LAURICELLA HYPERGEOMETRIC SERIES OVER FINITE FIELDS

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Abstract. In this paper we give a finite field analogue of the Lauricella hypergeometric series and obtain some transformation and reduction formulae and several generating functions for the Lauricella hypergeometric series over finite fields. These generalize some known results of Li *et al* as well as several other well-known results.

Contents

1. Introduction

Let q be a power of a prime. Then $\mathbb{F}_q$ and $\widehat{\mathbb{F}}_q^*$ are denoted the finite field of q elements and the group of multiplicative characters of $\mathbb{F}_q^*$ respectively. Setting $\chi(0)=0$ for all characters, we extend the domain of all characters χ of $\mathbb{F}_q^*$ to $\mathbb{F}_q$. Let $\overline{\chi}$ and ε denote the inverse of χ and the trivial character respectively. See [3] and [9, Chapter 8] for more information about characters.

Following [2], we define the generalized hypergeometric function as

$$ {}_{n+1}F_n\left(\begin{matrix} a_0, a_1, \ldots, a_n \\ b_1, \ldots, b_n \end{matrix} \middle| x\right) := \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_n)_k}{k!(b_1)_k \cdots (b_n)_k} x^k, $$

where $(z)_k$ is the Pochhammer symbol given by

$$ (z)_0 = 1,\ (z)_k = z(z+1)\cdots(z+k-1) \text{ for } k \geq 1. $$

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It was Greene who in [8] developed the theory of hypergeometric functions over finite fields and established a number of transformation and summation identities for hypergeometric series over finite fields which are analogues to those in the classical case. Greene, in particular, introduced the notation

$$ {}_2F_1\left(\begin{matrix}A,B\\C\end{matrix}\middle| x\right)^G = \varepsilon(x)\frac{BC(-1)}{q}\sum_y B(y)\overline{B}C(1-y)\overline{A}(1-xy) $$

for $A,B,C\in\widehat{\mathbb{F}}_q$ and $x\in\mathbb{F}_q$, that is a finite field analogue of the integral representation of Gauss hypergeometric series [2]:

$$ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^b(1-t)^{c-b}(1-tx)^{-a}\frac{dt}{t(1-t)}, $$

and defined the finite field analogue of the binomial coefficient as

$$ \binom{A}{B}^G = \frac{B(-1)}{q}J(A,\overline{B}), $$

where $J(\chi,\lambda)$ is the Jacobi sum given by

$$ J(\chi,\lambda) = \sum_u \chi(u)\lambda(1-u). $$

For more details about the finite field analogue of the generalized hypergeometric functions, please see [6, 7, 12].

In this paper, for the sake of simplicity, we define the finite field analogue of the binomial coefficient and the classic Gauss hypergeometric series by

$$ \binom{A}{B} = q\binom{A}{B}^G = B(-1)J(A,\overline{B}). $$

and

$$ {}_2F_1\left(\begin{matrix}A,B\\C\end{matrix}\middle| x\right) = q\cdot {}_2F_1\left(\begin{matrix}A,B\\C\end{matrix}\middle| x\right)^G = \varepsilon(x)BC(-1)\sum_y B(y)\overline{B}C(1-y)\overline{A}(1-xy), $$

respectively.

There are many interesting double hypergeometric functions in the field of hypergeometric functions. Among these functions, the Appell series F_1 may be one of the most important functions:

$$ F_1(a;b,b';c;x,y) = \sum_{m,n\geq 0}\frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)_{m+n}}x^m y^n,\ \ |x|<1,\ \ |y|<1. $$

See [2, 5, 14] for more material about the Appell series.

Inspired by Greene's work, Li *et al* [11] gave a finite field analogue of the Appell series F_1 and established some transformation and reduction formulas and the generating functions for the function over finite fields. In that paper, the finite field analogue of the Appell series F_1 was given by

$$ F_1(A;B,B';C;x,y) = \varepsilon(xy)AC(-1)\sum_u A(u)\overline{A}C(1-u)\overline{B}(1-ux)\overline{B'}(1-uy). $$

It was Lauricella [10] who in 1893 generalized the Appell series F_1 to the Lauricella hypergeometric series $F_D^{(n)}$ that is defined by

$$F_D^{(n)}\left(\begin{matrix}a;b_1,\cdots,b_n\\ c\end{matrix}\middle|x_1,\cdots,x_n\right):=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}m_1!\cdots m_n!}x_1^{m_1}\cdots x_n^{m_n}.$$

It is clear that

$$F_1(a;b,b';c;x,y)=F_D^{(2)}\left(\begin{matrix}a;b,b'\\ c\end{matrix}\middle|x,y\right)\text{ and }{}_2F_1\left(\begin{matrix}b,a\\ c\end{matrix}\middle|x\right)=F_D^{(1)}\left(\begin{matrix}a;b\\ c\end{matrix}\middle|x\right).$$

so the Lauricella hypergeometric series $F_D^{(n)}$ is an n-variable extension of the Appell series F_1 and the hypergeometric function ${}_2F_1$.

Motivated by the work of Greene [8] and Li *et al* [11], we give a finite field analogue of the Lauricella hypergeometric series. Since the Lauricella hypergeometric series $F_D^{(n)}$ has a integral representation

$$F_D^{(n)}\left(\begin{matrix}a;b_1,\cdots,b_n\\ c\end{matrix}\middle|x_1,\cdots,x_n\right)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1\frac{u^{a-1}(1-u)^{c-a-1}}{(1-x_1u)^{b_1}\cdots(1-x_nu)^{b_n}}du,$$

we give the finite field analogue of the Lauricella hypergeometric series in the following form:

$$\begin{aligned}&F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ C\end{matrix}\middle|x_1,\cdots,x_n\right)\\ &\quad=\varepsilon(x_1\cdots x_n)AC(-1)\sum_u A(u)\overline{A}C(1-u)\overline{B_1}(1-x_1u)\cdots\overline{B_n}(1-x_nu),\end{aligned}$$

where $A,B_1,\cdots,B_n,C_1,\cdots,C_n\in\widehat{\mathbb{F}}_q$, $x_1,\cdots,x_n\in\mathbb{F}_q$ and the sum ranges over all the elements of $\mathbb{F}_q$. In the above definition, the factor $\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}$ is dropped to obtain simpler results. We choose the factor $\varepsilon(x_1\cdots x_n)AC(-1)$ to get a better expression in terms of binomial coefficients. From the definition of the Lauricella hypergeometric series over finite fields, we know that

$$F_1(A;B,B';C;x,y)=F_D^{(2)}\left(\begin{matrix}A;B,B'\\ C\end{matrix}\middle|x,y\right)\text{ and }{}_2F_1\left(\begin{matrix}A,B\\ C\end{matrix}\middle|x\right)=F_D^{(1)}\left(\begin{matrix}B;A\\ C\end{matrix}\middle|x\right).$$

Then the Lauricella hypergeometric series over finite fields can be regarded as an n-variable extension of the finite field analogues of the Appell series F_1 and the hypergeometric function ${}_2F_1$.

The following theorem gives another expression for the Lauricella hypergeometric series over finite fields.

Theorem 1.1. *For $A,B_1,\cdots,B_n,C\in\widehat{\mathbb{F}}_q$ and $x_1,\cdots,x_n\in\mathbb{F}_q$, we have*

$$\begin{aligned}&F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ C\end{matrix}\middle|x_1,\cdots,x_n\right)\\ &\quad=\frac{1}{(q-1)^n}\sum_{\chi_1,\cdots,\chi_n}\binom{A\chi_1\cdots\chi_n}{C\chi_1\cdots\chi_n}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_n(x_n),\end{aligned}$$

where each sum ranges over all multiplicative characters of $\mathbb{F}_q$.

From Theorem 1.1 or the definition, we know that the Lauricella hypergeometric series over finite fields

$$F_D^{(n)}\left(\begin{matrix}A; B_1, \cdots, B_n \\ C\end{matrix}\middle| x_1, \cdots, x_n\right)$$

is invariant under permutation of the subscripts $1, 2, \cdots, n$, namely, it is invariant under permutation of the B's and x's together.

The aim of this paper is to give several transformation and reduction formulas and the generating functions for the Lauricella hypergeometric series over finite fields. We know that the Lauricella hypergeometric series over finite fields is an n-variable extension of the finite field analogues of the Appell series F_1 and the hypergeometric function ${}_2F_1$. So most of the results in this paper are generalizations of certain results in [11] and some other well-known results. For example, [11, Theorem 1.3] and [8, Theorem 3.6] are special cases of Theorem 1.1.

We will give our proof of Theorem 1.1 in the next section. Several transformation and reduction formulae for the Lauricella hypergeometric series over finite fields will be given in Section 3. The last section is devoted to some generating functions for the Lauricella hypergeometric series over finite fields.

2. Proof of Theorem 1.1

To carry out our study, we need some auxiliary results which will be used in the sequel.

The results in the following proposition follows readily from some properties of Jacobi sums.

Proposition 2.1. *If* $A, B \in \widehat{\mathbb{F}}_q$, *then*

$$\binom{A}{B} = \binom{A}{A\overline{B}}, \tag{2.1}$$

$$\binom{A}{B} = \binom{\overline{B}}{\overline{A}} AB(-1), \tag{2.2}$$

$$\binom{A}{\varepsilon} = \binom{A}{A} = -1 + (q-1)\delta(A), \tag{2.3}$$

where $\delta(\chi)$ *is a function on characters given by*

$$\delta(\chi) = \begin{cases} 1 & \text{if } \chi = \varepsilon \\ 0 & \text{otherwise} \end{cases}.$$

The following result is the finite field analogue of the well-known identity

$$\binom{a}{b}\binom{c}{a} = \binom{c}{b}\binom{c-b}{a-b}.$$

Proposition 2.2. (See [8, (2.15)]) *For* $A, B, C \in \widehat{\mathbb{F}}_q$, *we have*

$$\binom{A}{B}\binom{C}{A} = \binom{C}{B}\binom{C\overline{B}}{A\overline{B}} - (q-1)B(-1)\delta(A) + (q-1)AB(-1)\delta(B\overline{C}).$$

The following result is also very important in the derivation of Theorem 1.1.

Theorem 2.1. (Binomial theorem over finite fields, see [8, (2.10)]) *For $A \in \widehat{\mathbb{F}}_q$ and $x \in \mathbb{F}_q$, we have*

$$\overline{A}(1-x) = \delta(x) + \frac{1}{q-1}\sum_{\chi}\binom{A\chi}{\chi}\chi(x),$$

where the sum ranges over all multiplicative characters of $\mathbb{F}_q$ and $\delta(x)$ is a function on $\mathbb{F}_q$ given by

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. From the binomial theorem over finite fields, we know that for $1 \leq j \leq n$,

$$\overline{B_j}(1 - x_j u) = \delta(x_j u) + \frac{1}{q-1}\sum_{\chi_j}\binom{B_j\chi_j}{\chi_j}\chi_j(x_j u).$$

Then, by the fact that $\varepsilon(x_j)\delta(x_j u)A(u) = 0$ for $1 \leq j \leq n$,

$$\begin{aligned}
&F_D^{(n)}\left(\begin{matrix} A; B_1, \cdots, B_n \\ C \end{matrix}\middle| x_1, \cdots, x_n\right) \\
&= \varepsilon(x_1 \cdots x_n) AC(-1) \sum_u A(u)\overline{A}C(1-u) \\
&\cdot \left(\delta(x_1 u) + \frac{1}{q-1}\sum_{\chi_1}\binom{B_1\chi_1}{\chi_1}\chi_1(x_1 u)\right) \cdots \left(\delta(x_n u) + \frac{1}{q-1}\sum_{\chi_n}\binom{B_n\chi_n}{\chi_n}\chi_n(x_n u)\right) \\
&= \frac{AC(-1)}{(q-1)^n}\sum_{\chi_1, \cdots, \chi_n}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_n(x_n)\sum_u A\chi_1\cdots\chi_n(u)\overline{A}C(1-u) \\
&= \frac{1}{(q-1)^n}\sum_{\chi_1, \cdots, \chi_n}\binom{A\chi_1\cdots\chi_n}{A\overline{C}}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_n(x_n),
\end{aligned}$$

which, by (2.1), implies that

$$\begin{aligned}
&F_D^{(n)}\left(\begin{matrix} A; B_1, \cdots, B_n \\ C \end{matrix}\middle| x_1, \cdots, x_n\right) \\
&\quad = \frac{1}{(q-1)^n}\sum_{\chi_1, \cdots, \chi_n}\binom{A\chi_1\cdots\chi_n}{C\chi_1\cdots\chi_n}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_n(x_n).
\end{aligned}$$

This completes the proof of Theorem 1.1. □

3. An integral formula and its finite field analogue

In this section we derive an integral formula for the Lauricella hypergeometric series $F_D^{(n)}$, which relating $F_D^{(n)}$ to $F_D^{(n-1)}$, and then give its finite field analogue. In addition, a finite field analogue of a summation formula for the Lauricella hypergeometric series $F_D^{(n)}$ is also deduced.

The following theorem gives a result which is analogous to [8, (3.12)].

Theorem 3.1. *If $a, b_1, \cdots, b_n, c$ and $x_1, \cdots, x_n$ are complex numbers with $Re(b_1) > 0$ and $Re(b_2) > 0$, then*

$$B(b_1, b_2)F_D^{(n)}\left(\begin{matrix} a; b_1, \cdots, b_n \\ c \end{matrix}\middle| x_1, \cdots, x_n\right)$$
$$= \int_0^1 u^{b_1-1}(1-u)^{b_2-1}F_D^{(n-1)}\left(\begin{matrix} a; b_1+b_2, b_3, \cdots, b_n \\ c \end{matrix}\middle| ux_1+(1-u)x_2, x_3, \cdots, x_n\right) du,$$

where $B(x, y)$ is the beta integral given for $Re(x) > 0, Re(y) > 0$ by [1]

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Proof. It is easily seen from the binomial theorem that

$$(ux_1 + (1-u)x_2)^m = \sum_{0\le m_1\le m} \binom{m}{m_1} u^{m_1}(1-u)^{m-m_1}x_1^{m_1}x_2^{m-m_1}.$$

Then, by [1, Thoeorem 1.1.4]

$$\int_0^1 u^{b_1-1}(1-u)^{b_2-1}(ux_1+(1-u)x_2)^m du$$
$$= \sum_{0\le m_1\le m} \binom{m}{m_1} x_1^{m_1}x_2^{m-m_1} \int_0^1 u^{m_1+b_1-1}(1-u)^{m-m_1+b_2-1}du$$
$$= \sum_{0\le m_1\le m} \binom{m}{m_1} x_1^{m_1}x_2^{m-m_1} B(m_1+b_1, m-m_1+b_2)$$
$$= \frac{m!}{(b_1+b_2)_m}B(b_1, b_2) \sum_{0\le m_1\le m} \frac{(b_1)_{m_1}(b_2)_{m-m_1}}{m_1!(m-m_1)!} x_1^{m_1}x_2^{m-m_1}.$$

Using the above identity in the integral at the right side and making the substitution $m - m_1 \to m_2$, we obtain

$$\int_0^1 u^{b_1-1}(1-u)^{b_2-1}F_D^{(n-1)}\left(\begin{matrix} a; b_1+b_2, b_3, \cdots, b_n \\ c \end{matrix}\middle| ux_1+(1-u)x_2, x_3, \cdots, x_n\right) du$$
$$= \sum_{m=0}^{\infty}\sum_{m_3=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} \frac{(a)_{m+m_3+\cdots+m_n}(b_1+b_2)_m(b_3)_{m_3}\cdots(b_n)_{m_n}}{(c)_{m+m_3+\cdots+m_n}m!m_3!\cdots m_n!} x_3^{m_3}\cdots x_n^{m_n}$$
$$\cdot \int_0^1 u^{b_1-1}(1-u)^{b_2-1}(ux_1+(1-u)x_2)^m du$$
$$= B(b_1, b_2)\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_m\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}m_1!\cdots m_n!} x_1^{m_1}\cdots x_n^{m_n}$$
$$= B(b_1, b_2)F_D^{(n)}\left(\begin{matrix} a; b_1, \cdots, b_n \\ c \end{matrix}\middle| x_1, \cdots, x_n\right).$$

We comlete the proof of Theorem 3.1. □

Taking $n=2$ in Theorem 3.1, we have the following result.

Corollary 3.1. *If a, b_1, b_2, c and x_1, x_2 are complex numbers with $Re(b_1)>0$ and $Re(b_2)>0$, then*

$$B(b_1,b_2)F_1(a;b_1,b_2;c;x_1,x_2)=\int_0^1 u^{b_1-1}(1-u)^{b_2-1}{}_2F_1\left(\left.\begin{matrix}a, b_1+b_2\\ c\end{matrix}\right| ux_1+(1-u)x_2\right)du.$$

When $c=b_1+\cdots+b_n$, Theorem 3.1 reduces to [4, (7.8)].
We now give the finite field analogue of Theorem 3.1.

Theorem 3.2. *For $A, B_3, \cdots, B_n, C \in \widehat{\mathbb{F}}_q$, $B_1, B_2 \in \widehat{\mathbb{F}}_q\backslash\{\varepsilon\}$ and $x_1,\cdots,x_n \in \mathbb{F}_q$, we have*

$$\begin{aligned}
&\varepsilon(x_1x_2)\sum_u B_1(u)B_2(1-u)F_D^{(n-1)}\left(\left.\begin{matrix}A; B_1B_2, B_3,\cdots,B_n\\ C\end{matrix}\right| ux_1+(1-u)x_2, x_{3,}\cdots,x_n\right)\\
&=\binom{\overline{B_1B_2}}{\overline{B_1}}F_D^{(n)}\left(\left.\begin{matrix}A; B_1,\cdots,B_n\\ C\end{matrix}\right| x_1,\cdots,x_n\right)\\
&\quad-\varepsilon(x_1x_2)B_1(-1)\overline{B_1B_2}(x_1-x_2)F_D^{(n-2)}\left(\left.\begin{matrix}A\overline{B_1B_2}; B_3,\cdots,B_n\\ C\overline{B_1B_2}\end{matrix}\right| x_3,\cdots,x_n\right)\\
&\quad-B_1(x_2)B_2(-x_1)\overline{B_1B_2}(x_2-x_1)F_D^{(n-2)}\left(\left.\begin{matrix}A; B_3,\cdots,B_n\\ C\end{matrix}\right| x_3,\cdots,x_n\right).
\end{aligned}$$

Proof. It is easily known from the binomial theorem over finite fields that for $u, x_1 \in \mathbb{F}_q^*$, we have

$$\chi(ux_1+(1-u)x_2)=\frac{1}{q-1}\sum_{\chi_1}\binom{\chi}{\chi_1}\chi_1(ux_1)\chi\overline{\chi_1}((1-u)x_2).$$

Then

$$\begin{aligned}
&\varepsilon(x_1)\sum_u B_1(u)B_2(1-u)\chi(ux_1+(1-u)x_2)\\
&=\frac{1}{q-1}\sum_{\chi_1}\binom{\chi}{\chi_1}\chi_1(x_1)\chi\overline{\chi_1}(x_2)\sum_u B_1\chi_1(u)B_2\chi\overline{\chi_1}(1-u)\\
&=\frac{1}{q-1}\sum_{\chi_1}\binom{\chi}{\chi_1}\chi_1(x_1)\chi\overline{\chi_1}(x_2)B_2\chi\chi_1(-1)\binom{B_1\chi_1}{\overline{B_2}\overline{\chi}\chi_1}.
\end{aligned}$$

We use the above identity in the summation $\sum_u$ and replace $\chi\overline{\chi_1}$ by χ_2 to find

(3.1)
$$\begin{aligned}
&\varepsilon(x_1x_2)\sum_u B_1(u)B_2(1-u)F_D^{(n-1)}\left(\begin{matrix}A;B_1B_2,B_3,\cdots,B_n\\ C\end{matrix}\middle| ux_1+(1-u)x_2,x_{3,}\cdots,x_n\right)\\
&=\frac{\varepsilon(x_1x_2)}{(q-1)^{n-1}}\sum_{\chi,\chi_3,\cdots,\chi_n}\binom{A\chi\chi_3\cdots\chi_n}{C\chi\chi_3\cdots\chi_n}\binom{B_1B_2\chi}{\chi}\binom{B_3\chi_3}{\chi_3}\cdots\binom{B_n\chi_n}{\chi_n}\chi_3(x_3)\cdots\chi_n(x_n)\\
&\cdot\sum_u B_1(u)B_2(1-u)\chi(ux_1+(1-u)x_2)\\
&=\frac{1}{(q-1)^n}\sum_{\chi_1,\cdots,\chi_n}B_2\chi_2(-1)\binom{B_1\chi_1}{\overline{B_2}\overline{\chi_2}}\binom{B_1B_2\chi_1\chi_2}{\chi_1\chi_2}\binom{\chi_1\chi_2}{\chi_1}\\
&\cdot\binom{A\chi_1\cdots\chi_n}{C\chi_1\cdots\chi_n}\binom{B_3\chi_3}{\chi_3}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_n(x_n).
\end{aligned}$$

It follows from Proposition 2.2 that

$$\begin{aligned}\binom{B_1B_2\chi_1\chi_2}{\chi_1\chi_2}\binom{\chi_1\chi_2}{\chi_1}=\binom{B_1B_2\chi_1\chi_2}{\chi_1}\binom{B_1B_2\chi_2}{\chi_2}&-(q-1)\chi_1(-1)\delta(\chi_1\chi_2)\\
&+(q-1)\chi_2(-1)\delta(B_1B_2\chi_2),\end{aligned}$$

$$\binom{B_1\chi_1}{B_1B_2\chi_1\chi_2}\binom{B_1B_2\chi_1\chi_2}{\chi_1}=\binom{B_1\chi_1}{\chi_1}\binom{B_1}{B_1B_2\chi_2}-(q-1)\chi_1(-1)\delta(B_1B_2\chi_1\chi_2)$$

and

$$\binom{B_2\chi_2}{B_1B_2\chi_2}\binom{B_1B_2\chi_2}{\chi_2}=\binom{B_2\chi_2}{\chi_2}\binom{B_2}{B_1B_2}-(q-1)\chi_2(-1)\delta(B_1B_2\chi_2).$$

Then, by (2.1) and (2.2)

$$\begin{aligned}
&\binom{B_1\chi_1}{\overline{B_2}\overline{\chi_2}}\binom{B_1B_2\chi_1\chi_2}{\chi_1\chi_2}\binom{\chi_1\chi_2}{\chi_1}=\binom{B_1\chi_1}{B_1B_2\chi_1\chi_2}\binom{B_1B_2\chi_1\chi_2}{\chi_1}\binom{B_1B_2\chi_2}{\chi_2}\\
&-(q-1)\chi_1(-1)\delta(\chi_1\chi_2)\binom{B_1\chi_1}{B_1B_2\chi_1\chi_2}+(q-1)\chi_2(-1)\delta(B_1B_2\chi_2)\binom{B_1\chi_1}{B_1B_2\chi_1\chi_2}\\
&=B_1B_2\chi_2(-1)\binom{B_1\chi_1}{\chi_1}\binom{B_2\chi_2}{B_1B_2\chi_2}\binom{B_1B_2\chi_2}{\chi_2}-(q-1)\chi_1(-1)\delta(B_1B_2\chi_1\chi_2)\binom{B_1B_2\chi_2}{\chi_2}\\
&-(q-1)\chi_1(-1)\delta(\chi_1\chi_2)\binom{B_1\chi_1}{B_1B_2\chi_1\chi_2}+(q-1)\chi_2(-1)\delta(B_1B_2\chi_2)\binom{B_1\chi_1}{B_1B_2\chi_1\chi_2}\\
&=B_2\chi_2(-1)\binom{B_1\chi_1}{\chi_1}\binom{B_2\chi_2}{\chi_2}\binom{\overline{B_1B_2}}{\overline{B_1}}-(q-1)\chi_1(-1)\delta(B_1B_2\chi_1\chi_2)\binom{B_1B_2\chi_2}{\chi_2}\\
&-(q-1)\chi_1(-1)\delta(\chi_1\chi_2)\binom{B_1\chi_1}{B_1B_2\chi_1\chi_2},
\end{aligned}$$

where in the last step we have cancelled two terms

$$-(q-1)\delta(B_1B_2\chi_2)B_1B_2(-1)\binom{B_1\chi_1}{\chi_1} \text{ and } (q-1)\chi_2(-1)\delta(B_1B_2\chi_2)\binom{B_1\chi_1}{B_1B_2\chi_1\chi_2}.$$

Applying the above identity in (3.1), we obtain

(3.2)

$$\begin{aligned}
&\varepsilon(x_1x_2)\sum_u B_1(u)B_2(1-u)F_D^{(n-1)}\left(\begin{matrix}A;B_1B_2,B_3,\cdots,B_n\\ C\end{matrix}\middle| ux_1+(1-u)x_2,x_{3,}\cdots,x_n\right)\\
&=\binom{\overline{B_1B_2}}{\overline{B_1}}\frac{1}{(q-1)^n}\sum_{\chi_1,\cdots,\chi_n}\binom{A\chi_1\cdots\chi_n}{C\chi_1\cdots\chi_n}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_n(x_n)\\
&-\frac{\overline{B_1}(-x_1)\overline{B_2}(x_1)}{(q-1)^n}\sum_{\chi_3,\cdots,\chi_n}\binom{A\overline{B_1B_2}\chi_3\cdots\chi_n}{C\overline{B_1B_2}\chi_3\cdots\chi_n}\binom{B_3\chi_3}{\chi_3}\cdots\binom{B_n\chi_n}{\chi_n}\chi_3(x_3)\cdots\chi_n(x_n)\\
&\cdot\sum_{\chi_2}\binom{B_1B_2\chi_2}{\chi_2}\chi_2\left(\frac{x_2}{x_1}\right)\\
&-\frac{B_2(-1)\varepsilon(x_2)}{(q-1)^{n-1}}\sum_{\chi_3,\cdots,\chi_n}\binom{A\chi_3\cdots\chi_n}{C\chi_3\cdots\chi_n}\binom{B_3\chi_3}{\chi_3}\cdots\binom{B_n\chi_n}{\chi_n}\chi_3(x_3)\cdots\chi_n(x_n)\\
&\cdot\sum_{\chi_1}\binom{B_1\chi_1}{\overline{B_2}\chi_1}\chi_1\left(\frac{x_1}{x_2}\right).
\end{aligned}$$

From [8, (3.11)] we know that for any $A,\ B\in\widehat{\mathbb{F}}_q$ and $x\in\mathbb{F}_q$,

$$\sum_\chi\binom{A\chi}{B\chi}\chi(x)=\overline{B}(x)\sum_\chi\binom{A\overline{B}\chi}{\chi}\chi(x)=(q-1)\overline{B}(x)\overline{A}B(1-x). \tag{3.3}$$

Using (3.3) in (3.2) and simplifying yields

$$\begin{aligned}
&\varepsilon(x_1x_2)\sum_u B_1(u)B_2(1-u)F_D^{(n-1)}\left(\begin{matrix}A;B_1B_2,B_3,\cdots,B_n\\ C\end{matrix}\middle| ux_1+(1-u)x_2,x_{3,}\cdots,x_n\right)\\
&=\binom{\overline{B_1B_2}}{\overline{B_1}}F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ C\end{matrix}\middle| x_1,\cdots,x_n\right)\\
&-\varepsilon(x_1x_2)B_1(-1)\overline{B_1B_2}(x_1-x_2)F_D^{(n-2)}\left(\begin{matrix}A\overline{B_1B_2};B_3,\cdots,B_n\\ C\overline{B_1B_2}\end{matrix}\middle| x_3,\cdots,x_n\right)\\
&-B_1(x_2)B_2(-x_1)\overline{B_1B_2}(x_2-x_1)F_D^{(n-2)}\left(\begin{matrix}A;B_3,\cdots,B_n\\ C\end{matrix}\middle| x_3,\cdots,x_n\right).
\end{aligned}$$

This concludes the proof of Theorem 3.2. □

Putting $n=2$ in Theorem 3.2, we arrive at

Corollary 3.2. *For $A, C \in \widehat{\mathbb{F}}_q$, $B, B' \in \widehat{\mathbb{F}}_q \backslash \{\varepsilon\}$ and $x, y \in \mathbb{F}_q$, we have*

$$\begin{aligned}&\varepsilon(xy)\sum_u B(u)B'(1-u)_2F_1\left(\left.\begin{matrix}BB', A\\ C\end{matrix}\right| ux+(1-u)y\right)\\ &=\binom{\overline{BB'}}{\overline{B}}F_1(A;B,B';C;x,y)-\varepsilon(xy)B(-1)\overline{BB'}(x-y)\\ &-B(y)B'(-x)\overline{BB'}(y-x).\end{aligned}$$

Using the binomial theorem in the integral representation for the Lauricella hypergeometric series and then simplifying, we can get a summation formula connecting $F_D^{(n)}$ and $F_D^{(n-1)}$.

Theorem 3.3. *If $a, b_1, \cdots, b_n, c$ and $x_1, \cdots, x_n$ are complex numbers with $Re(a) > 0$ and $Re(c-a) > 0$, then*

$$F_D^{(n)}\left(\left.\begin{matrix}a; b_1, \cdots, b_n\\ c\end{matrix}\right| x_1, \cdots, x_n\right)=\sum_{k=0}^{\infty}\frac{(a)_k(b_n)_k}{k!(c)_k}x_n^kF_D^{(n-1)}\left(\left.\begin{matrix}a+k; b_1, \cdots, b_{n-1}\\ c+k\end{matrix}\right| x_1, \cdots, x_{n-1}\right).$$

A finite field analogue for the above theorem also holds.

Theorem 3.4. *For $A, B_1, \cdots, B_n, C \in \widehat{\mathbb{F}}_q$ and $x_1, \cdots, x_n \in \mathbb{F}_q$, we have*

$$\begin{aligned}&F_D^{(n)}\left(\left.\begin{matrix}A; B_1, \cdots, B_n\\ C\end{matrix}\right| x_1, \cdots, x_n\right)\\ &=\frac{1}{q-1}\sum_\chi\binom{B_n\chi}{\chi}\chi(x_n)F_D^{(n-1)}\left(\left.\begin{matrix}A\chi; B_1, \cdots, B_{n-1}\\ C\chi\end{matrix}\right| x_1, \cdots, x_{n-1}\right).\end{aligned}$$

Proof. It follows from the binomial theorem over finite fields that

$$\overline{B_n}(1-x_nu)=\delta(x_nu)+\frac{1}{q-1}\sum_\chi\binom{B_n\chi}{\chi}\chi(x_nu).$$

Using the above identity in the definition of the Lauricella hypergeometric series over finite fields and by the fact that $\varepsilon(x_n)A(u)\delta(x_nu)=0$, we have

$$\begin{aligned}&F_D^{(n)}\left(\left.\begin{matrix}A; B_1, \cdots, B_n\\ C\end{matrix}\right| x_1, \cdots, x_n\right)\\ &=\frac{\varepsilon(x_1\cdots x_{n-1})AC(-1)}{q-1}\sum_\chi\binom{B_n\chi}{\chi}\chi(x_nu)\\ &\cdot\sum_u A(u)\overline{A}C(1-u)\overline{B_1}(1-x_1u)\cdots\overline{B_{n-1}}(1-x_{n-1}u)\\ &=\frac{1}{q-1}\sum_\chi\binom{B_n\chi}{\chi}\chi(x_n)\varepsilon(x_1\cdots x_{n-1})AC(-1)\\ &\cdot\sum_u A\chi(u)\overline{A}C(1-u)\overline{B_1}(1-x_1u)\cdots\overline{B_{n-1}}(1-x_{n-1}u),\end{aligned}$$

from which the result follows. This finishes the proof of Theorem 3.4. □

We set $n = 2$ in Theorem 3.4 to get

Corollary 3.3. *For $A, B, B', C \in \widehat{\mathbb{F}}_q$ and $x, y \in \mathbb{F}_q$, we have*

$$F_1(A; B, B'; C; x, y) = \frac{1}{q-1} \sum_{\chi} \binom{B'\chi}{\chi} \chi(y) {}_2F_1\left(\begin{matrix} B, A\chi \\ C\chi \end{matrix} \middle| x \right).$$

[11, Corollary 1.1, (1.5)] is a special case of Corollary 3.3.

4. Reduction and Transformation formulae

In this section we give some reduction and transformation formulae for the Lauricella hypergeometric series over finite fields.

From the definition of the Lauricella hypergeometric series $F_D^{(n)}$, we know that

$$F_D^{(n)}\left(\begin{matrix} a; b_1, \cdots, b_{n-1}, 0 \\ c \end{matrix} \middle| x_1, \cdots, x_n \right) = F_D^{(n-1)}\left(\begin{matrix} a; b_1, \cdots, b_{n-1} \\ c \end{matrix} \middle| x_1, \cdots, x_{n-1} \right).$$

We now give a finite field analogue of the above identity.

Theorem 4.1. *For $A, B_1, \cdots, B_{n-1}, C \in \widehat{\mathbb{F}}_q$ and $x_1, \cdots, x_n \in \mathbb{F}_q$, we have*

$$\begin{aligned} F_D^{(n)}&\left(\begin{matrix} A; B_1, \cdots, B_{n-1}, \varepsilon \\ C \end{matrix} \middle| x_1, \cdots, x_n \right) = \varepsilon(x_n) F_D^{(n-1)}\left(\begin{matrix} A; B_1, \cdots, B_{n-1} \\ C \end{matrix} \middle| x_1, \cdots, x_{n-1} \right) \\ &- \varepsilon(x_1 \cdots x_{n-1}) B_1 \cdots B_{n-1} \overline{C}(x_n) \overline{A}C(1-x_n) \overline{B_1}(x_n - x_1) \cdots \overline{B_{n-1}}(x_n - x_{n-1}). \end{aligned}$$

Proof. It is clear that the result holds for $x_n = 0$. We now consider the case $x_n \neq 0$. By (3.3),

$$\sum_{\chi_n} \binom{A\chi_1 \cdots \chi_n}{C\chi_1 \cdots \chi_n} \chi_n(x_n) = (q-1) \overline{C}\overline{\chi_1} \cdots \overline{\chi_{n-1}}(x_n) \overline{A}C(1-x_n), \tag{4.1}$$

which, by (3.3), implies that

$$\begin{aligned} &\sum_{\chi_1, \cdots, \chi_{n-1}} \binom{B_1\chi_1}{\chi_1} \cdots \binom{B_{n-1}\chi_{n-1}}{\chi_{n-1}} \chi_1(x_1) \cdots \chi_{n-1}(x_{n-1}) \sum_{\chi_n} \binom{A\chi_1 \cdots \chi_n}{C\chi_1 \cdots \chi_n} \chi_n(x_n) \\ &= (q-1) \overline{C}(x_n) \overline{A}C(1-x_n) \sum_{\chi_1} \binom{B_1\chi_1}{\chi_1} \chi_1\left(\frac{x_1}{x_n}\right) \cdots \sum_{\chi_{n-1}} \binom{B_{n-1}\chi_{n-1}}{\chi_{n-1}} \chi_{n-1}\left(\frac{x_{n-1}}{x_n}\right) \\ &= (q-1)^n \varepsilon(x_1 \cdots x_{n-1}) B_1 \cdots B_{n-1} \overline{C}(x_n) \overline{A}C(1-x_n) \overline{B_1}(x_n - x_1) \cdots \overline{B_{n-1}}(x_n - x_{n-1}). \end{aligned}$$

This, together with Theorem 1.1 and (2.3), gives

$$\begin{aligned}
&F_D^{(n)}\left(\left.\begin{matrix}A; B_1, \cdots, B_{n-1}, \varepsilon\\ C\end{matrix}\right| x_1, \cdots, x_n\right)\\
&= \frac{1}{(q-1)^n}\sum_{\chi_1,\cdots,\chi_n}\binom{A\chi_1\cdots\chi_n}{C\chi_1\cdots\chi_n}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_{n-1}\chi_{n-1}}{\chi_{n-1}}\binom{\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_n(x_n)\\
&= -\frac{1}{(q-1)^n}\sum_{\chi_1,\cdots,\chi_{n-1}}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_{n-1}\chi_{n-1}}{\chi_{n-1}}\chi_1(x_1)\cdots\chi_{n-1}(x_{n-1})\sum_{\chi_n}\binom{A\chi_1\cdots\chi_n}{C\chi_1\cdots\chi_n}\chi_n(x_n)\\
&\ + \frac{1}{(q-1)^{n-1}}\sum_{\chi_1,\cdots,\chi_{n-1}}\binom{A\chi_1\cdots\chi_{n-1}}{C\chi_1\cdots\chi_{n-1}}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_{n-1}\chi_{n-1}}{\chi_{n-1}}\chi_1(x_1)\cdots\chi_{n-1}(x_{n-1})\\
&= F_D^{(n-1)}\left(\left.\begin{matrix}A; B_1, \cdots, B_{n-1}\\ C\end{matrix}\right| x_1, \cdots, x_{n-1}\right)\\
&\ - \varepsilon(x_1\cdots x_{n-1})B_1\cdots B_{n-1}\overline{C}(x_n)\overline{A}C(1-x_n)\overline{B_1}(x_n-x_1)\cdots\overline{B_{n-1}}(x_n-x_{n-1}).
\end{aligned}$$

This finishes the proof of Theorem 4.1. □

When $n = 2$, Theorem 4.1 reduces to [11, Theorem 3.1]. When $n = 1$, Theorem 4.1 reduces to [8, Corollary 3.16, (i)].

It is easily seen from the definition of the Lauricella hypergeometric series $F_D^{(n)}$ that

$$F_D^{(n)}\left(\left.\begin{matrix}a; b_1, \cdots, b_n\\ a\end{matrix}\right| x_1, \cdots, x_n\right) = (1-x_1)^{-b_1}\cdots(1-x_n)^{-b_n}.$$

We also deduce the finite field analogue of the above formula.

Theorem 4.2. *For $A, B_1, \cdots, B_n \in \widehat{\mathbb{F}_q}$ and $x_1, \cdots, x_n \in \mathbb{F}_q$, we have*

$$\begin{aligned}
F_D^{(n)}\left(\left.\begin{matrix}A; B_1, \cdots, B_n\\ A\end{matrix}\right| x_1, \cdots, x_n\right) &= -\varepsilon(x_1\cdots x_n)\overline{B_1}(1-x_1)\cdots\overline{B_n}(1-x_n)\\
&\quad + B_n(-1)\overline{A}(x_n)F_D^{(n-1)}\left(\left.\begin{matrix}A; B_1, \cdots, B_{n-1}\\ A\overline{B_n}\end{matrix}\right| \frac{x_1}{x_n}, \cdots, \frac{x_{n-1}}{x_n}\right).
\end{aligned}$$

Proof. It follows from Theorem 1.1, (2.3), (2.2) and (3.3) that

$$\begin{aligned}
&F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ A\end{matrix}\middle| x_1,\cdots,x_n\right)\\
&=\frac{1}{(q-1)^n}\sum_{\chi_1,\cdots,\chi_n}\binom{A\chi_1\cdots\chi_n}{A\chi_1\cdots\chi_n}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_n(x_n)\\
&=-\frac{1}{(q-1)^n}\sum_{\chi_1}\binom{B_1\chi_1}{\chi_1}\chi_1(x_1)\cdots\sum_{\chi_n}\binom{B_n\chi_n}{\chi_n}\chi_n(x_n)\\
&\ +\frac{1}{(q-1)^{n-1}}\sum_{A\chi_1\cdots\chi_n=\varepsilon}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_n(x_n)\\
&=-\varepsilon(x_1\cdots x_n)\overline{B_1}(1-x_1)\cdots\overline{B_n}(1-x_n)+\frac{B_n(-1)\overline{A}(x_n)}{(q-1)^{n-1}}\\
&\ \cdot\sum_{\chi_1,\cdots,\chi_{n-1}}\binom{A\chi_1\cdots\chi_{n-1}}{A\overline{B_n}\chi_1\cdots\chi_{n-1}}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_{n-1}\chi_{n-1}}{\chi_{n-1}}\chi_1\left(\frac{x_1}{x_n}\right)\cdots\chi_{n-1}\left(\frac{x_{n-1}}{x_n}\right)\\
&=-\varepsilon(x_1\cdots x_n)\overline{B_1}(1-x_1)\cdots\overline{B_n}(1-x_n)\\
&\ +B_n(-1)\overline{A}(x_n)F_D^{(n-1)}\left(\begin{matrix}A;B_1,\cdots,B_{n-1}\\ A\overline{B_n}\end{matrix}\middle|\frac{x_1}{x_n},\cdots,\frac{x_{n-1}}{x_n}\right).
\end{aligned}$$

This concludes the proof of Theorem 4.2. □

Setting $n=2$, we obtain the following result relating the Appell series F_1 over finite fields to the Gaussian hypergeometric series ${}_2F_1$.

Corollary 4.1. *For $A,B,B'\in\widehat{\mathbb{F}}_q$ and $x,y\in\mathbb{F}_q$, we have*

$$\begin{aligned}
F_1(A;B,B';A;x,y)&=-\varepsilon(xy)\overline{B}(1-x)\overline{B'}(1-y)+B'(-1)\overline{A}(y){}_2F_1\left(\begin{matrix}B,A\\ A\overline{B'}\end{matrix}\middle|\frac{x}{y}\right)\\
&=-\varepsilon(xy)\overline{B}(1-x)\overline{B'}(1-y)+B(-1)\overline{A}(x){}_2F_1\left(\begin{matrix}B',A\\ A\overline{B}\end{matrix}\middle|\frac{y}{x}\right).
\end{aligned}$$

When $n=1$, Theorem 4.2 reduces to [8, Corollary 3.16, (iv)].

The following theorem involves a transformation formula for the Lauricella hypergeometric series over finite fields.

Theorem 4.3. *For $A,B_1,\cdots,B_n,C\in\widehat{\mathbb{F}}_q$, $x_1,\cdots,x_n\in\mathbb{F}_q$, we have*

$$\begin{aligned}
&\varepsilon((1-x_1)\cdots(1-x_n))F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ C\end{matrix}\middle| x_1,\cdots,x_n\right)\\
&=\varepsilon(x_1\cdots x_n)B_1\cdots B_n(-1)F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ AB_1\cdots B_n\overline{C}\end{matrix}\middle|1-x_1,\cdots,1-x_n\right).
\end{aligned}$$

Proof. Making the substitution $u = \frac{v}{v-1}$ in the definition of the Lauricella hypergeometric series over finite fields gives

$$\begin{aligned}
&\varepsilon((1-x_1)\cdots(1-x_n))F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ C\end{matrix}\middle| x_1,\cdots,x_n\right)\\
&=\varepsilon(x_1\cdots x_n(1-x_1)\cdots(1-x_n))C(-1)\\
&\cdot\sum_v A(v)B_1\cdots B_n\overline{C}(1-v)\overline{B_1}(1-(1-x_1)v)\cdots\overline{B_n}(1-(1-x_n)v)\\
&=\varepsilon(x_1\cdots x_n)B_1\cdots B_n(-1)F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ AB_1\cdots B_n\overline{C}\end{matrix}\middle| 1-x_1,\cdots,1-x_n\right).
\end{aligned}$$

This completes the proof of Theorem 4.3. □

Taking $n=2$ in Theorem 4.3, we deduce a transformation formula for the Appell series F_1 over finite fields.

Corollary 4.2. *For $A,B,B',C\in\widehat{\mathbb{F}}_q$, $x,y\in\mathbb{F}_q$, we have*

$$\varepsilon((1-x)(1-y))F_1(A;B,B';C;x,y)=\varepsilon(xy)BB'(-1)F_1(A;B,B';ABB'\overline{C};1-x,1-y).$$

When $n=1$ and $x=x_1\in\mathbb{F}_q\backslash\{0,1\}$, Theorem 4.3 reduces to [8, Theorem 4.4, (i)].

From the integral representation for the Lauricella hypergeometric series $F_D^{(n)}$ we can easily obtain

$$\begin{aligned}
&F_D^{(n)}\left(\begin{matrix}a;b_1,\cdots,b_n\\ c\end{matrix}\middle| x_1,\cdots,x_n\right)\\
&=(1-x_1)^{-b_1}\cdots(1-x_n)^{-b_n}F_D^{(n)}\left(\begin{matrix}c-a;b_1,\cdots,b_n\\ c\end{matrix}\middle| \frac{x_1}{x_1-1},\cdots,\frac{x_n}{x_n-1}\right).
\end{aligned}$$

We give a transformation formula for the Lauricella hypergeometric series over finite fields which can be regarded as the finite field analogue of the above identity.

Theorem 4.4. *For $A,B_1,\cdots,B_n,C\in\widehat{\mathbb{F}}_q$ and $x_1,\cdots,x_n\in\mathbb{F}_q\backslash\{1\}$, we have*

$$\begin{aligned}
&F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ C\end{matrix}\middle| x_1,\cdots,x_n\right)\\
&=C(-1)\overline{B_1}(1-x_1)\cdots\overline{B_n}(1-x_n)F_D^{(n)}\left(\begin{matrix}\overline{A}C;B_1,\cdots,B_n\\ C\end{matrix}\middle| \frac{x_1}{x_1-1},\cdots,\frac{x_n}{x_n-1}\right).
\end{aligned}$$

Proof. The result follows from the definition of the Lauricella hypergeometric series over finite fields and Making the substitution $u=1-v$. □

When $n=2$, Theorem 4.4 reduces to [11, Theorem 3.2, (3.6)]. When $n=1$, Theorem 4.4 reduces to [8, Theorem 4.4, (ii)] for $x\neq 1$.

Theorem 4.5. *For $A,B_1,\cdots,B_n,C\in\widehat{\mathbb{F}}_q$, $x_1,\cdots,x_{n-1}\in\mathbb{F}_q$ and $x_n\in\mathbb{F}_q\backslash\{1\}$, we have*

$$\begin{aligned}
&\varepsilon((x_n-x_1)\cdots(x_n-x_{n-1}))F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ C\end{matrix}\middle| x_1,\cdots,x_n\right)\\
&=\varepsilon(x_1\cdots x_{n-1})\overline{A}(1-x_n)F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_{n-1},\overline{B_1}\cdots\overline{B_n}C\\ C\end{matrix}\middle| \frac{x_n-x_1}{x_n-1},\cdots,\frac{x_n-x_{n-1}}{x_n-1},\frac{x_n}{x_n-1}\right).
\end{aligned}$$

Proof. Making the substitution $u=\frac{v}{1-x_n+x_nv}$ in the the definition of the Lauricella hypergeometric series over finite fields, we have

$$\begin{aligned}
&\varepsilon((x_n-x_1)\cdots(x_n-x_{n-1}))F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\C\end{matrix}\middle|x_1,\cdots,x_n\right)\\
&=\varepsilon(x_1\cdots x_n(x_n-x_1)\cdots(x_n-x_{n-1}))AC(-1)\sum_u A(u)\overline{A}C(1-u)\overline{B_1}(1-x_1u)\cdots\overline{B_n}(1-x_nu)\\
&=\varepsilon(x_1\cdots x_n(x_n-x_1)\cdots(x_n-x_{n-1}))AC(-1)\overline{A}(1-x_n)\\
&\cdot\sum_v A(v)\overline{A}C(1-v)\overline{B_1}\left(1-\frac{x_n-x_1}{x_n-1}v\right)\cdots\overline{B_{n-1}}\left(1-\frac{x_n-x_{n-1}}{x_n-1}v\right)B_1\cdots B_n\overline{C}\left(1-\frac{x_n}{x_n-1}v\right)\\
&=\varepsilon(x_1\cdots x_{n-1})\overline{A}(1-x_n)F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_{n-1},\overline{B_1}\cdots\overline{B_n}C\\C\end{matrix}\middle|\frac{x_n-x_1}{x_n-1},\cdots,\frac{x_n-x_{n-1}}{x_n-1},\frac{x_n}{x_n-1}\right),
\end{aligned}$$

from which we complete the proof of Theorem 4.5. □

From Theorem 4.5 and Theorem 4.1, we can easily obtain the following reduction formula for the Lauricella hypergeometric series over finite fields.

Corollary 4.3. *For $A,B_1,\cdots,B_n,\in\widehat{\mathbb{F}}_q$, $x_1,\cdots,x_{n-1}\in\mathbb{F}_q$ and $x_n\in\mathbb{F}_q\backslash\{1\}$, we have*

$$\begin{aligned}
&\varepsilon((x_n-x_1)\cdots(x_n-x_{n-1}))F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\B_1\cdots B_n\end{matrix}\middle|x_1,\cdots,x_n\right)\\
&=\varepsilon(x_1\cdots x_n)\overline{A}(1-x_n)F_D^{(n-1)}\left(\begin{matrix}A;B_1,\cdots,B_{n-1}\\B_1\cdots B_n\end{matrix}\middle|\frac{x_n-x_1}{x_n-1},\cdots,\frac{x_n-x_{n-1}}{x_n-1}\right)\\
&-\varepsilon((x_n-x_1)\cdots(x_n-x_{n-1}))\overline{B_1}(-x_1)\cdots\overline{B_n}(-x_n).
\end{aligned}$$

Actually, the formula in Corollary 4.3 can be considered as a finite field analogue of the following reduction formula for the Lauricella hypergeometric series (see G. Mingari Scarpello and D. Ritelli [13]):

$$F_D^{(n)}\left(\begin{matrix}a;b_1,\cdots,b_n\\b_1+\cdots+b_n\end{matrix}\middle|x_1,\cdots,x_n\right)=\frac{1}{(1-x_n)^a}F_D^{(n-1)}\left(\begin{matrix}a;b_1,\cdots,b_{n-1}\\b_1+\cdots+b_n\end{matrix}\middle|\frac{x_1-x_n}{1-x_n},\cdots,\frac{x_{n-1}-x_n}{1-x_n}\right).$$

When $n=2$, Theorem 4.5 reduces to [11, Theorem 3.2, (3.7) and (3.9)]. When $n=1$, Theorem 4.5 reduces to [8, Theorem 4.4, (iii)] for $x\neq1$.

Theorem 4.6. *For $A,B_1,\cdots,B_n,C\in\widehat{\mathbb{F}}_q$ and $x_1,\cdots,x_n\in\mathbb{F}_q\backslash\{1\}$, we have*

$$\begin{aligned}
&\varepsilon((x_n-x_1)\cdots(x_n-x_{n-1}))F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\C\end{matrix}\middle|x_1,\cdots,x_n\right)\\
&=\varepsilon(x_1\cdots x_{n-1})C(-1)\overline{AB_n}C(1-x_n)\overline{B_1}(1-x_1)\cdots\overline{B_{n-1}}(1-x_{n-1})\\
&\cdot F_D^{(n)}\left(\begin{matrix}\overline{A}C;B_1,\cdots,B_{n-1},\overline{B_1}\cdots\overline{B_n}C\\C\end{matrix}\middle|\frac{x_n-x_1}{1-x_1},\cdots,\frac{x_n-x_{n-1}}{1-x_{n-1}},x_n\right)
\end{aligned}$$

Proof. Making another substitution $u = \frac{1-v}{1-vx_n}$ in the the definition of the Lauricella hypergeometric series over finite fields, we get

$$
\begin{aligned}
&\varepsilon((x_n-x_1)\cdots(x_n-x_{n-1}))F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ C\end{matrix}\middle| x_1,\cdots,x_n\right)\\
&=\varepsilon(x_1\cdots x_n(x_n-x_1)\cdots(x_n-x_{n-1}))AC(-1)\overline{AB_n}C(1-x_n)\overline{B_1}(1-x_1)\cdots\overline{B_{n-1}}(1-x_{n-1})\\
&\cdot\sum_v \overline{A}C(v)A(1-v)\overline{B_1}\left(1-\frac{x_n-x_1}{1-x_1}v\right)\cdots\overline{B_{n-1}}\left(1-\frac{x_n-x_{n-1}}{1-x_{n-1}}v\right)B_1\cdots B_n\overline{C}(1-x_nv)\\
&=\varepsilon(x_1\cdots x_{n-1})C(-1)\overline{AB_n}C(1-x_n)\overline{B_1}(1-x_1)\cdots\overline{B_{n-1}}(1-x_{n-1})\\
&\cdot F_D^{(n)}\left(\begin{matrix}\overline{A}C;B_1,\cdots,B_{n-1},\overline{B_1}\cdots\overline{B_n}C\\ C\end{matrix}\middle| \frac{x_n-x_1}{1-x_1},\cdots,\frac{x_n-x_{n-1}}{1-x_{n-1}},x_n\right).
\end{aligned}
$$

This completes the proof of Theorem 4.6. □

Similarly, we can get another reduction formula.

Corollary 4.4. *For $A, B_1, \cdots, B_n \in \widehat{\mathbb{F}}_q$ and $x_1, \cdots, x_n \in \mathbb{F}_q\backslash\{1\}$, we have*

$$
\begin{aligned}
&\varepsilon((x_n-x_1)\cdots(x_n-x_{n-1}))F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ B_1\cdots B_n\end{matrix}\middle| x_1,\cdots,x_n\right)\\
&=\varepsilon(x_1\cdots x_n)B_1\cdots B_n(-1)\overline{A}B_1\cdots B_{n-1}(1-x_n)\overline{B_1}(1-x_1)\cdots\overline{B_{n-1}}(1-x_{n-1})\\
&\cdot F_D^{(n-1)}\left(\begin{matrix}\overline{A}B_1\cdots B_n;B_1,\cdots,B_{n-1}\\ B_1\cdots B_n\end{matrix}\middle| \frac{x_n-x_1}{1-x_1},\cdots,\frac{x_n-x_{n-1}}{1-x_{n-1}}\right)\\
&-\varepsilon((x_n-1)(x_n-x_1)\cdots(x_n-x_{n-1}))\overline{B_1}(-x_1)\cdots\overline{B_n}(-x_n).
\end{aligned}
$$

When $n = 2$, Theorem 4.6 reduces to [11, (3.8) and (3.10)]. When $n = 1$, Theorem 4.6 reduces to [8, Theorem 4.4, (iv)] for $x \neq 1$.

5. Evaluations

In this section we give some evaluations for the Lauricella hypergeometric series over finite fields.

From the definition of the Lauricella hypergeometric series over finite fields, we can easily deduce the following results.

Theorem 5.1. *For $A, B_1, \cdots, B_n, C \in \widehat{\mathbb{F}}_q$ and $x, x_1, \cdots, x_{n-1} \in \mathbb{F}_q$, we have*

$$
F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ C\end{matrix}\middle| x,\cdots,x\right) = {}_2F_1\left(\begin{matrix}B_1\cdots B_n, A\\ C\end{matrix}\middle| x\right),
$$

$$
F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ C\end{matrix}\middle| x_1,\cdots,x_{n-1},1\right) = B_n(-1)F_D^{(n-1)}\left(\begin{matrix}A;B_1,\cdots,B_{n-1}\\ \overline{B_n}C\end{matrix}\middle| x_1,\cdots,x_{n-1}\right).
$$

In particular,

$$
F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ C\end{matrix}\middle| 1,\cdots,1\right) = B_1\cdots B_n(-1)\binom{A}{\overline{B_1}\cdots\overline{B_n}C}. \tag{5.1}
$$

When $n=1$, (5.1) reduces to [8, Theorem 4.9]:

$$ {}_2F_1\left(\begin{matrix}A,B\\ C\end{matrix}\middle|1\right)=A(-1)\binom{B}{\overline{A}C}. \tag{5.2}$$

Corollary 5.1. *For $A,B_1,\cdots,B_n\in\widehat{\mathbb{F}}_q$ and $x\in\mathbb{F}_q$, we have*

$$F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ A\end{matrix}\middle|x,\cdots,x\right)=-\varepsilon(x)\overline{B_1}\cdots\overline{B_n}(1-x)+B_1\cdots B_n(-1)\overline{A}(x)\binom{A}{B_1\cdots B_n}.$$

Proof. We can take $x_1=\cdots=x_n=x$ in Theorem 4.2 and using (5.1) to get the result. Alternatively, we take $C=A$ in the first identity of Theorem 5.1 and use (2.1)–(2.3) and (3.3):

$$\begin{aligned}F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ A\end{matrix}\middle|x,\cdots,x\right)&={}_2F_1\left(\begin{matrix}B_1\cdots B_n,A\\ A\end{matrix}\middle|x\right)\\ &=\frac{1}{q-1}\sum_{\chi}\binom{B_1\cdots B_n\chi}{\chi}\chi(x)+\overline{A}(x)\binom{\overline{A}B_1\cdots B_n}{\overline{A}}\\ &=-\varepsilon(x)\overline{B_1}\cdots\overline{B_n}(1-x)+B_1\cdots B_n(-1)\overline{A}(x)\binom{A}{B_1\cdots B_n}\end{aligned}$$

to complete the proof of Corollary 5.1. □

Setting $n=2$ in Corollary 5.1, we are led to

Corollary 5.2. *For $A,B,B'\in\widehat{\mathbb{F}}_q$ and $x\in\mathbb{F}_q$, we have*

$$F_1(A;B,B';A;x,x)=-\varepsilon(x)\overline{BB'}(1-x)+BB'(-1)\overline{A}(x)\binom{A}{BB'}.$$

From Theorem 5.1 we can obtain another result.

Corollary 5.3. *For $A,B_1,\cdots,B_n\in\widehat{\mathbb{F}}_q$ and $x\in\mathbb{F}_q$, we have*

$$\begin{aligned}F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ B_1\cdots B_n\end{matrix}\middle|x,\cdots,x\right)=&\binom{A}{B_1\cdots B_n}\varepsilon(x)\overline{A}(1-x)-\overline{B_1}\cdots\overline{B_n}(-x)\\ &+(q-1)B_1\cdots B_n(-1)\delta(1-x)\delta(A).\end{aligned}$$

Proof. It follows from [8, Corollary 3.16, (iii)] that

$${}_2F_1\left(\begin{matrix}A,B\\ A\end{matrix}\middle|x\right)=\binom{B}{A}\varepsilon(x)\overline{B}(1-x)-\overline{A}(-x)+(q-1)A(-1)\delta(1-x)\delta(B)$$

for $A,B\in\widehat{\mathbb{F}}_q$ and $x\in\mathbb{F}_q$. Then, putting $C=B_1\cdots B_n$ in the first identity of Theorem 5.1, we have

$$\begin{aligned}F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\ B_1\cdots B_n\end{matrix}\middle|x,\cdots,x\right)&={}_2F_1\left(\begin{matrix}B_1\cdots B_n,A\\ B_1\cdots B_n\end{matrix}\middle|x\right)\\ &=\binom{A}{B_1\cdots B_n}\varepsilon(x)\overline{A}(1-x)-\overline{B_1}\cdots\overline{B_n}(-x)\\ &\quad+(q-1)B_1\cdots B_n(-1)\delta(1-x)\delta(A),\end{aligned}$$

from which the result follows. □

Letting $n = 2$ in Corollary 5.3 gives

Corollary 5.4. *For $A, B, B' \in \widehat{\mathbb{F}}_q$ and $x \in \mathbb{F}_q$, we have*

$$F_1(A; B, B'; BB'; x, x) = \binom{A}{BB'}\varepsilon(x)\overline{A}(1-x) - \overline{BB'}(-x) + (q-1)BB'(-1)\delta(1-x)\delta(A).$$

6. Generating functions

In this section, we establish several generating functions for the Lauricella hypergeometric series over finite fields.

Theorem 6.1. *For $A, B_1, \cdots, B_n, C \in \widehat{\mathbb{F}}_q, x_1, \cdots, x_n \in \mathbb{F}_q$ and $t \in \mathbb{F}_q \backslash \{1\}$, we have*

$$\begin{aligned}
&\sum_\theta \binom{A\overline{C}\theta}{\theta} F_D^{(n)}\left(\left.\begin{matrix} A\theta; B_1, \cdots, B_n \\ C \end{matrix}\right| x_1, \cdots, x_n\right)\theta(t) \\
&= \varepsilon(t)\overline{A}(1-t)F_D^{(n)}\left(\left.\begin{matrix} A; B_1, \cdots, B_n \\ C \end{matrix}\right| \frac{x_1}{1-t}, \cdots, \frac{x_n}{1-t}\right) \\
&\quad - \varepsilon(x_1\cdots x_n)\overline{A}C(-t)\overline{B_1}(1-x_1)\cdots\overline{B_n}(1-x_n).
\end{aligned}$$

Proof. Making the substitution $u = \frac{v}{1-t}$, we have

$$\begin{aligned}
&\varepsilon(tx_1\cdots x_n)AC(-1)\sum_{u\neq 1} A(u)\overline{A}C(1-u+ut)\overline{B_1}(1-x_1u)\cdots\overline{B_n}(1-x_nu) \\
&= \varepsilon(tx_1\cdots x_n)AC(-1)\sum_{u} A(u)\overline{A}C(1-u+ut)\overline{B_1}(1-x_1u)\cdots\overline{B_n}(1-x_nu) \\
&\quad - \varepsilon(x_1\cdots x_n)\overline{A}C(-t)\overline{B_1}(1-x_1)\cdots\overline{B_n}(1-x_n) \\
&= \varepsilon(tx_1\cdots x_n)AC(-1)\overline{A}(1-t)\sum_{v} A(v)\overline{A}C(1-v)\overline{B_1}\left(1-\frac{x_1}{1-t}v\right)\cdots\overline{B_n}\left(1-\frac{x_n}{1-t}v\right) \\
&\quad - \varepsilon(x_1\cdots x_n)\overline{A}C(-t)\overline{B_1}(1-x_1)\cdots\overline{B_n}(1-x_n) \\
&= \varepsilon(t)\overline{A}(1-t)F_D^{(n)}\left(\left.\begin{matrix} A; B_1, \cdots, B_n \\ C \end{matrix}\right| \frac{x_1}{1-t}, \cdots, \frac{x_n}{1-t}\right) \\
&\quad - \varepsilon(x_1\cdots x_n)\overline{A}C(-t)\overline{B_1}(1-x_1)\cdots\overline{B_n}(1-x_n).
\end{aligned}$$

This combines the binomial theorem over finite fields to yield

$$\begin{aligned}
&\sum_{\theta}\binom{A\overline{C}\theta}{\theta}F_D^{(n)}\left(\begin{matrix}A\theta;B_1,\cdots,B_n\\C\end{matrix}\middle|x_1,\cdots,x_n\right)\theta(t)\\
&=\varepsilon(x_1\cdots x_n)AC(-1)\sum_{\theta,u}\binom{A\overline{C}\theta}{\theta}A(u)\overline{A}C(1-u)\theta(-ut)\overline{\theta}(1-u)\overline{B_1}(1-x_1u)\cdots\overline{B_n}(1-x_nu)\\
&=\varepsilon(x_1\cdots x_n)AC(-1)\sum_{u\neq 1}A(u)\overline{A}C(1-u)\overline{B_1}(1-x_1u)\cdots\overline{B_n}(1-x_nu)\sum_{\theta}\binom{A\overline{C}\theta}{\theta}\theta\left(-\frac{ut}{1-u}\right)\\
&=\varepsilon(tx_1\cdots x_n)AC(-1)\sum_{u\neq 1}A(u)\overline{A}C(1-u)\overline{A}C\left(1+\frac{ut}{1-u}\right)\overline{B_1}(1-x_1u)\cdots\overline{B_n}(1-x_nu)\\
&=\varepsilon(tx_1\cdots x_n)AC(-1)\sum_{u\neq 1}A(u)\overline{A}C(1-u+ut)\overline{B_1}(1-x_1u)\cdots\overline{B_n}(1-x_nu)\\
&=\varepsilon(t)\overline{A}(1-t)F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_n\\C\end{matrix}\middle|\frac{x_1}{1-t},\cdots,\frac{x_n}{1-t}\right)\\
&-\varepsilon(x_1\cdots x_n)\overline{A}C(-t)\overline{B_1}(1-x_1)\cdots\overline{B_n}(1-x_n),
\end{aligned}$$

which ends the proof of Theorem 6.1. □

Theorem 6.1 reduces to [11, Theorem 4.1] when $n=2$.

Setting $n=1$ in Theorem 6.1, we get a generating function for the Gaussian hypergeometric series ${}_2F_1$.

Corollary 6.1. *For $A,B,C\in\widehat{\mathbb{F}}_q, x\in\mathbb{F}_q$ and $t\in\mathbb{F}_q\backslash\{1\}$, we have*

$$\sum_{\theta}\binom{B\overline{C}\theta}{\theta}{}_2F_1\left(\begin{matrix}A,B\theta\\C\end{matrix}\middle|x\right)\theta(t)=\varepsilon(t)\overline{B}(1-t){}_2F_1\left(\begin{matrix}A,B\\C\end{matrix}\middle|\frac{x}{1-t}\right)-\varepsilon(x)\overline{B}C(-t)\overline{A}(1-x).$$

We also give another generating function for the Lauricella hypergeometric series over finite fields.

Theorem 6.2. *For $A,B_1,\cdots,B_n,C\in\widehat{\mathbb{F}}_q$ and $x_1,\cdots,x_n,t\in\mathbb{F}_q$, we have*

$$\begin{aligned}
&\sum_{\theta}\binom{B_n\theta}{\theta}F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_{n-1},B_n\theta\\C\end{matrix}\middle|x_1,\cdots,x_n\right)\theta(t)\\
&=(q-1)\varepsilon(t)\overline{B_n}(1-t)F_D^{(n)}\left(\begin{matrix}A;B_1,\cdots,B_{n-1},B_n\\C\end{matrix}\middle|x_1,\cdots,x_{n-1},\frac{x_n}{1-t}\right)\\
&-(q-1)\varepsilon(x_1\cdots x_{n-1})\overline{B_n}(-t)B_1\cdots B_{n-1}\overline{C}(x_n)\overline{A}C(1-x_n)\overline{B_1}\left(x_n-x_1\right)\cdots\overline{B_{n-1}}\left(x_n-x_{n-1}\right).
\end{aligned}$$

Proof. It is obvious that the result holds for $x_n=0$. We now consider the case $x_n\neq 0$. It follows from [8, Corollary 3.16, (iii)] that

$$\sum_{\theta}\binom{B_n\theta}{\theta}\binom{B_n\chi_n\theta}{B_n\theta}\theta(t)=(q-1)\left(\varepsilon(t)\overline{B_n}\overline{\chi_n}(1-t)\binom{B_n\chi_n}{\chi_n}-\overline{B_n}(-t)\right).$$

Then, by (4.1),

$$\begin{aligned}
&\sum_{\theta}\binom{B_n\theta}{\theta}F_D^{(n)}\left(\left.\begin{matrix}A;B_1,\cdots,B_{n-1},B_n\theta\\ C\end{matrix}\right|x_1,\cdots,x_n\right)\theta(t)\\
&=\frac{1}{(q-1)^n}\sum_{\chi_1,\cdots,\chi_n}\binom{A\chi_1\cdots\chi_n}{C\chi_1\cdots\chi_n}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_{n-1}\chi_{n-1}}{\chi_{n-1}}\chi_1(x_1)\cdots\chi_n(x_n)\\
&\cdot\sum_{\theta}\binom{B_n\theta}{\theta}\binom{B_n\chi_n\theta}{B_n\theta}\theta(t)\\
&=\frac{\varepsilon(t)\overline{B_n}(1-t)}{(q-1)^{n-1}}\sum_{\chi_1,\cdots,\chi_n}\binom{A\chi_1\cdots\chi_n}{C\chi_1\cdots\chi_n}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_{n-1}(x_{n-1})\chi_n\left(\frac{x_n}{1-t}\right)\\
&-\frac{\overline{B_n}(-t)}{(q-1)^{n-1}}\sum_{\chi_1,\cdots,\chi_{n-1}}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_{n-1}\chi_{n-1}}{\chi_{n-1}}\chi_1(x_1)\cdots\chi_{n-1}(x_{n-1})\sum_{\chi_n}\binom{A\chi_1\cdots\chi_n}{C\chi_1\cdots\chi_n}\chi_n(x_n)\\
&=(q-1)\varepsilon(t)\overline{B_n}(1-t)F_D^{(n)}\left(\left.\begin{matrix}A;B_1,\cdots,B_{n-1},B_n\\ C\end{matrix}\right|x_1,\cdots,x_{n-1},\frac{x_n}{1-t}\right)\\
&-\frac{\overline{B_n}(-t)\overline{C}(x_n)\overline{A}C(1-x_n)}{(q-1)^{n-2}}\sum_{\chi_1}\binom{B_1\chi_1}{\chi_1}\chi_1\left(\frac{x_1}{x_n}\right)\cdots\sum_{\chi_{n-1}}\binom{B_{n-1}\chi_{n-1}}{\chi_{n-1}}\chi_{n-1}\left(\frac{x_{n-1}}{x_n}\right)\\
&=(q-1)\varepsilon(t)\overline{B_n}(1-t)F_D^{(n)}\left(\left.\begin{matrix}A;B_1,\cdots,B_{n-1},B_n\\ C\end{matrix}\right|x_1,\cdots,x_{n-1},\frac{x_n}{1-t}\right)\\
&-(q-1)\varepsilon(x_1\cdots x_{n-1})\overline{B_n}(-t)\overline{C}(x_n)\overline{A}C(1-x_n)\overline{B_1}\left(1-\frac{x_1}{x_n}\right)\cdots\overline{B_{n-1}}\left(1-\frac{x_{n-1}}{x_n}\right),
\end{aligned}$$

from which the result follows. This finishes the proof of Theorem 6.2. □

When $n=2$, Theorem 6.2 reduces to [11, Theorem 4.2].

Taking $n=1$ in Theorem 6.2, we can easily obtain another generating function for the Gaussian hypergeometric series ${}_2F_1$.

Corollary 6.2. *For $A,B,C\in\widehat{\mathbb{F}}_q$ and $x,t\in\mathbb{F}_q$, we have*

$$\begin{aligned}
\sum_{\theta}\binom{A\theta}{\theta}{}_2F_1\left(\left.\begin{matrix}A\theta,B\\ C\end{matrix}\right|x\right)\theta(t)&=(q-1)\varepsilon(t)\overline{A}(1-t){}_2F_1\left(\left.\begin{matrix}A,B\\ C\end{matrix}\right|\frac{x}{1-t}\right)\\
&-(q-1)\overline{A}(-t)\overline{C}(x)\overline{B}C(1-x).
\end{aligned}$$

The following theorem involves another generating function for the Lauricella hypergeometric series over finite fields.

Theorem 6.3. *For $A, B_1, \cdots, B_n, C \in \widehat{\mathbb{F}}_q$ and $x_1, \cdots, x_n, t \in \mathbb{F}_q$, we have*

$$\begin{aligned}&\sum_{\theta} \binom{A\overline{C}\theta}{\theta} F_D^{(n)}\left(\begin{matrix}A; B_1, \cdots, B_n \\ C\overline{\theta}\end{matrix}\middle| x_1, \cdots, x_n\right)\theta(t)\\ &= (q-1)\varepsilon(t)\overline{C}(1+t)F_D^{(n)}\left(\begin{matrix}A; B_1, \cdots, B_n \\ C\end{matrix}\middle| \frac{x_1}{1+t}, \cdots, \frac{x_n}{1+t}\right)\\ &\quad - (q-1)\overline{A}C(-t)\varepsilon(x_1\cdots x_n)\overline{B_1}(1-x_1)\cdots\overline{B_n}(1-x_n).\end{aligned}$$

Proof. It is easily seen from [8, Corollary 3.16, (iii)] and (2.1), (2.2) that

$$\begin{aligned}\sum_{\theta}\binom{A\overline{C}\theta}{\theta}\binom{\overline{C}\overline{\chi_1}\cdots\overline{\chi_n}\theta}{A\overline{C}\theta}\theta(-t) &= (q-1)\binom{A\chi_1\cdots\chi_n}{C\chi_1\cdots\chi_n}AC(-1)\varepsilon(t)\overline{C}\overline{\chi_1}\cdots\overline{\chi_n}(1+t)\\ &\quad - (q-1)\overline{A}C(t).\end{aligned}$$

Combining (2.1), (2.2) and the above identity, we obtain

$$\begin{aligned}&\sum_{\theta}\binom{A\overline{C}\theta}{\theta}F_D^{(n)}\left(\begin{matrix}A; B_1, \cdots, B_n \\ C\overline{\theta}\end{matrix}\middle| x_1, \cdots, x_n\right)\theta(t)\\ &= \frac{AC(-1)}{(q-1)^n}\sum_{\theta,\chi_1,\cdots,\chi_n}\binom{A\overline{C}\theta}{\theta}\binom{\overline{C}\overline{\chi_1}\cdots\overline{\chi_n}\theta}{A\overline{C}\theta}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_n(x_n)\theta(-t)\\ &= \frac{AC(-1)}{(q-1)^n}\sum_{\chi_1,\cdots,\chi_n}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1(x_1)\cdots\chi_n(x_n)\sum_{\theta}\binom{A\overline{C}\theta}{\theta}\binom{\overline{C}\overline{\chi_1}\cdots\overline{\chi_n}\theta}{A\overline{C}\theta}\theta(-t)\\ &= \frac{\varepsilon(t)\overline{C}(1+t)}{(q-1)^{n-1}}\sum_{\chi_1,\cdots,\chi_n}\binom{A\chi_1\cdots\chi_n}{C\chi_1\cdots\chi_n}\binom{B_1\chi_1}{\chi_1}\cdots\binom{B_n\chi_n}{\chi_n}\chi_1\left(\frac{x_1}{1+t}\right)\cdots\chi_n\left(\frac{x_n}{1+t}\right)\\ &\quad - \frac{\overline{A}C(-t)}{(q-1)^{n-1}}\sum_{\chi_1}\binom{B_1\chi_1}{\chi_1}\chi_1(x_1)\cdots\sum_{\chi_n}\binom{B_n\chi_n}{\chi_n}\chi_n(x_n)\\ &= (q-1)\varepsilon(t)\overline{C}(1+t)F_D^{(n)}\left(\begin{matrix}A; B_1, \cdots, B_n \\ C\end{matrix}\middle| \frac{x_1}{1+t}, \cdots, \frac{x_n}{1+t}\right)\\ &\quad - (q-1)\overline{A}C(-t)\varepsilon(x_1\cdots x_n)\overline{B_1}(1-x_1)\cdots\overline{B_n}(1-x_n).\end{aligned}$$

This concludes the proof of Theorem 6.3. □

Putting $n = 2$ in Theorem 6.3, we get the following result which is a generating function for the finite field analogue of the Appell series F_1.

Corollary 6.3. *For $A, B, B', C \in \widehat{\mathbb{F}}_q$ and $x, y, t \in \mathbb{F}_q$, we have*

$$\begin{aligned}\sum_{\theta}\binom{A\overline{C}\theta}{\theta}F_1(A; B, B'; C\overline{\theta}; x, y)\theta(t) &= (q-1)\varepsilon(t)\overline{C}(1+t)F_1\left(A; B, B'; C; \frac{x}{1+t}, \frac{y}{1+t}\right)\\ &\quad - (q-1)\overline{A}C(-t)\varepsilon(xy)\overline{B}(1-x)\overline{B'}(1-y).\end{aligned}$$

Letting $n = 1$ in Theorem 6.3 yields a generating function for the Gaussian hypergeometric series ${}_2F_1$.

Corollary 6.4. *For $A, B, C \in \widehat{\mathbb{F}}_q$ and $x, t \in \mathbb{F}_q$, we have*

$$\sum_{\theta} \binom{B\overline{C}\theta}{\theta} {}_2F_1\left(\begin{matrix} A, B \\ C\overline{\theta} \end{matrix} \middle| x\right) \theta(t) = (q-1)\varepsilon(t)\overline{C}(1+t) {}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \middle| \frac{x}{1+t}\right) - (q-1)\overline{B}C(-t)\varepsilon(x)\overline{A}(1-x).$$

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