SPECTRA OF A NEW JOIN IN DUPLICATION GRAPH

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Abstract. The Duplication graph DG of a graph G, is obtained by inserting new vertices corresponding to each vertex of G and making the vertex adjacent to the neighbourhood of the corresponding vertex of G and deleting the edges of G. Let G₁ and G₂ be two graph with vertex sets V(G₁) and V(G₂) respectively. The DG - vertex join of G₁ and G₂ is denoted by G₁ ⊔ G₂ and it is the graph obtained from DG₁ and G₂ by joining every vertex of V(G₁) to every vertex of V(G₂). The DG - add vertex join of G₁ and G₂ is denoted by G₁ ⊕ G₂ and is the graph obtained from DG₁ and G₂ by joining every additional vertex of DG₁ to every vertex of V(G₂). In this paper we determine the A - spectra and L - spectra of the two new joins of graphs for a regular graph G₁ and an arbitrary graph G₂. As an application we give the number of spanning tree, the Kirchhoff index and Laplace energy invariant of the new join. Also we obtain some infinite family of new class of integral graphs.

Keywords: Spectrum, co-spectral graphs, Join of graphs, spanning tree, Kirchhoff index, Laplace energ-like invariant

AMS Subject Classification (2010) : 05C50

1. Introduction

All graphs described in this paper are simple and undirected. Let G be a graph with vertex set V(G) = {v₁, v₂, · · · , vₙ}. The adjacency matrix of G, denoted by A(G) = (aᵢⱼ)n×n is an n × n symmetric matrix with

\[
a_{ij} = \begin{cases} 
1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\
0 & \text{otherwise}
\end{cases}
\]

Let dᵢ be the degree of the vertex vᵢ in G and D(G) = diag(d₁, d₂, · · · , dₙ) be the diagonal matrix of G. The Laplacian matrix is defined as L(G) = D(G) − A(G). The characteristic polynomial of A(G) is defined as \( f_{A}(x) = \det(xI_n - A) \) where Iₙ is the identity matrix of order n. The roots of the characteristic equation of A(G) are called the eigenvalues of G. It is denoted by λ₁(G) ≥ λ₂(G) ≥ · · · ≥ λₙ(G). It is called the A - Spectrum of G. The eigen values of L(G) is denoted by 0 = µ₁(G) ≤ µ₂(G) ≤ · · · ≤ µₙ(G) and it is called the L - Spectrum of G. Since A(G) and L(G) are real and symmetric, their eigen values are all real numbers. A graph is A - integral, if the A - spectrum consists only of integers [4, 14]. Two graphs are said to be A - Cospectral if they have the same A - spectrum.

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The characteristic polynomial and spectra of graphs help to investigate some properties of graphs such as energy [8, 16], number of spanning trees [18, 9, 17], the Kirchhoff index [2, 5, 11], Laplace energy like invariants [7] etc.

The first result on Laplacian matrix was discovered by Kirchhoff, which appeared in a paper published in the year 1847 is related to electrical network. There exists a vast literature that studies the Laplacian eigen values and their relationship with various properties of graphs [12, 13]. Most of the studies of the Laplacian eigen values has naturally concentrated on external non trivial eigen values. Gutman et al. [16] discovered the connection between photoelectron spectra of standard hydrocarbons and the Laplacian eigen values of the underlying molecular graphs.

In the first section we define DG - vertex join and DG - add vertex join of two graphs and discuss some important results, which are found essential to prove the results given in the subsequent sections. In the third section we find the A - spectrum and the L - spectrum of the new join and prove some related results. As an application, we find the number of spanning trees, Kirchhoff index and Laplacian - energy like invariant. Fourth section contains a discussion on some infinite family of integral graphs.

2. Preliminaries

In a paper published in 1973 on duplicate graphs, which appeared in the Journal of Indian Mathematical Society, Sampathkumar [10] defined duplicate graphs. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. Take another set $U = \{u_1, u_2, ..., u_n\}$. Make $u_i$ adjacent to all the vertices in $N(v_i)$, the neighbourhood set of $v_i$, in $G$ for each $i$ and remove the edges of $G$ only. The resulting graph is called the duplication graph of $G$ and is denoted by $DG$. The following result tells us an easy way to find the determinant of a bigger matrix using the determinant of relatively smaller matrices.

**Proposition 2.1.** [1] Let $M_1, M_2, M_3, M_4$ be respectively $p \times p, p \times q, q \times p, q \times q$ matrix with $M_1$ and $M_4$ are invertible then

$$
\det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_1)\det(M_4 - M_3M_1^{-1}M_2)
$$

$$
= \det(M_4)\det(M_1 - M_2M_4^{-1}M_3)
$$

where $M_4 - M_3M_1^{-1}M_2$ and $M_1 - M_2M_4^{-1}M_3$ are called the Schur complements of $M_1$ and $M_4$ respectively.

Let $G$ be a graph on $n$ vertices, with the adjacency matrix $A$. The characteristic matrix $xI - A$ of $A$ has determinant $\det(xI - A) = f_G(A : x) \neq 0$, so is invertible. The $A$ - coronal [6], $\Gamma_A(x)$ of $G$ is defined to be the sum of the entries of the matrix $(xI - A)^{-1}$. This can be calculated as

$$
\Gamma_A(x) = 1_n^T(xI - A)^{-1}1_n
$$

**Lemma 2.2.** [6] Let $G$ be $r$ - regular on $n$ vertices. Then

$$
\Gamma_A(x) = \frac{n}{x - r}
$$
Since for any graph $G$ with $n$ vertices, each row sum of the Laplacian matrix $L(G)$ is equal to 0, we have

$$\Gamma_L(x) = \frac{n}{x}$$

**Lemma 2.3.** [6] Let $G$ be the bipartite graph $K_{pq}$, where $p + q = n$. Then

$$\Gamma_A(x) = \frac{nx + 2pq}{x^2 - pq}$$

**Proposition 2.4.** [15] Let $A$ be an $n \times n$ real matrix, and $J_{s\times t}$ denote the $s \times t$ matrix with all entries equal to one. Then

$$\det(A + \alpha J_{n \times n}) = \det(A) + \alpha I_n^T \text{adj}(A) I^n.$$ 

Here $\alpha$ is a real number and $\text{adj}(A)$ is the adjugate matrix of $A$.

**Corollary 2.5.** [15] Let $A$ be an $n \times n$ real matrix. Then

$$\det(xI_n - A - \alpha J_{n \times n}) = (1 - \alpha \Gamma_A(x)) \det(xI_n - A).$$

**Definition 2.6.** Let $G_1$ be a graph on $n_1$ vertices and $m_1$ edges. $G_2$ be an arbitrary graph on $n_2$ vertices. The $DG$ – vertex join of $G_1$ and $G_2$ is denoted by $G_1 \sqcup G_2$ and is the graph obtained from $DG_1$ and $G_2$ by joining every vertex of $V(G_1)$ to every vertex of $V(G_2)$. Where $DG_1$ is the duplication graph of $G_1$.

**Definition 2.7.** The $DG$ – addvertex join of $G_1$ and $G_2$ is denoted by $G_1 \Join G_2$ and is the graph obtained from $DG_1$ and $G_2$ by joining the additional vertices of $DG_1$ corresponding to the vertices of $G_1$ with every vertex of $V(G_2)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{$C_4 \sqcup K_2$}
\end{figure}
Figure 2 : $C_4 \cong K_2$

3. Spectrum of $G_1 \sqcup G_2$

**Theorem 3.1.** Let $G_1$ be an $r_1$-regular graph on $n_1$ vertices and $m_1$ edges. $G_2$ be an arbitrary graph on $n_2$ vertices. Then, the Characteristic polynomial of $G_1 \sqcup G_2$ is

$$f_{G_1 \sqcup G_2}(A : x) = (x^2 - n_1x \Gamma_{A_2}(x) - r_1^2) \prod_{i=2}^{n_2}(x - \lambda_i(G_2)) \prod_{i=2}^{n_1}(x^2 - \lambda_i(G_1)^2)$$

**Proof.** The adjacency matrix of $G_1 \sqcup G_2$ is

$$A = \begin{bmatrix} 0 & A_1 & J_{n_1 \times n_2} \\ A_1 & 0_{n_1} & 0_{n_1 \times n_2} \\ J_{n_2 \times n_1} & 0_{n_2 \times n_1} & A_2 \end{bmatrix}$$

where $A_1$ and $A_2$ are the adjacency matrix of $G_1$ and $G_2$ respectively and $J$ is a matrix with each entries 1.

The Characteristic polynomial of $G_1 \sqcup G_2 =$

$$f_{G_1 \sqcup G_2}(A : x) = \begin{vmatrix} xI_{n_1} - A_1 & -J \\ -A_1 & xI_{n_2} - J \\ -J & 0_{n_2 \times n_1} - xI_{n_2} - A_2 \end{vmatrix}$$

$$= det(xI_{n_2} - A_2) \det S$$

where

$$S = \begin{pmatrix} xI_{n_1} - A_1 \\ -A_1 & xI_{n_1} \end{pmatrix} - \begin{pmatrix} -J_{n_2 \times n_2} \\ 0_{n_2 \times n_1} \end{pmatrix} (xI_{n_2} - A_2)^{-1} \begin{pmatrix} -J_{n_2 \times n_1} & 0 \end{pmatrix}$$
\[
\begin{pmatrix}
xI - \Gamma A_2(x)J_{n_1 \times n_1} & -A_1 \\
-A_1 & xI \\
\end{pmatrix}
\]
\[
det S = det(xI) \det \left( (xI - \Gamma A_2(x)J - \frac{A_1^2}{x} \right) \\
= x^{n_1} det \left( xI - \Gamma A_2(x)J - \frac{A_1^2}{x} \right) \\
= x^{n_1} det \left( xI - \frac{A_1^2}{x} - \Gamma A_2(x)J \right) \\
= x^{n_1} det \left( xI - \frac{A_1^2}{x} \right) \left( 1 - \Gamma A_2(x) \Gamma \frac{A_1^2}{x} \right)
\]

\(G_1\) is \(r_1\) - regular and the row sum of \(A_1^2\) is \(r_2^2\)

\[
\Gamma \frac{A_1^2}{x} = \frac{n_1}{x - r_1^2} \\
= \frac{n_1 x}{x^2 - r_1^2}
\]

\[
det S = x^{n_1} det \left( xI - \frac{A_1^2}{x} \right) \left( 1 - \frac{n_1 x}{x^2 - r_1^2} \Gamma A_2(x) \right) \\
= det(x^2 I - A^2) \left( \frac{x^2 - r_2^2 - n_1 x \Gamma A_2(x)}{x^2 - r_1^2} \right)
\]

Hence
\[
det(xI - A) = (x^2 - n_1 x \Gamma A_2(x) - r_1^2) \prod_{i=1}^{n_2} (x - \lambda_i(G_2)) \prod_{i=2}^{n_1} (x^2 - \lambda_i(G_1)^2)
\]

\(\square\)

Corollary 3.2. Let \(G_1\) be an \(r_1\) - regular graph on \(n_1\) vertices, \(G_2\) be \(r_2\) - regular graph on \(n_2\) vertices. Then the \(A\) – Spectrum of \(G_1 \cup G_2\) consists of

(i) \(\lambda_i(G_2)\), for \(i = 2, 3, ..., n_2\)
(ii) \(\pm \lambda_i(G_1)\), for \(i = 2, 3, ..., n_1\)
(iii) Three roots of the equation
\[
x^3 - r_2 x^2 - (n_1 n_2 + r_1^2) x + r_2^2 r_1
\]

Proof. If \(G_2\) is \(r_2\) - regular then

\[
\Gamma A_2(x) = \frac{n_2}{x - r_2}
\]

We get
\[
det(xI - A) = (x^3 - r_2 x^2 - (n_1 n_2 + r_1^2) x + r_2^2 r_1)
\]

\[
\prod_{i=2}^{n_2} (x - \lambda_i(G_2)) \prod_{i=2}^{n_1} (x^2 - \lambda_i(G_1)^2)
\]

\(\square\)

Corollary 3.3. Let \(G_1\) be an \(r_1\) - regular graph on \(n_1\) vertices, \(A\) – Spectrum of \(G_1 \cup K_n\) consists of

(i) 0, repeats \(n_2\) times
(ii) \(\pm \lambda_i(G_1)\), for \(i = 2, 3, ..., n_1\)
(iii) \(\pm \sqrt{n_1 n_2 + r_1^2}\)
Corollary 3.4. Let $G_1$ be an $r_1$-regular graph on $n_1$ vertices. A–Spectrum of $G_1 \cup K_{pq}$ consists of
(i) $0$, repeats $p + q - 2$ times
(ii) $\pm \lambda_i(G_1)$, for $i = 2, 3, ..., n_1$
(iii) Four roots of the equation
$$x^4 - (pq + r_1^2 + n_1p + n_1q)x^2 - 2pqn_1x + r_1^2pq = 0$$

3.1. Laplacian Spectrum of $G_1 \cup G_2$.

Theorem 3.5. Let $G_1$ be an $r_1$-regular graph on $n_1$ vertices and $m_1$ edges. $G_2$ be an arbitrary graph on $n_2$ vertices. Then,

$$f_{G_1 \cup G_2}(L : x) = x(x^2 - (n_1 + n_2 + 2r_1)x + r_1(2n_1 + n_2)) \prod_{i=2}^{n_1}(x - n_1 - \mu_i(G_2))$$

$$\prod_{i=2}^{n_1}(x^2 - (2r_1 + n_2)x + n_2r_1 + r_1^2 - \lambda_i(G_1)^2)$$

Proof. The Laplace adjacency matrix of $G_1 \cup G_2$ is

$$L = \begin{bmatrix}
(r_1 + n_2)I & -A_1 & J_{n_1 \times n_2} \\
-A_1 & r_1I & 0_{n_1 \times n_2} \\
-J_{n_2 \times n_1} & 0_{n_1 \times n_2} & n_1I_{n_2} + L_2
\end{bmatrix}$$

where $L_2$ is the Laplacian adjacency matrix of $G_2$.

The Laplacian Characteristic polynomial of $G_1 \cup G_2 = f_{G_1 \cup G_2}(L : x) = det((x - n_1)I_{n_2} - L_2) detS$

where

$$S = \begin{pmatrix}
(x - r_1 - n_2)I_{n_1} & A_1 \\
A_1 & (x - r_1)I_{n_1}
\end{pmatrix} - \begin{pmatrix}
J & 0
\end{pmatrix} ((x - n_1)I_{n_1} - L_2)^{-1} \begin{pmatrix}
J \\
0
\end{pmatrix}$$

$$= \left( (x - r_1 - n_2)I - A_1 \right) \left( \Gamma_{L_2} (x - n_1)J_{n_1 \times n_1} 0 \right)$$

$$= \left( (x - r_1 - n_2)I - \Gamma_{L_2} (x - n_1)J \right) \left( A_1 \right)$$

$$det S = (x - r_1)^{n_1} det \left( (x - r_1 - n_2)I - \Gamma_{L_2} (x - n_1)J - \frac{A_1^2}{x - r_1} \right)$$

By corollary 2.7

$$det S = (x - r_1)^{n_1} det \left( (x - r_1 - n_2)I - \frac{A_1^2}{x - r_1} \right)$$

$$\left( 1 - \Gamma_{L_2} (x - n_1) \frac{A_1^2}{x - r_1} (x - r_1 - n_2) \right)$$

$$= det ((x - r_1 - n_2)(x - r_1)I - A_1^2) \left( 1 - \Gamma_{L_2} (x - n_1) \frac{A_1^2}{x - r_1} (x - r_1 - n_2) \right)$$
Since $G_1$ is $r_1$ regular graph, the row sum of $A^2_i \over x-r_i$ is $\frac{r_1^2}{x-r_1}$.
Therefore

$$\Gamma \frac{A^2}{x-r_1} (x - r_1 - n_2) = \frac{n_1(x - r_1)}{x^2 - (2r_1 + n_2)x + n_2r_1}$$

$$1 - \Gamma \frac{A^2}{x-r_1} (x - r_1 - n_2) = \frac{x(x^2 - (n_1 + n_2 + 2r_1)x + r_1(2n_1 + n_2))}{(x - n_1)(x^2 - (2r_1 + n_2)x + n_2r_1)}$$

Hence

$$f_{G_1 \cup G_2}(L : x) = x(x^2 - (n_1 + n_2 + 2r_1)x + r_1(2n_1 + n_2)) \prod_{i=2}^{n_2}(x - n_1 - \mu_i(G_2))$$

$$\prod_{i=2}^{n_1}(x^2 - (2r_1 + n_2)x + n_2r_1 + r_1^2 - \lambda_i(G_1)^2)$$

\[\square\]

Let $t(G)$ denote the number of spanning tree of the graph $G$, the total number of distinct spanning subgraphs of $G$ that are trees. The number of spanning trees of the graph describe the network which is one of the natural characteristics of its reliability. If $G$ is a connected graph with $n$ vertices and the Laplacian spectrum $0 = \mu_1(G) \leq \mu_2(G), \cdots , \leq \mu_n(G)$ then [17]

$$t(G) = \frac{\prod_{i=2}^{n}(\mu_2(G)\mu_3(G)\cdots\mu_n(G))}{n}$$

**Corollary 3.6.** Let $G_1$ be an $r_1$ - regular graph on $n_1$ vertices and $G_2$ be an arbitrary graph on $n_2$ vertices. Then

$$t(G_1 \cup G_2) = \frac{r_1(2n_1 + n_2)\prod_{i=2}^{n_1}(n_1 + \mu_i(G_2))\prod_{i=2}^{n_2}(r_1^2 + n_2r_1 - \lambda_i^2(G_1))}{2n_1 + n_2}$$

**Proof.** By Theorem 3.5 the roots of $f_{G_1 \cup G_2}(L : x)$ are as follows

1. 0
2. $n_1 + \mu_i(G_2)$ for $i = 2, 3, \ldots, n_2$
3. Two roots say $x_1$ and $x_2$ of the equation $x^2 - (n_1 + n_2 + 2r_1)x + r_1(2n_1 + n_2)$
4. Two roots say $x_{i1}$ and $x_{i2}$ of the equation $x^2 - (2r_1 + n_2)x + n_2r_1 + r_1^2 - \lambda_i(G_1)^2$

for $i = 2, 3, \ldots, n_2$

For case (iii) $x_1x_2 = r_1(2n_1 + n_2)$
For case (iv) $x_{i1}x_{i2} = n_2r_1 + r_1^2 - \lambda_i(G_1)^2$, $i = 2, 3, \ldots, n_2$

Then

$$t(G_1 \cup G_2) = \frac{r_1(2n_1 + n_2)\prod_{i=2}^{n_1}(n_1 + \mu_i(G_2))\prod_{i=2}^{n_2}(r_1^2 + n_2r_1 - \lambda_i^2(G_1))}{2n_1 + n_2}$$

\[\square\]

Another Laplacian spectrum based on graph invariant was defined by Liu and Liu [3] called the Laplacian - energy - like invariant.

The Laplacian - energy - like invariant(LEL) of a graph $G$ of $n$ vertices is defined as $LEL(G) = \sum_{i=2}^{n} \sqrt{\mu_i}$

**Corollary 3.7.** Let $G_1$ be an $r_1$ - regular graph on $n_1$ vertices and $G_2$ be an arbitrary graph on $n_2$ vertices. Then Laplace - energy - like invariant

$$LEL = \left( n_1 + n_2 + 22^{1/2} + 2\sqrt{r_1(2n_1 + n_2)} \right)^{1/2} + \sum_{i=2}^{n_2} (n_1 + \mu_i(G_1)^2)^{1/2}$$

$$+ \sum_{i=2}^{n_1} \left( \frac{22^{1/2} + \sqrt{r_1^2 + n_2r_1 - \lambda_i^2(G_1)^2}}{r_1^2 + n_2r_1 - \lambda_i^2(G_1)^2} \right)^{1/2}$$
Proof. Using the Theorem 3.5 and Corollary 3.6 we have
\[ \sqrt{x_1} + \sqrt{x_2} = \left( x_1 + x_2 + 2\sqrt{x_1x_2} \right)^{1/2} \]
\[ = \left( n_1 + n_2 + 2\sqrt{r_1(2n_1 + n_2)} \right)^{1/2} \]
\[ \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} = \frac{\sqrt{x_1} + \sqrt{x_2}}{2\sqrt{x_1x_2}} \]
\[ = \frac{x_1 + x_2 + \sqrt{x_1x_2}}{x_1x_2} \]
\[ = \left( \frac{2r_1 + n_2 + \sqrt{r_1^2 + n_2 - \lambda_1(G_1)^2}}{r_1^2 + n_2 + \lambda_1(G_1)^2} \right)^{1/2} \]
Hence the required result is obtained using the formula for LEL.

Klein [5] propounder of resistance distance defined electric resistance in network corresponding to the considered graph as the resistance distance between any two adjacent nodes is 1 ohm. The sum of the resistance distance between all pairs of the vertices of a graph is conceived as a new graph invariant. The electric resistance is calculated by means of the Kirchhoff laws called Kirchhoff index.

Kirchhoff index of a connected graph \( G \) with \( n \) (\( n \geq 2 \)) vertices is defined as
\[ Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i} \]

**Corollary 3.8.** Let \( G_1 \) be an \( r_1 \)-regular graph on \( n_1 \) vertices. \( G_2 \) be an arbitrary graph on \( n_2 \) vertices. Then
\[ Kf(G_1 \cup G_2) = (2n_1 + n_2) \left[ \frac{n_1 + n_2 + 2r_1}{r_1(2n_1 + n_2)} + \sum_{i=2}^{n_2} \frac{1}{\mu_i(G_2)} + \sum_{i=2}^{n_1} \frac{2r_1 + n_2}{r_1^2 + n_2 r_1 - \lambda_i(G_1)^2} \right] \]

**Proof.** Using Theorem 3.5, Corollary 3.7 and the formula for Kirchhoff index we obtain the required result.

**3.2. Spectra of \( DG - addvertex \) graph.**

**Proposition 3.9.** Let \( G_1 \) be an \( r_1 \)-regular graph on \( n_1 \) vertices and \( G_2 \) be an arbitrary graph on \( n_2 \) vertices then \( G_1 \cup G_2 \) and \( G_1 \bowtie G_2 \) are A-Cospectral

**Proof.** We can prove that the characteristic polynomial of \( G_1 \cup G_2 \) and \( G_1 \bowtie G_2 \) are same.

**Proposition 3.10.** Let \( G_1 \) be an \( r_1 \)-regular graph on \( n_1 \) vertices and \( G_2 \) be an arbitrary graph on \( n_2 \) vertices then \( G_1 \cup G_2 \) and \( G_1 \bowtie G_2 \) are L-Cospectral

**4. Infinite Families of Integral Graphs**

The following properties give a necessary and sufficient condition for \( DG - vertex \) join and \( DG - addvertex \) join of \( G_1 \) and \( G_2 \) to be integral.

**Proposition 4.1.** Let \( G_1 \) be \( r_1 \)-regular graph on \( n_1 \) vertices and \( G_2 \) be \( r_2 \)-regular graph on \( n_2 \) vertices. \( G_1 \cup G_2 \) (respectively \( G_1 \bowtie G_2 \)) is an integral graph if and only if \( G_1 \) and \( G_2 \) are integral graphs and the roots of \( x^3 - r_2 x^2 - (n_1 n_2 + r_1^2)x + r_1^2 r_2 \) are integers.

In particular if \( G_2 = \overline{K_n} \) (totally disconnected) then \( r_2 = 0 \) then \( G_1 \cup G_2 \) (respectively \( G_1 \bowtie G_2 \)) is integral iff \( G_1 \) is an integral graph and \( n_1 n_2 + r_1^2 \) is a perfect square.
Figure 3: $K_4 \sqcup \overline{K_4}$ with spectrum $\{-5, -1^3, 0^4, 1^3, 5\}$

**Proposition 4.2.** Let $G_1$ be $r_1$-regular graph on $n_1$. $G_1 \sqcup K_{pq}$ (respectively $G_1 \ltimes K_{pq}$) is an integral graph if and only if $G_1$ is an integral graph and the roots of $x^4 - (pq + r_1^2 + n_1 p + n_1 q)x^2 - 2pqn_1 x + r_1^2 pq$ are integers.

**References**


