A Relation Between n-square and m-square Matrix Vector Bases of the Same Dimension
- Relating Spin Matrices and Components

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If a vector basis is made up of matrices, the number of matrices must equal the dimension of the basis and they must be linearly independent.

Thus, each of \( \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \) and \( \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \) form a basis of dimension 2.

Since the matrices of these bases are both 2-square, it is a simple matter to determine a transformation function between the two representations of the same space.

Things are more complicated when the basis sets contain square matrices of differing sizes (that is: \( n \)-square and \( m \)-square, \( m \neq n \)).

Define the augmentation function on two matrices with the same number of columns as follows:

\[
\text{aug}(A, B) = \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{where A & B are block matrices of the resulting matrix:}
\]

\[
\begin{pmatrix} A \\ B \end{pmatrix} = \text{aug}(A, B)
\]

Thus, if \( A, B, C \) all have the same number of columns:

\[
\text{aug}(\text{aug}(A, B), C) = \begin{pmatrix} A \\ B \\ C \end{pmatrix}
\]

exists.

Thus, if \( \text{aug}(A, B) \) exists and \( B \) is a square matrix, then \( \text{aug}(A^T, B) \) exists, and:

\[
\text{aug}(A, B) = \begin{pmatrix} A \\ B \end{pmatrix} \implies \text{aug}(A^T, B) = \begin{pmatrix} A^T : B \end{pmatrix}
\]

(since \( A^T \) has the same number of rows as \( B \)).

Similarly:

If \( \text{aug}(A, B) \) exists and \( B \) is a square matrix, then \( \text{aug}(A^T, B) \) exists, and:

\[
\text{aug}(\text{aug}(A, B), C) = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \implies \text{aug}(\text{aug}(A^T, B), C^T) = \begin{pmatrix} A^T : B : C^T \end{pmatrix}
\]

Define: \( \theta_{m \times n} \) as a \( m \times n \)-zero matrix (with matrix entries all zero).

Define \( \theta_n \) as the \( n \)-square zero (additive identity) matrix (with matrix entries all zero).

Define \( I_n \) as the \( n \)-square (multiplicative) identity matrix (\( \delta \)).

Thus, for \( n \)-square matrix \( A : \text{aug}(\theta_{m \times n}; A) = \begin{pmatrix} \theta_{m \times n} \\ A \end{pmatrix} \implies \text{aug}(\theta_{m \times n}; A) = \begin{pmatrix} \theta_{m \times n} : A \end{pmatrix} \)

Similarly:

Thus, if \( \text{aug}(A, B) \) exists and \( A \) is a square matrix, then \( \text{aug}(A, B^T) \) exists, and:

\[
\text{aug}(A, B) = \begin{pmatrix} A \\ B \end{pmatrix} \implies \text{aug}(A, B^T) = \begin{pmatrix} A : B^T \end{pmatrix}
\]

(since \( B^T \) has the same number of rows as \( A \)).

Thus, for \( n \)-square matrix \( A : \text{aug}(A; \theta_{m \times n}) = \begin{pmatrix} A \\ \theta_{m \times n} \end{pmatrix} \implies \text{aug}(A; \theta_{m \times n}) = \begin{pmatrix} A : \theta_{m \times n} \end{pmatrix} \)

Thus, the following matrices may always be formed:

\[
\text{aug}(\theta_{m \times n}; L) = \begin{pmatrix} \theta_{m \times n} \\ L \end{pmatrix}; \quad \text{aug}(\theta_{m \times n}; L) = \begin{pmatrix} \theta_{m \times n} : L \end{pmatrix}
\]

\[
\text{aug}(I_n; \theta_{m \times n}) = \begin{pmatrix} I_n \\ \theta_{m \times n} \end{pmatrix}; \quad \text{aug}(I_n; \theta_{m \times n}) = \begin{pmatrix} I_n : \theta_{m \times n} \end{pmatrix}
\]

**Theorem 1:**

For \( n \)-square matrix basis \( \langle A_i \rangle \) & \( m \)-square matrix basis \( \langle B_j \rangle \) each of dimension \( M \):

a relation exists between the components of bases \( \langle A_i \rangle \) and \( \langle B_j \rangle \).
proof:

Without loss of generality, let: \( n < m \):

\[ n < m \Rightarrow \exists k \in \mathbb{N} \text{ such that: } n + k = m \]

Form the matrices \((n + 1)-\text{square matrices:}\)

\[
\text{aug}(0_{I,n};I_1) = \begin{pmatrix} 0_{1,n} \\ I_1 \end{pmatrix} \quad \text{&} \quad \text{aug}(0_{I,n};I_m) = \begin{pmatrix} 0_{1,n} \\ I_m \end{pmatrix} ; \forall i, 1 \leq i \leq n 
\]

\[
\text{aug}(I_n;0_{I,m}) = \begin{pmatrix} I_n \\ 0_{1,m} \end{pmatrix} \quad \text{&} \quad \text{aug}(I_n;0_{I,n}) = \begin{pmatrix} I_n \\ 0_{1,n} \end{pmatrix} ; \forall i, 1 \leq i \leq n
\]

and the set:

\[
\left\{ \begin{pmatrix} 0_{1,n} \\ I_1 \end{pmatrix} \right\} A_i \left( \begin{pmatrix} 0_{1,n} \\ I_m \end{pmatrix} \right) + \left\{ \begin{pmatrix} I_n \\ 0_{1,m} \end{pmatrix} \right\} A_i \left( \begin{pmatrix} I_n \\ 0_{1,n} \end{pmatrix} \right) ; \forall i, 1 \leq i \leq n
\]

This is a set of \((n + 1)-\text{square matrices; the first term of which has zeros appended to the left column and top row of each } A_i , \text{ and the second term of which has zeros appended to the right column and bottom row of each } A_i \).

Application of this process \(k\)-times results in a set of \((n + k)-\text{square matrices; i.e. } m\)-square matrices. Whenever they form a \(M\)-dimensional basis a transformation exists relating the components of bases \(\langle A_i \rangle\) and \(\langle B_i \rangle \).

Since each of the sets \( \left\{ \begin{pmatrix} 0_{1,n} \\ I_1 \end{pmatrix} \right\} A_i \left( \begin{pmatrix} 0_{1,n} \\ I_m \end{pmatrix} \right) \) and \( \left\{ \begin{pmatrix} I_n \\ 0_{1,m} \end{pmatrix} \right\} A_i \left( \begin{pmatrix} I_n \\ 0_{1,n} \end{pmatrix} \right) \); \( \forall i, 1 \leq i \leq n \)

are each sets of \(m\)-square matrices and \(\langle B_i \rangle \) is a basis of \(m\)-square matrices; then each may be expressed as a linear combination of the base vector matrices of \(\langle B_i \rangle\) - yielding a relationship between \(\langle A_i \rangle\) and \(\langle B_i \rangle \).

\(\square\)

Examples:

example 1:

For \( n = 2 \), using the \(2 \times 2 \text{ SU}(2)\) spin-\(\frac{1}{2}\) Pauli matrices \([1]\), with \( M = 3 \):

\[
S_{1,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \quad S_{2,2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma_2 \quad S_{3,2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3
\]

and the \( n = 3 \) \(3 \times 3\) spin-1 matrices \([1]\), with \( M = 3 \):

\[
S_{1,3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad S_{2,3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_{3,3} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

Performing the theorem 1 operations on the \(S_{1,2}\) yield:

\[
\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}
\]

So:

\[
S_{1,3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \right]
\]

\[
= \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} S_{1,2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} S_{2,2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right]
\]

\[
= \frac{1}{\sqrt{2}} \sum_{i=2}^{3} \begin{pmatrix} \delta_{i,2} \\ \delta_{i,3} \end{pmatrix} S_{1,2} \begin{pmatrix} 0_{1,2} \\ 0_{1,3} \end{pmatrix}
\]

\[
S_{2,3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \right]
\]
Thus, for:

$$Z = S_{1,2}Z^1_+ + S_{1,2}Z^1_+ + S_{1,2}Z^1_+ + S_{1,3}Z^1_+ + S_{1,3}Z^1_+$$

we get:

$$Z = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} Z^1_+ + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
\[
\begin{align*}
&\left[ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} S_{1,2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} S_{3,2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] Z^3 \\
&= S_{1,2} \sqrt{2} Z^1 + S_{3,2} \sqrt{2} Z^3 + S_{1,3} Z^1 + S_{2,3} Z^2 + S_{3,3} Z^3 \\
&\Rightarrow \\
&\begin{cases}
Z_l = \sqrt{2} Z^1 \\
Z_l = \sqrt{2} Z^3 \\
Z_l = Z^1
\end{cases}
\end{align*}
\]

example 2:

For \( n = 2 \), using the \( 2 \times 2 \) SU(2) spin-\( \frac{1}{2} \) Pauli matrices [1], with \( M = 3 \):

\[
S_{1,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad S_{2,2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma_2, \quad S_{3,2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3
\]

and the \( n = 4 \times 4 \) spin-\( \frac{1}{2} \) matrices [1], with \( M = 3 \):

\[
S_{i,4} = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad S_{i,3} = \frac{1}{2} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 0 & -2 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}
\]

Again, performing the theorem 1 operations on the \( S_{i,2} \) yield:

\[
\begin{align*}
&\begin{cases}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{cases} \\
\begin{cases}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{cases} \\
\begin{cases}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{cases}
\end{align*}
\]

Performing them to these, yields:

\[
\begin{align*}
&\begin{cases}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{cases} \\
\begin{cases}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{cases} \\
\begin{cases}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{cases}
\end{align*}
\]

So:

\[
\begin{align*}
S_{i,4} &= \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \sqrt{3} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \sqrt{3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \sqrt{3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]
Thus, a better way is found via a generalization of the above techniques. Any matrix may be constructed by appending \( a_{h,2} \) \( \ldots \) \( I_2 \) \( \ldots \) \( S_{j,2} \) \( (0_{2^r,2} : 1_2 : 0_{2^r}) \) + \( \sum_{h=0}^{h} a_{h,2} \) \( \ldots \) \( I_2 \) \( \ldots \) \( S_{j,2} \) \( (0_{2^r,2} : 1_2 : 0_{2^r}) \)

\[
S_{j,4} = \sum_{h=0}^{h} a_{h,2} \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -3
\end{bmatrix}
+ \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & -3
\end{bmatrix}
+ \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

So, this technique works; but the factors become different than the usual ones obtained from the commutation rules. Thus, a better way is found via a generalization of the above techniques.

Any matrix may be constructed by appending \( n \)-tuples: \( \mathbf{a}_n = (a_1, a_2, \ldots, a_n) \)

\[
aug(\ldots \aug(\aug(a_1, a_2), a_3), \ldots, a_n) = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{bmatrix}
\]

So:

\[
0_{a_{1,2}} \quad \text{and} \quad \begin{bmatrix}
1 \\
0
\end{bmatrix} : 1_{a_{3,2}} : \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

exist.

and, so:

\[
S_{j,4} = \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -3
\end{bmatrix}
+ \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3
\end{bmatrix}
+ \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

So:

\[
S_{2,4} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

So:

\[
S_{2,4} = \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

So:

\[
S_{2,4} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\sum_{n=0}^{\frac{1}{2}} a_n \begin{pmatrix}
\theta_{1,2} \\
(1,0) \\
\vdots \\
\theta_{2,2:n-2} \\
(0,1) \\
\theta_{n,2}
\end{pmatrix} = S_{n,2} \left( \theta_{1,0} : (1,0) : \theta_{2,2:n-2} : (0,1) : \theta_{2,0} \right)
\]

where \(a_n = s - h\), \(s = \frac{n - 1}{2} \Leftrightarrow n = 2s + 1\), \((h \in \{0,1\})\)

Note: \(s\) is the spin quantum number.

Example 3:

For \(n = 2\), using the 2 \(\times\) 2 \(\text{SU}(2)\) spin-\(\frac{1}{2}\) Pauli matrices \([1]\), with \(M = 3\):

\[
S_{1,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad S_{0,2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma_2, \quad S_{3,2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3
\]

and the \(n = 6 \times 6\) spin-\(\frac{3}{2}\) matrices \([1]\), with \(M = 3\):

\[
S_{1,6} = \frac{1}{2} \begin{pmatrix}
0 & \sqrt{3} & 0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & 2\sqrt{3} & 0 & 0 & 0 \\
0 & 2\sqrt{3} & 0 & 3 & 0 & 0 \\
0 & 0 & 3 & 0 & 2\sqrt{3} & 0 \\
0 & 0 & 0 & 2\sqrt{3} & 0 & \sqrt{3} \\
0 & 0 & 0 & 0 & \sqrt{3} & 0
\end{pmatrix},
\]

\[
S_{2,6} = \frac{1}{2} \begin{pmatrix}
0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & -2\sqrt{3} & 0 & 0 & 0 \\
0 & 2\sqrt{3} & 0 & -3 & 0 & 0 \\
0 & 0 & 3 & 0 & -2\sqrt{3} & 0 \\
0 & 0 & 0 & 2\sqrt{3} & 0 & -\sqrt{3} \\
0 & 0 & 0 & 0 & -\sqrt{3} & 0
\end{pmatrix},
\]

\[
S_{3,6} = \frac{1}{2} \begin{pmatrix}
5 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & -5
\end{pmatrix},
\]

So:

\[
S_{1,6} = \frac{1}{2} \left\{ \sqrt{3} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + 2\sqrt{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \right\} + 3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
= \frac{1}{2} \left\{ \sqrt{3} \begin{pmatrix} \theta_{1,2} \\ (1,0) \\ \vdots \\ \theta_{2,2:n-2} \\ (0,1) \\ \theta_{n,2} \end{pmatrix} + S_{1,2} \left( \theta_{1,0} : (1,0) : \theta_{2,2:n-2} : (0,1) : \theta_{2,0} \right) \right\} +
\]

\[
+ 2\sqrt{3} \begin{pmatrix} \theta_{1,2} \\ (1,0) \\ \vdots \\ \theta_{2,2:n-2} \\ (0,1) \\ \theta_{n,2} \end{pmatrix} + S_{1,2} \left( \theta_{1,0} : (1,0) : \theta_{2,2:n-2} : (0,1) : \theta_{2,0} \right) +
\]
\[
\begin{align*}
S_{1,2} &= \left( \begin{array}{ccc}
0_{2,2} \\
(1,0) \\
\vdots \\
0_{0,2} \\
0_{2,2}
\end{array} \right) + \left( \begin{array}{cc}
0_{0,2} \\
(0,1) \\
\vdots \\
0_{2,2}
\end{array} \right) = 3 \left( \begin{array}{cc}
0_{2,2} \circ (1,0) \circ 0_{2,2} \circ (0,1) \circ 0_{2,2}
\end{array} \right)
\end{align*}
\]

\[
= \sum_{h=0}^{2} a_h \left( \begin{array}{ccc}
0_{2,2} \\
(1,0) \\
\vdots \\
0_{0,2} \\
0_{2,2}
\end{array} \right)
\left( \begin{array}{ccc}
0_{2,2} \\
(1,0) \\
\vdots \\
0_{0,2} \\
0_{2,2}
\end{array} \right) \left( \begin{array}{ccc}
0_{2,2} \\
(1,0) \\
\vdots \\
0_{0,2} \\
0_{2,2}
\end{array} \right) \left( \begin{array}{ccc}
0_{2,2} \\
(1,0) \\
\vdots \\
0_{0,2} \\
0_{2,2}
\end{array} \right) \left( \begin{array}{ccc}
0_{2,2} \\
(1,0) \\
\vdots \\
0_{0,2} \\
0_{2,2}
\end{array} \right)
\]

\[
a_h = \frac{1}{2} \sqrt{(h+1)(2s+1)-(h+2)} \quad (h \in \{0,1,2\}) \quad (j \in \{1,2\})
\]

\[
S_{1,3} = \frac{1}{2} \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right) + \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right) = \frac{1}{2} \left( \begin{array}{ccc}
0_{1,2} \circ (1,0) \circ 0_{1,2} \\
(1,0) \\
\vdots \\
0_{0,2} \\
0_{1,2}
\end{array} \right) + \left( \begin{array}{ccc}
0_{0,2} \\
(0,1) \\
\vdots \\
0_{0,2} \\
0_{2,2}
\end{array} \right) \left( \begin{array}{ccc}
0_{1,2} \circ (1,0) \circ 0_{1,2} \\
(1,0) \\
\vdots \\
0_{0,2} \\
0_{1,2}
\end{array} \right) \left( \begin{array}{ccc}
0_{1,2} \circ (1,0) \circ 0_{1,2} \\
(1,0) \\
\vdots \\
0_{0,2} \\
0_{1,2}
\end{array} \right) \left( \begin{array}{ccc}
0_{1,2} \circ (1,0) \circ 0_{1,2} \\
(1,0) \\
\vdots \\
0_{0,2} \\
0_{1,2}
\end{array} \right)
\]

\[
a_h = s - h \quad (h \in \{0,1,2\})
\]

**Example 4:**

For \(n = 2\), using the \(2 \times 2\) SU(2) spin-\(\frac{1}{2}\) Pauli matrices \([1]\) & \(I_2\), with \(M = 4\):

\[
S_{1,2} = \left( \begin{array}{cc}
0 & 1 \\
1 & 0 \\
\end{array} \right) = \sigma_1 \quad S_{2,2} = \left( \begin{array}{cc}
0 & -1 \\
1 & 0 \\
\end{array} \right) = \sigma_2
\]

\[
S_{1,3} = \left( \begin{array}{cc}
1 & 0 \\
0 & -1 \\
\end{array} \right) = \sigma_3 \quad S_{4,2} = \left( \begin{array}{cc}
1 & 0 \\
0 & 1 \\
\end{array} \right) = I
\]

and the \(n = 4\) \(4 \times 4\) quaternion matrices \([2]\), with \(M = 4\):
If a homomorphism is desired beyond the vector space isomorphism, a suitable \( m \times m \) matrix linear transformation matrix may be applied (see [3]).
References


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