphic to $SU(2)$ group . As the generalization of the three Pauli matrices to five $SO(5)$ Dirac matrices Γ_{a}, it should satisfying the Clifford algebra.

$$\left\{\gamma^{a},\gamma^{b}\right\}=2\delta^{ab} \tag{48}$$

where the given matrices are described as quaternion i.e.

$$
\begin{aligned}
\gamma^1 &= \begin{pmatrix} 0 & e_1 \\ -e_1 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} \\
\gamma^2 &= \begin{pmatrix} 0 & e_2 \\ -e_2 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \\
\gamma^3 &= \begin{pmatrix} 0 & e_3 \\ -e_3 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix} \\
&\gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (49)
\end{aligned}
$$

So the second Hopf spinor may be expressed as

$$
\begin{aligned}
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \sqrt{\frac{1+x_5}{2}} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \\
\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} &= \sqrt{\frac{1}{2\,(1+x_5)}}\,(x_4 + x_i e_i) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}; \qquad (50)
\end{aligned}
$$

where φ_1, φ_2 is the first Hopf spinor and $e_i = -i\sigma_i$. In this case, the normalized lowest Landau level(LLL) [39] is written as

$$
< \psi,\, n_i \mid m_1, m_2, m_3, m_4 >= \sqrt{\frac{P!}{m_1!\, m_2!\, m_3!\, m_4!}} \psi_1^{m_1} \psi_2^{m_2} \psi_3^{m_3} \psi_4^{m_4} \qquad (51)
$$

with integers $m_1+m_2+m_3+m_4 = P$. So the form of the single particle wave function described in equation (51) helps us to calculate the many-body wave function. So the simplest case has also been obtained for $N = d(P,0)$, (N particle density), when the lowest level is completely filled. In this case, the filling factor is describes as $\nu \equiv \frac{N}{d(P,0)} = 1$, this result corresponds to the integer quantum Hall effect. While for the case of fractional quantum Hall effect the many body wave function is written like $\Phi_m = \Phi^m\,(x_1, ...x_N)$ with odd integer m, and then the filling fraction $\nu = \frac{d(P,0)}{d(mP,0)} \approx \frac{1}{m^3}$, leading to fractional quantum Hall effect. Therefore the degeneracy of the lowest $SO(5)$ for fractional quantum Hall effect is describes

$$
d\,(mP,\,0) = \frac{1}{3!}\,(mP+1)\,(mP+2)\,(mP+3) \rightarrow \frac{1}{6} m^3 P^3 \qquad (52)
$$

7 Discussion

We have concluded this paper by discussing the eigen states of four dimensional quantum Hall system. Thus the second Hopf map($S^7 - S^4$) has been constructed for division algebra by replacing the one imaginary unit of complex algebra to three imaginary units ($e_{1,}e_{2,}e_3$) of quaternion algebra connected well with Pauli spin matrices. So we have discussed the second Hopf map for our quantum Hall system(QHE) by equation (46), which is realized in terms of four component quaternion spinors given by equation (47). As such, we agree with Zhang et al that an incompressible quantum spin liquid involves N fermions for which the simplest case is described by $SO(5)$ group of quaternions. The $SO(5)$ Dirac matrices has been discussed by equation (49) and accordingly the Hopf spinor are discussed in equation (50). The normalized lowest Landau level(LLL) for four dimensional quantum Hall system has also been described in equation (51). It is shown that for N fermions for $N = d(P, 0)$, (N particle density), as investigated when the lowest level is completely filled, for which the filling factor becomes unity (an integer). So our result resembles with the result Zhang et.al [14, 15], K. Hasebe [39] corresponding to the integer quantum Hall effect, while the case of fractional quantum Hall effect where the many body wave function $\Phi_m = \Phi^m (x_1, ...x_N)$ with odd integer m, corresponding to legitimate fermionic wave function in the lowest $SO(5)$ in terms of quaternion. Therefore the degeneracy of the lowest $SO(5)$ level in this case given by equation (52). While the particle number still $N = d(P, 0)$ and the filling factor turns out to be $\nu = \frac{d(P,0)}{d(mP,0)} \approx \frac{1}{m^3}$ and is inversely proportional to third power of odd integer m. Here it should be noted that as the correction functions for $m = 1$ case can be computed exactly, it is plausible that the $m > 1$ case has similar correlations, for an incompressible liquid. However, the effective parameters need to be rescaled properly in the fractional case. This incompressible liquid supports fractionalized charge excitation with charge m^{-3}. Such a state may be described by a wave function of the form $\mathit{\Phi}^{m-1}\mathit{\Phi}_h$(where $\mathit{\Phi}_h$ is the wave function of the integer case). So it may be concluded that the four dimensional Hall system provides a macroscopic number of degenerate states and at appropriate integer or fractional filling fractions this system forms an incompressible quantum liquid.

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