Regular Rational Diophantine Sextuples

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A polynomial equation in six variables is given that generalises the definition of **regular** rational Diophantine triples, quadruples and quintuples to **regular** rational Diophantine sextuples. The definition can be used to extend a rational Diophantine quintuple to a **weak** rational Diophantine sextuple. In some cases a regular sextuple is a full rational Diophantine sextuple. Ten examples of this are provided.

Introduction

A rational Diophantine m-tuple is a set of m positive rational numbers $\{a_1, \dots, a_m\}$ such that the product of any two is one less than a rational number squared.

$$a_i a_j + 1 = {x_{ij}}^2, i \neq j, \quad a_{i,} \, x_{ij} \in \mathbb{Q}$$

The problem of finding such m-tuples was originally introduced in the third century AD by Diophantus of Alexandria who was able to find triples and quadruples of such numbers [1]. Diophantus was interested in solving a variety of algebraic problems in rational numbers. In this case it is not clear why he choose such an esoteric problem without any natural motivation but it has turned out to be a rich subject connecting Fibonacci numbers [2] elliptic curves [3] and algebraic invariants [4] while providing many conjectures, generalisations, and of course some results.

During the renaissance Pierre de Fermat reinvented Diophantine number theory as the search for solutions in integers rather than rationals and provided the sequence $1,3,8,120$ as the first Diophantine quadruple in positive integers [5]. It was not until the twentieth century that Baker and Davenport showed that no fifth integer can be added to Fermat's sequence to make a Diophantine quintuple [6]. However there are many such Diophantine quadruples and it is an outstanding problem to determine whether any such Diophantine quintuple exists. It is now known that no Diophantine sextuple exists in integers and there is a bound on the number of possible quintuples [7,8].

After so much progress on the problem in integers, focus is returning to the problem in rationals. Euler discovered that a fifth rational can be added to Fermat's sequence to give the following rational Diophantine quintuple (the Fermat-Euler sequence) [9]

$$1, 3, 8, 120, \frac{777480}{8288641}$$

No sixth rational that extends this sequence further has been found, nor has any alternative value for the fifth rational. However, rational Diophantine quintuples are also very abundant and there are now known examples of rational Diophantine sextuples such as [10,11,12]

$$\frac{96}{847}, \frac{287}{484}, \frac{1209}{700}, \frac{129}{28}, \frac{147}{25}, \frac{847}{4}$$

Very recently some infinite families of sextuples have been found [13,14,15] but no substantial progress has been made towards finding a rational Diophantine septuple with seven fractions or proving their non-existence.

Regular Diophantine m-tuples

If the existence of Diophantine m-tuples were a pseudo-random process where the probability of a positive integer N being square is $N^{-\frac{1}{2}}$, how many of them would be expected? For an m-tuple of height H (height being the largest numerator or denominator) the probability of it being a rational Diophantine m-tuple would be of order $H^{-m(m-1)}$ and the number of m-tuples of this size is of order H^{2m-1}. The expected number of rational Diophantine m-tuples would therefore be given by an integral of order $\int H^{3m-m^2-1} dH$.

This integral diverges logarithmically for $m = 3$ and converges rapidly for $m > 3$. This means infinitely many rational Diophantine triples would be expected but they should be rare. Rational Diophantine quadruples and larger m-tuples would only be finite in number, if they existed at all. Only Diophantine pairs ($m = 2$) should exist in large numbers. In reality m-tuples are not pseudo-random in this way and it is only the pairs that follow this prediction. There is also an abundance of m-tuples up to at least $m = 6$. This means that there must be some principle at work that makes them more common than the pseudo-random argument suggests.

This unexpected plenitude of rational Diophantine m-tuples can in part be explained by the existence of symmetric polynomial equations which can be solved to extend rational Diophantine m-tuples to rational Diophantine $(m + 1)$-tuples for $= 2,3,4$.

Given two distinct positive rational numbers $a, b \in \mathbb{Q}$ such that $ab + 1 = x^2,\ x \in \mathbb{Q}$ (called a rational Diophantine pair), a third rational number c can be defined in two ways to make a rational Diophantine triple using the formula

$$c = a + b \pm 2x$$

This is equivalent to the polynomial formula

$$P(a, b, c) = (a + b - c)^2 - 4(ab + 1) = 0$$

When expanded, this expression is found to be symmetric under permutations of a, b, and c which means it can also be written as

$$(a + c - b)^2 = 4(ac + 1)$$

$$(b + c - a)^2 = 4(bc + 1)$$

Therefore, given the rational Diophantine pair $\{a, b\}$, c can be found as a solution to $P(a, b, c) = 0$ and then $ac + 1$ and $bc + 1$ will be squares giving the rational Diophantine triple $\{a, b, c\}$ When using the minus sign to give $c = a + b - 2x$ the triple can fail to be valid because c may be zero or negative or a repetition of a or b, but when using the plus sign $c = a + b + 2x$, c is always positive and distinct from a and b, so a valid rational Diophantine triple is always formed.

A rational Diophantine triple that satisfies the equation $P(a, b, c) = 0$ is said to be **regular** and one that does not is **irregular** [10]. There are many examples of either in both rationals and positive integers.

Similar polynomials exist for regular quadruples and quintuples. For quadruples the polynomial is defined by [16]

$$P(a, b, c, d) = (a + b - c - d)^2 - 4(ab + 1)(cd + 1)$$

Again this is symmetric under permutations of the four variables. It is quadratic in each variable individually but is quartic overall due to the inclusion of the term $-4abcd$

The equation $P(a, b, c, d) = 0$ can be solved for d by completing the square and finding that the discriminant factorizes giving,

$$P(a, b, c, d) = (2abc + a + b + c - d)^2 - 4(ab + 1)(bc + 1)(ac + 1)$$

This shows that if $\{a, b, c\}$ is a Diophantine triple, then the equation can be solved for d giving two solutions at least one of which is positive and not equal to a, b or c. $\{a, b, c, d\}$ will then be a rational Diophantine quadruple [17] (e.g. $cd + 1$ is a square when $ab + 1$ is a square because of the defining equation) It satisfies $P(a, b, c, d) = 0$ so we call it **regular**.

For quintuples the corresponding polynomial is defined by

$$P(a, b, c, d, e) =$$

$$(abcde + 2abc + a + b + c - d - e)^2 - 4(ab + 1)(ac + 1)(bc + 1)(de + 1)$$

Once again this can be solved for e to extend a rational Diophantine quadruple to a regular rational Diophantine quintuple [18]. This time the expression has a factor $(abcd - 1)^2$ in the denominator and it can fail in exceptional circumstances including when $\{a, b, c, d\}$ is regular and $abcd = 1$ (It is an interesting exercise to work out the general solution to this case.)

The polynomials are related by $P(a, b, c) = P(a, b, c, 0)$ and $P(a, b, c, d) = P(a, b, c, d, 0)$. For completeness $P(a, b) = P(a, b, 0) = (a - b)^2 - 4$ and $P(a) = P(a, 0) = a^2 - 4$

The Fermat-Euler sequence is then the positive solution to

$$a = 1, P(a, b) = P(a, b, c) = P(a, b, c, d) = P(a, b, c, d, e) = 0$$

Is there a similar polynomial for extending rational Diophantine quintuples to sextuples and beyond? This would require a polynomial $P(a, b, c, d, e, f)$ which is symmetric under permutations of all its arguments and whose discriminant as a quadratic in f factorises to four times the product of all squares formed in the remaining quintuple. To continue the sequence we also expect that $P(a,b,c,d,e) = P(a,b,c,d,e,0)$. Until now it has been assumed that no solution to this exists but in fact it does and is given as follows,

$$P(a,b,c,d,e,f)$$

$$=$$

$$(abcde + abcdf + abcef - abdef - acdef - bcdef$$

$$+2abc - 2def + a + b + c - d - e - f)^2$$

$$-4(ab+1)(ac+1)(bc+1)(de+1)(df+1)(ef+1)$$

A rational Diophantine sextuple will be called regular if it satisfies $P(a,b,c,d,e,f) = 0$.

Given any rational Diophantine quintuple $\{a,b,c,d,e\}$, this equation can be solved for f with two roots except in special cases. The weakness of this extension method compared to those for smaller m-tuples is that $\{a,b,c,d,e,f\}$ is often not a rational Diophantine sextuple. From the definition we only get that the product $(df+1)(ef+1)$ is a square and similarly when $\{d,e\}$ is replaced with other pairs of elements from the original quintuple. In other words the five products $(af+1),\dots,(ef+1)$ are squares multiplied by a single common factor.

For example this equation can be used to add a sixth element f to the Fermat-Euler sequence

$$1, 3, 8, 120, \frac{777480}{8288641}, \frac{2928955408242515137 20}{38325051691626892 6081}$$

This does not make it a Diophantine sextuple. The product of f with any of the previous five numbers is one less than a square divided by the denominator of f. Nevertheless, f is the natural next element in the sequence because it solves the equation $P(a,b,c,d,e,f) = 0$

Despite its failings as an equation for extending quintuples to sextuples, it does have some value in the theory of sextuples because some of the known examples of sextuples are in fact regular. Here are ten examples:

33/152 7360/5491 4275/2312 1209/152 19/2 1920/19
249/2048 3720/6241 715/384 369/128 38/3 920/3
2261/37752 29/78 989/1248 52793/24576 819/8 30447/104
6/65 68103/437320 616/65 10920/841 1955/104 975/8
17255/149784 247/408 1122/961 2138455/392088 1653240/106097 504/17
1695/23276 3/11 143/108 59840/14283 1335/44 29536/297
25900/690561 100/333 216/185 4004/1665 518/45 7344/185
232/1875 150/529 209/96 40672/1587 24864/625 3675/32
65455/411864 103/96 275575/185856 2445/968 213/8 252343/726
703560/5555449 3243/4576 248/143 913/416 7215/352 51510/143

For reference the polynomial $P(a,b,c,d,e,f)$ which has 105 terms when fully expanded can be conveniently written in symmetric form as:

$$P(a,b,c,d,e,f) =$$

$$(abcde)^2 + (abcdf)^2 + (abcef)^2 + (abdef)^2 + (acdef)^2 + (bcdef)^2 - 4(abcdef)^2$$

$$-2(abcdef + 2)\begin{pmatrix} abcd + abce + abcf + abde + abdf + abef + acde \\ +acdf + acef + adef + bcde + bcdf + bcef + bdef + cdef \end{pmatrix}$$

$$-2(2abcdef + 1)\begin{pmatrix} ab + ac + ad + ae + af + bc + bd + be \\ +bf + cd + ce + cf + de + df + ef \end{pmatrix} \qquad - 8abcdef$$

$$-2(a + b + c + d + e + f)(abcde + abcdf + abcef + abdef + acdef + bcdef)$$

$$+a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 4$$

Weak Diophantine m-tuples

The value of the equation for regular Diophantine sextuples can be understood a little better in the context of weak Diophantine m-tuples defined as follows.

A set of positive rational numbers $\{a_1, \dots, a_m\}$ is a **weak Diophantine m-tuple** if

$$(a_i a_j + 1)(a_i a_k + 1)(a_j a_k + 1) = x^2, x \in \mathbb{Q}, i < j < k$$

A weak Diophantine m-tuple up to $m = 6$ will be called regular when it satisfies the same polynomial equations that define rational Diophantine m-tuples as regular.

Here are some properties of weak Diophantine m-tuples:

A rational Diophantine m-tuple is also a weak Diophantine m-tuple

If $\{a_1, \dots, a_m\}$ is a weak Diophantine m-tuple then so is the set of its reciprocals $\left\{\frac{1}{a_1}, \dots, \frac{1}{a_m}\right\}$. This is because $\left(\frac{1}{a_i}\frac{1}{a_j} + 1\right)\left(\frac{1}{a_i}\frac{1}{a_k} + 1\right)\left(\frac{1}{a_j}\frac{1}{a_k} + 1\right) = \frac{(a_i a_j+1)(a_i a_k+1)(a_j a_k+1)}{(a_i a_j a_k)^2}$

A weak Diophantine triple $\{a,b,c\}$ can be extended to a weak Diophantine quadruple $\{a,b,c,d\}$ by solving $P(a,b,c,d) = 0$.

A weak Diophantine triple $\{a,b,c\}$ can also be extended to a weak Diophantine quadruple $\{a,b,c,d\}$ by solving $abcd = 1$.

A weak Diophantine m-tuple in positive integers is always a Diophantine m-tuple. Proof: any weak Diophantine triple in positive integers can be extended to a weak Diophantine quadruple in integers by solving $P(a,b,c,d) = 0$. However, it is known that any solution of this equation in positive integers is a Diophantine quadruple (proof is by infinite decent.) This implies that the weak Diophantine triple is a Diophantine triple. Since this applies to any triple in the weak Diophantine m-tuple it means that it is a Diophantine m-tuple.

A regular weak Diophantine quintuple is a regular rational Diophantine m-tuple. This follows from the defining equation for regular quintuples.

In general a weak Diophantine quadruple cannot be extended to a weak Diophantine quintuple using the equation for regular quintuples.

In a weak Diophantine quintuple, the product of 10 factors $D = \prod_{i<j}(a_i a_j + 1)$ is a square. This is because the product $\prod_{i<j<k}(a_i a_j + 1)(a_i a_k + 1)\left(a_j a_k + 1\right) = D^3$and since each triple factor is a square this makes D^3a square. Therefore D is a square.

Since D is a square, the equation for a regular Diophantine sextuple can usually be solved in rationals to extend a weak Diophantine quintuple to a sextuple. The defining equation for the polynomial equation then makes this a weak Diophantine sextuple.

If a weak Diophantine sextuple is regular then its reciprocal is also regular. This follows from the identity $P\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}, \frac{1}{f}\right) \times\ (abcdef)^2 = P(a, b, c, d, e, f)$ which can be verified from the definition.

The unusual case when $P(a, b, c, d, e, f)$ cannot be solved for f given a weak Diophantine quintuple $\{a, b, c, d, e\}$ is when the polynomial coefficient of f^2 is zero and the polynomial terms independent if f give zero. In other words when both the quintuple and its reciprocal are regular, but then they are both rational Diophantine quintuples. It is not known if there are any examples of regular rational Diophantine quintuples whose reciprocals are also regular rational Diophantine quintuples.

The Square Identities

A number of identities for the polynomials $P(a, b, ...)$ have already been given in the form

$$B^2P = A^2 - 4\Pi$$

Where A and B are polynomials and Π is a product of factors like $(ab + 1)$ etc.

In order to understand why the polynomial for regular sextuples does not always give full rational Diophantine sextuples and why there is no generalisation to septuples it is helpful to record the full list of these identities.

For quadruples upwards these identities do not exist for all possible products Π but the cases in which they do exist can be characterised as follows:

Partition the m-tuple into two subsets of variables X and Y. The products are formed in of two ways, by taking all factors $(ab + 1)$ where either a and b are in the same subset X or Y, or a and b are in different subsets X and Y. The product in the first case will be written as $\Pi\{X; Y\}$ and in the second case as $\Pi[X; Y]$. For example with four variables

$$\Pi\{a, b; p, q\} = (ab + 1)(pq + 1)$$

$$\Pi\{a, b, c; p\} = (ab + 1)(ac + 1)(bc + 1)$$

$$\Pi\{a, b, c, d\} = (ab + 1)(ac + 1)(ad + 1)(bc + 1)(bd + 1)(cd + 1)$$

$$\Pi[a, b; p, q] = (ap + 1)(bp + 1)(aq + 1)(bq + 1)$$

$$\Pi[a, b, c; p] = (ap + 1)(bp + 1)(cp + 1)$$

$$\Pi[a, b, c, d] = 1$$

For each of these products there is an identity written

$$B\{X; Y\}^2P(X, Y) = A\{X; Y\}^2 - 4\Pi\{X; Y\}$$

$$B[X; Y]^2P(X, Y) = A[X; Y]^2 - 4\Pi[X; Y]$$

The identities can be summarised in two tables. For the largest cases YES/NO are used to indicate where they exist or not.

$$A\{X;Y\}$$

		p	p q	p q r
	0	$-p$	$-p-q$	$-2pqr-p-q-r$
a	a	$a-p$	$a-p-q$	$a-2pqr-p-q-r$
a b	$a+b$	$a+b-p$	$a+b-p-q$	$abpqr-2pqr-p$ $-q-r-a-b$
a b c	$2abc+a+b+c$	$2abc+a+b+c-p$	$abcpq+2abc+a$ $+b+c-p-q$	$abc(pq+pr+qr)$ $-pqr(ab+bc+bc)$ $+2abc-2pqr$ $+a+b+c-p-q-r$
a b c d	$abcd(a+b+c+d)$ $+2abc+2abd+2bcd$ $+2acd+a+b+c+d$	$abcd(a+b+c+d)$ $+2abc+2abd+2bcd$ $+2acd+a+b+c$ $+d-p(1-abcd)^2$	NO	
a b c d e	YES	YES		
a b c d e f	NO			

The polynomials $B\{X;Y\}$ have not been shown. For the first three rows they are 1 and for the fourth row they are $(1-abcd)$.

How can we be sure that no solution exists for the sextuple case where indicated with “NO”? If a solution did exist in either of these cases then it could be used to show that the extension formula would always give a full rational Diophantine sextuple but the Fermat-Euler sequence is already a counterexample to that possibility.

The entries in the table have been written so that it is possible to move upwards or leftwards by setting one of the variables to zero. This means that no entries for septuple cases are possible because if they were it would be possible to move either up or left and provide a solution for one of the forbidden sextuple cases.

$$A[X;Y]$$

		p	p q
	2	$2-p^2$	$2-(p-q)^2$
a	2	$2-p^2+ap$	$2-(p-l)^2+ap+aq$
a b	2	$2-p^2+ap+bp$	$2-(p-q)^2+ap+bp+aq+bq+2abpq$
a b c	2	$2-p^2+ap+bp+cp$	$2-(p-q)^2+abcpq(p+q)$ $+(a+b+c)(p+q)$ $+2(ab+bc+ac)pq$
a b c d	2	$2-abcdp^2-p^2$ $+ap+bp+cp+dp$	$2-(p-q)^2+abcd(2p^2q^2-(p-q)^2)$ $+(abc+abd+acd+bcd)pq(p+q)$ $+(a+b+c+d)(p+q)$ $+2(ab+bc+ac+ad+bd+cd)pq$
a b c d e	2	NO	
a b c d e f	2		

The polynomial $B[X,Y]$ is zero for the first column (trivial case), p for the second column and $(p-q)$ for the third column.

Once again the “NO” case can’t exist because it would imply that extension gives full sextuples and no case for septuples fit. No fourth column can be added preserving the rule for moving up and left by setting variables to zero.

In summary, the identities for sextuples only exist with products $\Pi\{X;Y\}$ or $\Pi[X;Y]$ when the number of times each variable appears in the product is even. The extension formula therefore works to extend weak Diophantine m-tuples but does not normally succeed in extending rational Diophantine quintuples to full rational Diophantine sextuples. Nevertheless it only requires one of the new products to be one less than a square and they will all be. There are multiple instances where this happens and extension does then produce a full rational Diophantine sextuple.

Further Polynomial Generalisations

The polynomials that define regular m-tuples can in part be explained from the theory of elliptic curves [19,20], yet the full level of symmetry remains mysterious. Some further explanation arises from the observation that the polynomial equation for regular quadruples is a special case of Cayley's hyperdeterminant which generalises the 2 x 2 determinant to an expression for a 2 by 2 by 2 array. This is done in such a way as to extend its properties as a polynomial invariant and as a discriminant [4].

A polynomial generalising $P(a,b,c,d)$ can be defined as

$$H(a,b,c,d,k,l,m,n) = (ak + bl - cm - dn)^2 - 4(ab + nm)(cd + kl)$$

Then Cayley's hyperdeterminant for a three dimensional array of numbers a_{ijk} is given by

$$Det\left(a_{ijk}\right) = H(-a_{000}, a_{110}, a_{101}, a_{011}, -a_{111}, a_{001}, a_{010}, a_{100})$$

The equation for regular Diophantine quadruples can be recovered from

$$P(a,b,c,d) = H(a,b,c,d,1,1,1,1) = H(1,1,1,1,a,b,c,d)$$

The following identity can also be verified

$$\begin{aligned} n^2 H(a,b,c,d,k,l,m,n) \\ &= (2abc + ank + bnl + cnm - dn^2)^2 - 4(ab + nm)(ac + nl)(bc + nk) \end{aligned}$$

This shows that the quadratic discriminant for Cayley's hyperdeterminant when treated as a quadratic in any one of its variables factorises into three factors which are 2 by 2 determinants. This had never been noted before the comparison with the formula for regular Diophantine quadruples had been made.

When reduced to expressions for regular Diophantine quadruples this identity for the hyperdeterminant yields two cases one from each of the two tables above for $A[a,b;c,d]$ and $A\{a,b;c,d\}$

Given that $P(a,b,c,d)$ generalises to $P(a,b,c,d,e)$ and then to $P(a,b,c,d,e,f)$, it is natural to investigate whether $H(a,b,c,d,k,l,m,n)$ also generalises to expressions in more variables which reduce to the expressions for regular quintuples and sextuples. Cayley's hyperdeterminant can be generalises to invariants for multi-dimensional arrays of any size but it does not appear that any of these can be reduced as required.

Nevertheless, the generalisations do exist, but they are not invariants, discriminants or any other kind of previously recognised polynomials. Their origins and significance therefore remains mysterious and nothing more can be done other than to describe what they are.

The generalisation for $P(a,b,c,d,e)$ is a polynomial of degree ten in fifteen variables defined in terms of a simple block design. Fifteen variables can be grouped into six blocks of five such that each variable appears in two blocks. The polynomial is formed from just the products of each block and its compliment. I.e.

$$T_1 = abcde,\quad T_2 = asghk,\quad T_3 = bslmn,\quad T_4 = cglpq,\quad T_5 = dhmpr,\quad T_6 = eknqr$$

(Notice that this can also be regarded as the parametric solution to the problem of finding six square free integers whose product is a square number)

$$T_1{T'}_1 = T_2{T'}_2 = T_3{T'}_3 = T_4{T'}_4 = T_5{T'}_5 = T_6{T'}_6 = abcdeghklmnpqrs = \sqrt{T_1T_2T_3T_4T_5T_6}$$

Then the polynomial is given by

$$H(a,b,c,d,e,g,h,k,l,m,n,p,q,r,s) = \sum_i {T_i}^2 - 2\sum_{i<j} T_i T_j - 4\sum_i {T'}_i$$

Identities satisfied by this polynomial which reduce to the known identities for $P(a,b,c,d,e)$ include the following two,

$$r^2 H(a,b,c,d,e,g,h,k,l,m,n,p,q,r,s) =$$

$$(abcder + 2abcsgl + asghkr + bslmnr + cglpqr - dhmpr^2 - eknqr^2)^2$$

$$-4(abs + pqr)(acg + mnr)(bcl + hkr)(der + slg)$$

$$(hdmp - kenq)^2 H(a,b,c,d,e,g,h,k,l,m,n,p,q,r,s) =$$

$$\begin{aligned}(2lqngpksmh &+ 2bcdelmnpq + 2acdeghkpq + 2abdehkmns + abcd^2ehmp \\ &+ abcde^2knq + adgh^2kmps + aeghk^2nqs + bdhlm^2nps \\ &+ beklmn^2qs + cdghlmp^2q + cegklnpq^2 - r(eknq - dhmp)^2)^2\end{aligned}$$

$$-4(adh + lqn)(bdm + gqk)(cdp + snk)(aek + pml)(ben + gph)(ceq + smh)$$

The master generalisation of the equation for regular sextuples is a polynomial of degree 32 in 32 variables which can be defined by the following identity

$$p^2P^2H(a,\dots,Z) =$$

$$(abcdeFStuVwxYzrp^2P + abcdEfsTuvWxyZrp^2P + abcDefstUvwXyzRp^2P$$

$$+ aBCDEFstuvwxYZRp^2P + AbCDEFstuVWXyzrp^2P + ABcDEFSTUvwxyzrp^2P$$

$$- aBcdefSTUvwxYZRpP^2 - AbcdefSTUVWXyzrpP^2 - abCdefstuVWXYZRpP^2$$

$$- ABCdEFSTuVWxYZrpP^2 - ABCDeFStUVwXYzRpP^2 - ABCDEfsTUvWXyZRpP\text{^}2$$

$$+ 2abcDEFstuvwxyzrp^3 - 2ABCdefSTUVWXYZRP^3)^2$$

$$-4\,(abstup + ABSTUP)\,(acvwxp + ACVWXP)\,(bcyzrp + BCYZRP)$$

$$(deSVYP + DEsvyp)\,(dfTWZP + DFtwzp)\,(efUXRP + EFuxrp)$$

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