A Relation Between n-square and m-square Matrix Vector Bases of the Same Dimension
- Relating Spin Matrices and Components

If a vector basis is made up of matrices, the number of matrices must equal the dimension of the basis and they must be linearly independent.

Thus, each of $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$ form a basis of dimenson 2.

Since the matrices of these bases are both 2-square, it is a simple matter to determine a transformation function between the two representations of the same space.

Things are more complicated when the basis sets contain square matrices of differing sizes
(that is: n-square and m-square, $m \neq n$).

Define the augmentation function on two matrices with the same number of columns as follows:

$aug(\mathbf{A},\mathbf{B}) \equiv \begin{pmatrix} \mathbf{A} \\ \cdots \\ \mathbf{B} \end{pmatrix}$; where $\mathbf{A}$ & $\mathbf{B}$ are block matrices of the resulting matrix:

$$\begin{pmatrix} \mathbf{A} \\ \cdots \\ \mathbf{B} \end{pmatrix} = aug(\mathbf{A},\mathbf{B})$$

So, if $\mathbf{A},\mathbf{B},\mathbf{C}$ all have the same number of columns:

$aug(aug(\mathbf{A},\mathbf{B}),\mathbf{C}) = \begin{pmatrix} \mathbf{A} \\ \cdots \\ \mathbf{B} \\ \cdots \\ \mathbf{C} \end{pmatrix}$ exists.

Thus, if $aug(\mathbf{A},\mathbf{B})$ exists and $\mathbf{B}$ is a square matrix, then $aug(\mathbf{A}^T,\mathbf{B})$ exists, and:

$$aug(\mathbf{A},\mathbf{B}) = \begin{pmatrix} \mathbf{A} \\ \cdots \\ \mathbf{B} \end{pmatrix} \Rightarrow aug(\mathbf{A}^T,\mathbf{B}) = \left(\mathbf{A}^T \vdots \mathbf{B}\right)$$

(since $\mathbf{A}^T$ has the same number of rows as $\mathbf{B}$).

Similarly:

If $aug(\mathbf{A},\mathbf{B})$ exists and $\mathbf{B}$ is a square matrix, then $aug(\mathbf{A}^T,\mathbf{B})$ exists, and:

$$aug(aug(\mathbf{A},\mathbf{B}),\mathbf{C}) = \begin{pmatrix} \mathbf{A} \\ \cdots \\ \mathbf{B} \\ \cdots \\ \mathbf{C} \end{pmatrix} \Rightarrow aug(aug(\mathbf{A}^T,\mathbf{B}),\mathbf{C}^T) = \left(\mathbf{A}^T \vdots \mathbf{B} \vdots \mathbf{C}^T\right)$$

Define: $\mathbf{0}_{m,n}$ as a $m \times n$- zero matrix (with matrix entries all zero).
Define $\mathbf{0}_n$ as the n-square zero (additive identity) matrix (with matrix entries all zero).
Define $\mathbf{I}_n$ as the n-square (multiplicative) identity matrix (δ^i_j)

Thus, for n-square matrix $\mathbf{A}$: $aug(\mathbf{0}_{m,n};\mathbf{A}) = \begin{pmatrix} \mathbf{0}_{m,n} \\ \mathbf{A} \end{pmatrix} \Rightarrow aug(\mathbf{0}^T_{m,n};\mathbf{A}) = \left(\mathbf{0}_{n,m} \vdots \mathbf{A}\right)$

Similarly:

Thus, if $aug(\mathbf{A},\mathbf{B})$ exists and $\mathbf{A}$ is a square matrix, then $aug(\mathbf{A},\mathbf{B}^T)$ exists, and:

$$aug(\mathbf{A},\mathbf{B}) \equiv \begin{pmatrix} \mathbf{A} \\ \cdots \\ \mathbf{B} \end{pmatrix} \Rightarrow aug(\mathbf{A},\mathbf{B}^T) = \left(\mathbf{A} \vdots \mathbf{B}^T\right)$$

(since $\mathbf{B}^T$ has the same number of rows as $\mathbf{A}$).

Thus, for n-square matrix $\mathbf{A}$: $aug(\mathbf{A};\mathbf{0}_{m,n}) = \begin{pmatrix} \mathbf{A} \\ \mathbf{0}_{m,n} \end{pmatrix} \Rightarrow aug(\mathbf{A},\mathbf{0}^T_{m,n}) = \left(\mathbf{A} \vdots \mathbf{0}_{n,m}\right)$

Thus, the following matrices may always be formed:

$$aug(\mathbf{0}_{m,n};\mathbf{I}_n) = \begin{pmatrix} \mathbf{0}_{m,n} \\ \mathbf{I}_n \end{pmatrix} \quad ; \quad aug(\mathbf{0}^T_{m,n};\mathbf{I}_n) = \left(\mathbf{0}_{n,m} \vdots \mathbf{I}_n\right)$$

$$aug(\mathbf{I}_n;\mathbf{0}_{m,n}) = \begin{pmatrix} \mathbf{I}_n \\ \mathbf{0}_{m,n} \end{pmatrix} \quad ; \quad aug(\mathbf{I}_n,\mathbf{0}^T_{m,n}) = \left(\mathbf{I}_n \vdots \mathbf{0}_{n,m}\right)$$

Theorem 1:
For n-square matrix basis $\{\mathbf{A}_i\}$ & m-square matrix basis $\{\mathbf{B}_i\}$ each of dimension M :
a relation exists between the components of bases $\{\mathbf{A}_i\}$ and $\{\mathbf{B}_i\}$.

proof :

Without loss of generality, let: $n < m$:

$n < m \Rightarrow \exists k \in \mathbb{N}$ such that: $n + k = m$

Form the matrices $(n+1)$-square matrices:

$$aug(\mathbf{0}_{1,n};\mathbf{I}_n) = \begin{pmatrix} \mathbf{0}_{1,n} \\ \mathbf{I}_n \end{pmatrix} \quad \& \quad aug(\mathbf{0}_{1,n}^T;\mathbf{I}_n) = \left(\mathbf{0}_{n,1} \vdots \mathbf{I}_n\right); \forall i, 1 \le i \le n$$

$$aug(\mathbf{I}_n;\mathbf{0}_{1,n}) = \begin{pmatrix} \mathbf{I}_n \\ \mathbf{0}_{1,n} \end{pmatrix} \quad \& \quad aug(\mathbf{I}_n,\mathbf{0}_{1,n}^T) = \left(\mathbf{I}_n \vdots \mathbf{0}_{1,m}\right); \forall i, 1 \le i \le n$$

and the set:

$$\left\{ \begin{pmatrix} \mathbf{0}_{1,n} \\ \mathbf{I}_n \end{pmatrix} \mathbf{A}_i \left(\mathbf{0}_{n,1} \vdots \mathbf{I}_n\right) + \begin{pmatrix} \mathbf{I}_n \\ \mathbf{0}_{1,n} \end{pmatrix} \mathbf{A}_i \left(\mathbf{I}_n \vdots \mathbf{0}_{1,m}\right) \right\}; \forall i, 1 \le i \le n$$

This is a set of $(n+1)$-square matrices; the first term of which has zeros appended to the left column and top row of each $\mathbf{A}_i$, and the second term of which has zeros appended to the right column and bottom row of each $\mathbf{A}_i$.

Applcation of this process k-times results in a set of $(n+k)$-square matrices; i.e. m-square matrices. Whenever they form a M-dimensional basis a transformation exists relating the components of bases $\{\mathbf{A}_i\}$ and $\{\mathbf{B}_i\}$.

Since each of the sets $\left\{ \begin{pmatrix} \mathbf{0}_{1,n} \\ \mathbf{I}_n \end{pmatrix} \mathbf{A}_i \left(\mathbf{0}_{n,1} \vdots \mathbf{I}_n\right) \right\}$ & $\left\{ \begin{pmatrix} \mathbf{I}_n \\ \mathbf{0}_{1,n} \end{pmatrix} \mathbf{A}_i \left(\mathbf{I}_n \vdots \mathbf{0}_{1,m}\right) \right\}; \forall i, 1 \le i \le n$

are each sets of m-square matrices and $\{\mathbf{B}_i\}$ is a basis of m-square matrices; then each may be expressed as a linear combination of the base vector matrices of $\{\mathbf{B}_i\}$ - yielding a relationship between $\{\mathbf{A}_i\}$ and $\{\mathbf{B}_i\}$.

□

Examples:

example 1:

For $n = 2$, using the 2×2 SU(2) spin-$\frac{1}{2}$ Pauli matrices [1], with $M = 3$:

$$\mathbf{S}_{1,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \boldsymbol{\sigma}_1 \quad , \quad \mathbf{S}_{2,2} = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \boldsymbol{\sigma}_2 \quad , \quad \mathbf{S}_{3,2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \boldsymbol{\sigma}_3$$

and the $n = 3$ 3×3 spin-1 matrices [1], with $M = 3$:

$$\mathbf{S}_{1,3} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \mathbf{S}_{2,3} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \mathbf{S}_{3,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Performing the theorem 1 operations on the $\mathbf{S}_{i,2}$ yield:

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

So:

$$\mathbf{S}_{1,3} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\sum_{h=0}^{1}\begin{pmatrix} \mathbf{0}_{1-h,2} \\ \mathbf{I}_2 \\ \cdots \\ \mathbf{0}_{h,2} \end{pmatrix}\mathbf{S}_{1,2}\left(\mathbf{0}_{2,1-h} \vdots \mathbf{I}_2 \vdots \mathbf{0}_{2,h}\right)$$

$$\mathbf{S}_{2,3} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}_{2,2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{S}_{2,2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\sum_{h=0}^{1}\begin{pmatrix} \mathbf{0}_{1-h,2} \\ \mathbf{I}_2 \\ \cdots \\ \mathbf{0}_{h,2} \end{pmatrix}\mathbf{S}_{2,2}\left(\mathbf{0}_{2,1-h} \vdots \mathbf{I}_2 \vdots \mathbf{0}_{2,h}\right)$$

$$\mathbf{S}_{3,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}_{3,2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{S}_{3,2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \sum_{h=0}^{1}\begin{pmatrix} \mathbf{0}_{1-h,2} \\ \mathbf{I}_2 \\ \cdots \\ \mathbf{0}_{h,2} \end{pmatrix}\mathbf{S}_{3,2}\left(\mathbf{0}_{2,1-h} \vdots \mathbf{I}_2 \vdots \mathbf{0}_{2,h}\right)$$

are transformation equations between spin-$\frac{1}{2}$ and spin-1 bases.

Thus, for:

$$\mathbf{Z} = \mathbf{S}_{1,2}Z^1_{\frac{1}{2}} + \mathbf{S}_{2,2}Z^2_{\frac{1}{2}} + \mathbf{S}_{3,2}Z^3_{\frac{1}{2}} = \mathbf{S}_{1,3}Z^1_1 + \mathbf{S}_{2,3}Z^2_1 + \mathbf{S}_{3,3}Z^3_1$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{Z}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{Z}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\left[\mathbf{S}_{1,2}Z^1_{\frac{1}{2}} + \mathbf{S}_{2,2}Z^2_{\frac{1}{2}} + \mathbf{S}_{3,2}Z^3_{\frac{1}{2}}\right]\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} +$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\left[\mathbf{S}_{1,2}Z^1_{\frac{1}{2}} + \mathbf{S}_{2,2}Z^2_{\frac{1}{2}} + \mathbf{S}_{3,2}Z^3_{\frac{1}{2}}\right]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}Z^1_{\frac{1}{2}} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}_{2,2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}Z^2_{\frac{1}{2}} +$$

$$+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}_{3,2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}Z^3_{\frac{1}{2}} +$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}Z^1_{\frac{1}{2}} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{S}_{2,2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}Z^2_{\frac{1}{2}} +$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{S}_{3,2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}Z^3_{\frac{1}{2}}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}Z^1_{\frac{1}{2}} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}Z^1_{\frac{1}{2}} +$$

$$+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}_{2,2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}Z^2_{\frac{1}{2}} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{S}_{2,2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}Z^2_{\frac{1}{2}} +$$

$$+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}_{3,2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}Z^3_{\frac{1}{2}} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{S}_{3,2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}Z^3_{\frac{1}{2}}$$

$$= \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right]Z^1_{\frac{1}{2}} +$$

$$+ \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{S}_{2,2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{S}_{2,2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right]Z^2_{\frac{1}{2}} +$$

$$+\left[\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}\mathbf{S}_{3,2}\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\mathbf{S}_{3,2}\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\right]Z^3_{\frac{1}{2}}$$

$$= \mathbf{S}_{1,3}\sqrt{2}\,Z^1_{\frac{1}{2}} + \mathbf{S}_{2,3}\sqrt{2}\,Z^2_{\frac{1}{2}} + \mathbf{S}_{3,3}Z^3_{\frac{1}{2}} = \mathbf{S}_{1,3}Z^1_1 + \mathbf{S}_{2,3}Z^2_1 + \mathbf{S}_{3,3}Z^3_1$$

$$\Rightarrow \begin{cases} Z^1_1 = \sqrt{2}\,Z^1_{\frac{1}{2}} \\ Z^2_1 = \sqrt{2}\,Z^2_{\frac{1}{2}} \\ Z^3_1 = Z^3_{\frac{1}{2}} \end{cases}$$

example 2:

For $n = 2$, using the 2×2 SU(2) spin-$\frac{1}{2}$ Pauli matrices [1], with $M = 3$:

$$\mathbf{S}_{1,2} = \begin{pmatrix}0&1\\1&0\end{pmatrix} = \boldsymbol{\sigma}_1 \quad , \quad \mathbf{S}_{2,2} = i\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \boldsymbol{\sigma}_2 \quad , \quad \mathbf{S}_{3,2} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = \boldsymbol{\sigma}_3$$

and the $n = 4$ 4×4 spin-$\frac{3}{2}$ matrices [1], with $M = 3$:

$$\mathbf{S}_{1,4} = \frac{1}{2}\begin{pmatrix}0&\sqrt{3}&0&0\\\sqrt{3}&0&2&0\\0&2&0&\sqrt{3}\\0&0&\sqrt{3}&0\end{pmatrix} \quad , \quad \mathbf{S}_{2,3} = \frac{i}{2}\begin{pmatrix}0&-\sqrt{3}&0&0\\\sqrt{3}&0&-2&0\\0&2&0&-\sqrt{3}\\0&0&\sqrt{3}&0\end{pmatrix} \quad ,$$

$$\mathbf{S}_{3,3} = \frac{1}{2}\begin{pmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{pmatrix}$$

Again, performing the theorem 1 operations on the $\mathbf{S}_{i,2}$ yield:

$$\left\{\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}, i\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, i\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\right\}$$

Performing them to these, yields:

$$\left\{\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, i\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}, i\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, i\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}, i\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}\right\}$$

So:

$$\mathbf{S}_{1,4} = \frac{1}{2}\begin{pmatrix}0&\sqrt{3}&0&0\\\sqrt{3}&0&2&0\\0&2&0&\sqrt{3}\\0&0&\sqrt{3}&0\end{pmatrix} = \frac{1}{2}\left[2\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix} + \sqrt{3}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right]$$

$$= \frac{\sqrt{3}}{2}\left[\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix}0&0&1&0\\0&0&0&1\end{pmatrix} + \begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix}\right] +$$

$$+\left[\begin{pmatrix}0&0\\1&0\\0&1\\0&0\end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix}0&1&0&0\\0&0&1&0\end{pmatrix}+\begin{pmatrix}0&0\\1&0\\0&1\\0&0\end{pmatrix}\mathbf{S}_{1,2}\begin{pmatrix}0&1&0&0\\0&0&1&0\end{pmatrix}\right]$$

$$=\sum_{h=0}^{1}a_h\begin{pmatrix}\mathbf{0}_{2-h,2}\\\cdots\\\mathbf{I}_2\\\cdots\\\mathbf{0}_{h,2}\end{pmatrix}\mathbf{S}_{1,2}\left(\mathbf{0}_{2,2-h}\vdots\mathbf{I}_2\vdots\mathbf{0}_{2,h}\right)+\sum_{h=1}^{2}a_{2-h}\begin{pmatrix}\mathbf{0}_{2-h,2}\\\cdots\\\mathbf{I}_2\\\cdots\\\mathbf{0}_{h,2}\end{pmatrix}\mathbf{S}_{1,2}\left(\mathbf{0}_{2,2-h}\vdots\mathbf{I}_2\vdots\mathbf{0}_{2,h}\right)$$

$$=\sum_{h=0}^{1}a_h\begin{pmatrix}\mathbf{0}_{2-h,2}\\\cdots\\\mathbf{I}_2\\\cdots\\\mathbf{0}_{h,2}\end{pmatrix}\mathbf{S}_{1,2}\left(\mathbf{0}_{2,2-h}\vdots\mathbf{I}_2\vdots\mathbf{0}_{2,h}\right)+\sum_{h=0}^{1}a_h\begin{pmatrix}\mathbf{0}_{h,2}\\\cdots\\\mathbf{I}_2\\\cdots\\\mathbf{0}_{2-h,2}\end{pmatrix}\mathbf{S}_{1,2}\left(\mathbf{0}_{2,h}\vdots\mathbf{I}_2\vdots\mathbf{0}_{2,2-h}\right)$$

$$\mathbf{S}_{j,4}=\sum_{h=0}^{1}a_h\left[\begin{pmatrix}\mathbf{0}_{2-h,2}\\\cdots\\\mathbf{I}_2\\\cdots\\\mathbf{0}_{h,2}\end{pmatrix}\mathbf{S}_{j,2}\left(\mathbf{0}_{2,2-h}\vdots\mathbf{I}_2\vdots\mathbf{0}_{2,h}\right)+\begin{pmatrix}\mathbf{0}_{h,2}\\\cdots\\\mathbf{I}_2\\\cdots\\\mathbf{0}_{2-h,2}\end{pmatrix}\mathbf{S}_{j,2}\left(\mathbf{0}_{2,h}\vdots\mathbf{I}_2\vdots\mathbf{0}_{2,2-h}\right)\right]$$

$$a_h=\frac{1}{2}\sqrt{(h+1)[2(s+1)-(h+2)]}\quad,\quad(h\in\{0,1\})\quad(j\in\{1,2\})$$

$$\mathbf{S}_{3,4}=\frac{1}{2}\begin{pmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{pmatrix}=\frac{1}{2}\left[\begin{pmatrix}3&0&0&0\\0&-3&0&0\\0&0&3&0\\0&0&0&-3\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&4&0&0\\0&0&-4&0\\0&0&0&0\end{pmatrix}\right]$$

$$=\frac{1}{2}\left[3\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}+4\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}\right]$$

So, this technique works; but the factors become different than the usual ones obtained from the commutation rules. Thus, a better way is found via a generalization of the above techniques.

Any matrix may be constructed by appending n-tuples: $\mathbf{a}_j\equiv(a_{j1},a_{j2},\ldots,a_{jn})$

$$aug(\ldots(aug(aug(\mathbf{a}_1,\mathbf{a}_2),\mathbf{a}_3),\ldots),\mathbf{a}_m)=\begin{pmatrix}\mathbf{a}_1\\\cdots\\\mathbf{a}_2\\\cdots\\\vdots\\\mathbf{a}_m\end{pmatrix}.$$

So:

$$\begin{pmatrix}(1,0)\\\cdots\\\mathbf{0}_{m,2}\\\cdots\\(0,1)\end{pmatrix}\quad\text{and}\quad\left(\begin{pmatrix}1\\\cdots\\0\end{pmatrix}\vdots\mathbf{0}_{2,m}\vdots\begin{pmatrix}0\\\cdots\\1\end{pmatrix}\right)\quad\text{exist.}$$

and, so:

$$\mathbf{S}_{3,4}=\frac{1}{2}\begin{pmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{pmatrix}=\frac{1}{2}\left[\begin{pmatrix}3&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-3\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}\right]$$

$$=\frac{1}{2}\left[3\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}\right]$$

$$\mathbf{S}_{3,4}=\frac{1}{2}\left\{3\left[\begin{pmatrix}(1,0)\\\cdots\\\mathbf{0}_{2,2}\\\cdots\\(0,1)\end{pmatrix}\mathbf{S}_{3,2}\left(\begin{pmatrix}1\\\cdots\\0\end{pmatrix}\vdots\mathbf{0}_{2,2}\vdots\begin{pmatrix}0\\\cdots\\1\end{pmatrix}\right)\right]+\left[\begin{pmatrix}\mathbf{0}_{1,2}\\\cdots\\\mathbf{I}_2\\\cdots\\\mathbf{0}_{1,2}\end{pmatrix}\mathbf{S}_{3,2}\left(\mathbf{0}_{2,1}\vdots\mathbf{I}_2\vdots\mathbf{0}_{2,1}\right)\right]\right\}$$

$$\mathbf{S}_{3,4}=\frac{1}{2}\left\{3\left[\begin{pmatrix}(1,0)\\\cdots\\\mathbf{0}_{2,2}\\\cdots\\(0,1)\end{pmatrix}\mathbf{S}_{3,2}\left(\begin{pmatrix}1\\\cdots\\0\end{pmatrix}\vdots\mathbf{0}_{2,2}\vdots\begin{pmatrix}0\\\cdots\\1\end{pmatrix}\right)\right]+\left[\begin{pmatrix}\mathbf{0}_{1,2}\\\cdots\\\mathbf{I}_2\\\cdots\\\mathbf{0}_{1,2}\end{pmatrix}\mathbf{S}_{3,2}\left(\mathbf{0}_{2,1}\vdots\mathbf{I}_2\vdots\mathbf{0}_{2,1}\right)\right]\right\}$$

$$= \sum_{h=0}^{1} a_h \left[\begin{pmatrix} \mathbf{0}_{h,2} \\ \cdots \\ (1,0) \\ \cdots \\ \mathbf{0}_{2-2h,2} \\ \cdots \\ (0,1) \\ \cdots \\ \mathbf{0}_{h,2} \end{pmatrix} \mathbf{S}_{3,2} \left(\mathbf{0}_{2,h} \vdots (1,0) \vdots \mathbf{0}_{2,2-2h} \vdots (0,1) \vdots \mathbf{0}_{2,h} \right) \right]$$

$$a_h = s - h \quad , \quad (h \in \{0,1\})$$

Note: $s = \frac{n-1}{2} \Leftrightarrow n = 2s+1$

example 3:

For $n = 2$, using the 2×2 SU(2) spin-$\frac{1}{2}$ Pauli matrices [1], with $M = 3$:

$$\mathbf{S}_{1,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \boldsymbol{\sigma}_1 \quad , \quad \mathbf{S}_{2,2} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \boldsymbol{\sigma}_2 \quad , \quad \mathbf{S}_{3,2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \boldsymbol{\sigma}_3$$

and the $n = 6$ 6×6 spin-$\frac{5}{2}$ matrices [1], with $M = 3$:

$$\mathbf{S}_{1,6} = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{5} & 0 & 0 & 0 & 0 \\ \sqrt{5} & 0 & 2\sqrt{2} & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 2\sqrt{2} & 0 \\ 0 & 0 & 0 & 2\sqrt{2} & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 \end{pmatrix} ,$$

$$\mathbf{S}_{2,6} = \frac{i}{2} \begin{pmatrix} 0 & -\sqrt{5} & 0 & 0 & 0 & 0 \\ \sqrt{5} & 0 & -2\sqrt{2} & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & -2\sqrt{2} & 0 \\ 0 & 0 & 0 & 2\sqrt{2} & 0 & -\sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 \end{pmatrix} ,$$

$$\mathbf{S}_{3,6} = \frac{1}{2} \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{pmatrix}$$

So:

$$\mathbf{S}_{1,6} = \frac{1}{2} \left\{ \sqrt{5} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} + 2\sqrt{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

$$= \frac{1}{2} \left\{ \sqrt{5} \begin{pmatrix} \mathbf{0}_{0,2} \\ \cdots \\ (1,0) \\ \cdots \\ \mathbf{0}_{2,2} \\ \cdots \\ (0,1) \\ \cdots \\ \mathbf{0}_{0,2} \end{pmatrix} \mathbf{S}_{3,2} \left(\mathbf{0}_{2,0} \vdots (1,0) \vdots \mathbf{0}_{2,2} \vdots (0,1) \vdots \mathbf{0}_{2,0} \right) + \right.$$

$$+ 2\sqrt{2} \begin{pmatrix} \mathbf{0}_{1,2} \\ \cdots \\ (1,0) \\ \cdots \\ \mathbf{0}_{0,2} \\ \cdots \\ (0,1) \\ \cdots \\ \mathbf{0}_{1,2} \end{pmatrix} \mathbf{S}_{3,2} \left(\mathbf{0}_{2,1} \vdots (1,0) \vdots \mathbf{0}_{2,0} \vdots (0,1) \vdots \mathbf{0}_{2,1} \right) +$$

$$+3\begin{pmatrix}\mathbf{0}_{2,2}\\ \cdots\\ (1,0)\\ \cdots\\ \mathbf{0}_{0,2}\\ \cdots\\ (0,1)\\ \cdots\\ \mathbf{0}_{2,2}\end{pmatrix}\mathbf{S}_{3,2}\left(\mathbf{0}_{2,2}\vdots(1,0)\vdots\mathbf{0}_{2,0}\vdots(0,1)\vdots\ \mathbf{0}_{2,2}\right)\Big\}$$

$$=\sum_{h=0}^{2}a_h\left[\begin{pmatrix}\mathbf{0}_{h,2}\\ \cdots\\ (1,0)\\ \cdots\\ \mathbf{0}_{2-h,2}\\ \cdots\\ (0,1)\\ \cdots\\ \mathbf{0}_{h,2}\end{pmatrix}\mathbf{S}_{3,2}\left(\mathbf{0}_{2,h}\vdots(1,0)\vdots\mathbf{0}_{2,2-h}\vdots(0,1)\vdots\ \mathbf{0}_{2,h}\right)\right]$$

$$a_h=\frac{1}{2}\sqrt{(h+1)[2(s+1)-(h+2)]}\quad,\quad(h\in\{0,1,2\})\quad(j\in\{1,2\})$$

$$\mathbf{S}_{3,6}=\frac{1}{2}\left\{5\begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\end{pmatrix}+3\begin{pmatrix}0&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right\}$$

$$=\frac{1}{2}\Bigg\{5\begin{pmatrix}\mathbf{0}_{0,2}\\ \cdots\\ (1,0)\\ \cdots\\ \mathbf{0}_{4,2}\\ \cdots\\ (0,1)\\ \cdots\\ \mathbf{0}_{0,2}\end{pmatrix}\mathbf{S}_{3,2}\left(\mathbf{0}_{2,0}\vdots(1,0)\vdots\mathbf{0}_{2,4}\vdots(0,1)\vdots\ \mathbf{0}_{2,0}\right)+$$

$$+3\begin{pmatrix}\mathbf{0}_{1,2}\\ \cdots\\ (1,0)\\ \cdots\\ \mathbf{0}_{2,2}\\ \cdots\\ (0,1)\\ \cdots\\ \mathbf{0}_{1,2}\end{pmatrix}\mathbf{S}_{3,2}\left(\mathbf{0}_{2,1}\vdots(1,0)\vdots\mathbf{0}_{2,2}\vdots(0,1)\vdots\ \mathbf{0}_{2,1}\right)+$$

$$+3\begin{pmatrix}\mathbf{0}_{2,2}\\ \cdots\\ (1,0)\\ \cdots\\ \mathbf{0}_{0,2}\\ \cdots\\ (0,1)\\ \cdots\\ \mathbf{0}_{2,2}\end{pmatrix}\mathbf{S}_{3,2}\left(\mathbf{0}_{2,2}\vdots(1,0)\vdots\mathbf{0}_{2,0}\vdots(0,1)\vdots\ \mathbf{0}_{2,2}\right)\Big\}$$

$$=\sum_{h=0}^{2}a_h\left[\begin{pmatrix}\mathbf{0}_{h,2}\\ \cdots\\ (1,0)\\ \cdots\\ \mathbf{0}_{4-2h,2}\\ \cdots\\ (0,1)\\ \cdots\\ \mathbf{0}_{h,2}\end{pmatrix}\mathbf{S}_{3,2}\left(\mathbf{0}_{2,h}\vdots(1,0)\vdots\mathbf{0}_{2,4-2h}\vdots(0,1)\vdots\ \mathbf{0}_{2,h}\right)\right]$$

$$a_h=s-h\quad,\quad(h\in\{0,1,2\})$$

example 4:

For $n=2$, using the 2×2 SU(2) spin-$\frac{1}{2}$ Pauli matrices [1] & $\mathbf{I}_2$, with $M=4$:

$$\mathbf{S}_{1,2}=\begin{pmatrix}0&1\\1&0\end{pmatrix}=\boldsymbol{\sigma}_1\quad,\quad\mathbf{S}_{2,2}=i\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\boldsymbol{\sigma}_2\quad,$$

$$\mathbf{S}_{3,2}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\boldsymbol{\sigma}_3\quad,\quad\mathbf{S}_{4,2}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}_2$$

and the $n=4$ 4×4 quaternion matrices [2], with $M=4$:

$$\mathbf{q}_{1,4} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathbf{q}_{2,4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad ,$$

$$\mathbf{q}_{3,4} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad , \quad \mathbf{q}_{4,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_4$$

So:

$$\mathbf{q}_{1,4} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{0}_2 \\ \cdots \\ \mathbf{I}_2 \end{pmatrix} \boldsymbol{\sigma}_1 \left(\mathbf{I}_2 \vdots \mathbf{0}_2 \right) - \begin{pmatrix} \mathbf{I}_2 \\ \cdots \\ \mathbf{0}_2 \end{pmatrix} \boldsymbol{\sigma}_1 \left(\mathbf{0}_2 \vdots \mathbf{I}_2 \right)$$

$$\mathbf{q}_{2,4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} \mathbf{I}_2 \\ \cdots \\ \mathbf{0}_2 \end{pmatrix} \boldsymbol{\sigma}_2 \left(\mathbf{I}_2 \vdots \mathbf{0}_2 \right) - \begin{pmatrix} \mathbf{0}_2 \\ \cdots \\ \mathbf{I}_2 \end{pmatrix} \boldsymbol{\sigma}_2 \left(\mathbf{0}_2 \vdots \mathbf{I}_2 \right)$$

$$\mathbf{q}_{3,4} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{0}_2 \\ \cdots \\ \mathbf{I}_2 \end{pmatrix} \boldsymbol{\sigma}_3 \left(\mathbf{I}_2 \vdots \mathbf{0}_2 \right) - \begin{pmatrix} \mathbf{I}_2 \\ \cdots \\ \mathbf{0}_2 \end{pmatrix} \boldsymbol{\sigma}_3 \left(\mathbf{0}_2 \vdots \mathbf{I}_2 \right)$$

$$\mathbf{q}_{4,4} = \mathbf{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{I}_2 \\ \cdots \\ \mathbf{0}_2 \end{pmatrix} \mathbf{I}_2 \left(\mathbf{I}_2 \vdots \mathbf{0}_2 \right) + \begin{pmatrix} \mathbf{0}_2 \\ \cdots \\ \mathbf{I}_2 \end{pmatrix} \mathbf{I}_2 \left(\mathbf{0}_2 \vdots \mathbf{I}_2 \right)$$

If a homomorphism is desired beyond the vector space isomorphism, a suitable $m \times m$ matrix linear transformation matrix may be applied (see [3]).

References

[1] https://en.wikipedia.org/wiki/Spin_(physics)

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