AN APPELL SERIES OVER FINITE FIELDS

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Abstract. In this paper we give a finite field analogue of one of the Appell series and obtain some transformation and reduction formulae and the generating functions for the Appell series over finite fields.

1. Introduction

Let $\mathbb{F}_q$ denote the finite field of q elements and $\widehat{\mathbb{F}_q^*}$ the group of multiplicative characters of $\mathbb{F}_q^*$ where q is a power of a prime. We extend the domain of all characters χ of $\mathbb{F}_q^*$ to $\mathbb{F}_q$ by setting $\chi(0) = 0$ for all characters and denote $\overline{\chi}$ and ε as the inverse of χ and the trivial character respectively. See [2] and [7, Chapter 8] for more details about characters.

The generalized hypergeometric function is defined by [1]

$$ {}_{n+1}F_n\left(\begin{matrix} a_0, a_1, \ldots, a_n \\ b_1, \ldots, b_n \end{matrix}\middle| x\right) := \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_n)_k}{k!(b_1)_k \cdots (b_n)_k} x^k $$

where $(z)_k$ is the Pochhammer symbol given by

$$ (z)_0 = 1,\ (z)_k = z(z+1)\cdots(z+k-1) \text{ for } k \geq 1. $$

It was Greene [6] who in 1987 developed the theory of hypergeometric functions over finite fields and proved a number of transformation and summation identities for hypergeometric series over finite fields which are analogues to those in the classical case. In that paper, Greene introduced the notation

$$ {}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix}\middle| x\right)^G = \varepsilon(x)\frac{BC(-1)}{q}\sum_y B(y)\overline{B}C(1-y)\overline{A}(1-xy) $$

for $A, B, C \in \widehat{\mathbb{F}_q}$ and $x \in \mathbb{F}_q$, which is the finite field analogue of the integral representation of Gauss hypergeometric series [1]:

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^b(1-t)^{c-b}(1-tx)^{-a}\frac{dt}{t(1-t)}, $$

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and defined the finite field analogue of the binomial coefficient as

$$\binom{A}{B}^G = \frac{B(-1)}{q} J(A, \overline{B}),$$

where $J(\chi, \lambda)$ is the Jacobi sum given by

$$J(\chi, \lambda) = \sum_u \chi(u)\lambda(1-u).$$

For more information about the finite field analogue of the generalized hypergeometric functions, please see [4, 5, 9].

In this paper, for the sake of simplicity, we use the notation

$$\binom{A}{B} = q\binom{A}{B}^G = B(-1)J(A, \overline{B}).$$

Then the finite field analogue of the binomial theorem can be written in the form:

Theorem 1.1. (Binomial theorem, see [6, (2.5)]) *For any character $A \in \widehat{\mathbb{F}_q}$ and $x \in \mathbb{F}_q$, we have*

$$A(1+x) = \delta(x) + \frac{1}{q-1}\sum_\chi \binom{A}{\chi}\chi(x),$$

where the sum ranges over all multiplicative characters of $\mathbb{F}_q$ and $\delta(x)$ is a function on $\mathbb{F}_q$ given by

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}.$$

Furthermore, we define the finite field analogue of the classic Gauss hypergeometric series as

$${}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix}\middle| x\right) = q \cdot {}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix}\middle| x\right)^G = \varepsilon(x)BC(-1)\sum_y B(y)\overline{B}C(1-y)\overline{A}(1-xy).$$

Then by [6, Theorem 3.6],

$$ {}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix}\middle| x\right) = \frac{1}{q-1}\sum_\chi \binom{A\chi}{\chi}\binom{B\chi}{C\chi}\chi(x) \tag{1.1}$$

for any $A, B, C \in \widehat{\mathbb{F}_q}$ and $x \in \mathbb{F}_q$. Similarly, the finite field analogue of the generalized hypergeometric series for any $A_0, A_1, \cdots, A_n, B_1, \cdots, B_n \in \widehat{\mathbb{F}_q}$ and $x \in \mathbb{F}_q$ is defined by

$${}_{n+1}F_n\left(\begin{matrix} A_0, A_1, \cdots, A_n \\ B_1, \cdots, B_n \end{matrix}\middle| x\right) = \frac{1}{q-1}\sum_\chi \binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi}\cdots\binom{A_n\chi}{B_n\chi}\chi(x).$$

One of Greene's theorems is now stated in our notations.

Theorem 1.2. (See [6, Theorem 4.9])*For any characters $A, B, C \in \widehat{\mathbb{F}_q}$, we have*

$${}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix}\middle| 1\right) = A(-1)\binom{B}{\overline{A}C}. \tag{1.2}$$

There are many interesting double hypergeometric functions in the field of hypergeometric functions. Among these functions, Appell's four functions may be the most important functions:

$$F_1(a;b,b';c;x,y)=\sum_{m,n\geq 0}\frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)_{m+n}}x^my^n,\ |x|<1,\ |y|<1,$$
$$F_2(a;b,b';c,c';x,y)=\sum_{m,n\geq 0}\frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)_m(c')_n}x^my^n,\ |x|+|y|<1,$$
$$F_3(a,a';b,b';c;x,y)=\sum_{m,n\geq 0}\frac{(a)_m(a')_n(b)_m(b')_n}{m!n!(c)_{m+n}}x^my^n,\ |x|<1,\ |y|<1,$$
$$F_4(a;b;c,c';x,y)=\sum_{m,n\geq 0}\frac{(a)_{m+n}(b)_{m+n}}{m!n!(c)_m(c')_n}x^my^n,\ |x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}<1.$$

See [1, 3, 10] for more detailed material about Appell's functions.

Inspired by Greene's work, the second author *et al* in [8] gave a finite field analogue of the Appell series F_1 and obtained some transformation and reduction formulas and the generating functions for the function over finite fields. In that paper, the finite field analogue of the Appell series F_1 was given by

$$F_1(A;B,B';C;x,y)=\varepsilon(xy)AC(-1)\sum_u A(u)\overline{A}C(1-u)\overline{B}(1-ux)\overline{B'}(1-uy)$$

owing to the fact that the F_1 function has the integral representation in terms of a single integral [1, Chapter IX]:

$$F_1(a;b,b';c;x,y)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b}(1-uy)^{-b'}du$$

where $0<\Re(a)<\Re(c)$.

Motivated by the work of Greene [6] and the second author *et al* [8], we give a finite field analogue of the Appell series F_2. Since the Appell series F_2 has the following simple double integral representation [1, Chapter IX]:

$$\begin{aligned}F_2(a;b,b';c,c';x,y)&=\frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')}\\&\quad\cdot\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-u)^{c-b-1}(1-v)^{c'-b'-1}(1-ux-vy)^{-a}dudv,\end{aligned}$$

we now give the finite field analogue of F_2 in the following form:

$$\begin{aligned}&F_2(A;B,B';C,C';x,y)\\&\quad=\varepsilon(xy)BB'CC'(-1)\sum_{u,v}B(u)B'(v)\overline{B}C(1-u)\overline{B'}C'(1-v)\overline{A}(1-ux-vy),\end{aligned}$$

where $A,B,B',C,C'\in\widehat{\mathbb{F}_q},\ x,y\in\mathbb{F}_q$ and each sum ranges over all the elements of $\mathbb{F}_q$. In the above definition, the factor $\frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')}$ is dropped to obtain simpler results.

We choose the factor $\varepsilon(xy)\cdot BB'CC'(-1)$ to get a better expression in terms of binomial coefficients. In the following theorem we give another expression for $F_2(A;B,B';C,C';x,y)$.

Theorem 1.3. *For any $A,B,B',C,C'\in\widehat{\mathbb{F}_q}$ and $x,y\in\mathbb{F}_q$, we have*

$$F_2(A;B,B';C,C';x,y)=\frac{1}{(q-1)^2}\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{A\chi\lambda}{\lambda}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)$$
$$+\overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y)\binom{\overline{A}B}{B\overline{C}},$$

where each sum ranges over all multiplicative characters of $\mathbb{F}_q$.

From the definition of $F_2(A;B,B';C,C';x,y)$, Theorem 1.3 and (1.2), we can easily deduce the following results.

Proposition 1.1. *For any $A,B,B',C,C'\in\widehat{\mathbb{F}_q}$ and $x,y\in\mathbb{F}_q$, we have*

$$F_2(A;B,B';C,C';x,y)=F_2(A;B',B;C',C;y,x), \tag{1.3}$$
$$F_2(A;B,B';C,C';x,1)=B'C'(-1)\,{}_3F_2\left(\left.\begin{matrix}A,B,A\overline{C'}\\ C,AB'\overline{C'}\end{matrix}\right|x\right),$$
$$F_2(A;B,B';C,C';1,y)=BC(-1)\,{}_3F_2\left(\left.\begin{matrix}A,B',A\overline{C}\\ C',AB\overline{C}\end{matrix}\right|y\right).$$

The aim of this paper is to give several transformation and reduction formulas and the generating functions for the Appell series F_2 over finite fields. The facts that the Appell series F_2 does not have a single integral representation but has a double one and the F_1 has a single one led us to giving a finite field analogue for the Appell series F_2 which is more complicated than that for F_1 in [8]. Consequently, the results on the transformation and reduction formulas and the generating functions for the Appell series F_2 over finite fields are also more complicated than those in [8].

The proof of Theorem 1.3 will be given in the next section. We give several transformation and reduction formulae for $F_2(A;B,B';C,C';x,y)$ in Section 3. The last section is devoted to deducing some generating functions for $F_2(A;B,B';C,C';x,y)$.

2. Proof of Theorem 1.3

To carry out our study, we need some auxiliary results which will be used frequently in this paper.

The results in the following proposition follows readily from some properties of Jacobi sums.

Proposition 2.1. *If $A, B \in \widehat{\mathbb{F}_q}$, then*

$$\binom{A}{B} = \binom{A}{A\overline{B}}, \tag{2.1}$$

$$\binom{A}{B} = \binom{B\overline{A}}{B} B(-1), \tag{2.2}$$

$$\binom{A}{B} = \binom{\overline{B}}{\overline{A}} AB(-1), \tag{2.3}$$

$$\binom{A}{\varepsilon} = \binom{A}{A} = -1 + (q-1)\delta(A), \tag{2.4}$$

where $\delta(\chi)$ is a function on characters given by

$$\delta(\chi) = \begin{cases} 1 & \text{if } \chi = \varepsilon \\ 0 & \text{otherwise} \end{cases}.$$

The following result is also very important in the derivation of Theorem 1.3.

Proposition 2.2. *For any character $A \in \widehat{\mathbb{F}_q}$ and $x, y \in \mathbb{F}_q$, we have*

$$A(1+x+y) = \begin{cases} A(x) & \text{if } y = -1 \\ \delta(x)\delta(y) + \frac{1}{q-1}\left(\delta(x)\sum_\chi \binom{A}{\chi}\chi(y) + \delta(y)\sum_\chi \binom{A}{\chi}\chi(x)\right) & \\ \qquad + \frac{1}{(q-1)^2}\sum_{\chi,\lambda}\binom{A}{\chi}\binom{A\overline{\chi}}{\lambda}\chi(x)\lambda(y) & \text{if } y \neq -1 \end{cases},$$

where each sum ranges over all multiplicative characters of $\mathbb{F}_q$.

Proof. It is obvious that $A(1+x+y) = A(x)$ when $y = -1$. We only need to consider the case $y \neq -1$. When $y \neq -1$, by the binomial theorem, we have

$$\begin{aligned} A(1+x+y) &= A(1+y)A\left(1 + \frac{x}{1+y}\right) \\ &= \delta(x)A(1+y) + \frac{1}{q-1}\sum_\chi \binom{A}{\chi}\chi(x)A\overline{\chi}(1+y) \\ &= \delta(x)\delta(y) + \frac{1}{q-1}\left(\delta(x)\sum_\chi \binom{A}{\chi}\chi(y) + \delta(y)\sum_\chi \binom{A}{\chi}\chi(x)\right) \\ &\quad + \frac{1}{(q-1)^2}\sum_{\chi,\lambda}\binom{A}{\chi}\binom{A\overline{\chi}}{\lambda}\chi(x)\lambda(y). \end{aligned}$$

This completes the proof of Proposition 2.2. □

Actually, Proposition 2.2 can be considered as the finite field analogue of the trinomial theorem:

$$(1+x+y)^a = \sum_{i,j\geq 0}\binom{a}{i}\binom{a-i}{j}x^i y^j.$$

We now turn to our proof of Theorem 1.3

Proof of Theorem 1.3. It is clear that $F_2(A;B,B';C,C';x,y)=0$ for $y=0$. We now consider the case $y\neq 0$. When $y\neq 0$, if $v=y^{-1}$, then

$$\overline{A}(1-ux-vy)=\overline{A}(-ux);$$

if $v\neq y^{-1}$, then from Proposition 2.2, we have

$$\overline{A}(1-ux-vy)=\delta(ux)\delta(v)+\frac{1}{q-1}\left(\delta(ux)\sum_{\chi}\binom{\overline{A}}{\chi}\chi(-vy)+\delta(v)\sum_{\chi}\binom{\overline{A}}{\chi}\chi(-ux)\right)$$
$$+\frac{1}{(q-1)^2}\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A}\overline{\chi}}{\lambda}\chi(-ux)\lambda(-vy).$$

It is easily seen from the binomial theorem that

$$\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A}\overline{\chi}}{\lambda}\chi(-ux)\lambda(-1)=\sum_{\chi}\binom{\overline{A}}{\chi}\chi(-ux)\sum_{\lambda}\binom{\overline{A}\overline{\chi}}{\lambda}\lambda(-1)=0,$$

which implies that

$$\sum_{u\in\mathbb{F}_q}B(u)B'(y^{-1})\overline{B}C(1-u)\overline{B'}C'(1-y^{-1})\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A}\overline{\chi}}{\lambda}\chi(-ux)\lambda(-1)=0.$$

Then, by (2.2),

$$\begin{aligned}
&\sum_{u\in\mathbb{F}_q,v\neq y^{-1}}B(u)B'(v)\overline{B}C(1-u)\overline{B'}C'(1-v)\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A}\overline{\chi}}{\lambda}\chi(-ux)\lambda(-vy)\\
&=\sum_{u,v\in\mathbb{F}_q}B(u)B'(v)\overline{B}C(1-u)\overline{B'}C'(1-v)\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A}\overline{\chi}}{\lambda}\chi(-ux)\lambda(-vy)\\
&-\sum_{u\in\mathbb{F}_q}B(u)B'(y^{-1})\overline{B}C(1-u)\overline{B'}C'(1-y^{-1})\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A}\overline{\chi}}{\lambda}\chi(-ux)\lambda(-1)\\
&=\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{A\chi\lambda}{\lambda}\binom{\overline{C}\overline{\chi}}{B\overline{C}}\binom{\overline{C'}\overline{\lambda}}{B'\overline{C'}}\chi(x)\lambda(y).
\end{aligned}$$

Thus, by the fact that $\varepsilon(xy)\delta(ux)B(u)=\delta(v)B'(v)=0$, (2.1) and (2.2),

$$F_2(A;B,B';C,C';x,y)=\varepsilon(xy)BB'CC'(-1)\left(\sum_{u\in\mathbb{F}_q,v=y^{-1}}+\sum_{u\in\mathbb{F}_q,v\neq y^{-1}}\right)$$

$$=\overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y)\binom{\overline{A}B}{B\overline{C}}+BB'CC'(-1)\sum_{v\neq y^{-1}}B(u)B'(v)\overline{B}C(1-u)\overline{B'}C'(1-v)$$

$$\cdot\frac{1}{(q-1)^2}\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A}\overline{\chi}}{\lambda}\chi(-ux)\lambda(-vy)$$

$$=\frac{1}{(q-1)^2}\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{A\chi\lambda}{\lambda}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)+\overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y)\binom{\overline{A}B}{B\overline{C}}.$$

In view of the above, we complete the proof of Theorem 1.3. $\square$

3. Reduction and Transformation formulae

In this section we give some reduction and Transformation for $F_2(A;B,B';C,C';x,y)$. In order to derive these formulae we need some auxiliary results.

Proposition 3.1. (See [6, Corollary 3.16 and Theorem 3.15]) *For any $A,B,C,D\in\widehat{\mathbb{F}_q}$ and $x\in\mathbb{F}_q$, we have*

$$ {}_2F_1\left(\begin{matrix}A,\varepsilon\\ C\end{matrix}\middle| x\right)=\binom{C}{A}A(-1)\overline{C}(x)\overline{A}C(1-x)-C(-1)\varepsilon(x) \tag{3.1}$$
$$+(q-1)A(-1)\delta(1-x)\delta(\overline{A}C),$$

$$ {}_2F_1\left(\begin{matrix}A,B\\ A\end{matrix}\middle| x\right)=\binom{B}{A}\varepsilon(x)\overline{B}(1-x)-\overline{A}(-x) \tag{3.2}$$
$$+(q-1)A(-1)\delta(1-x)\delta(B),$$

$$ {}_3F_2\left(\begin{matrix}A,B,C\\ A,D\end{matrix}\middle| x\right)=\binom{B}{A}{}_2F_1\left(\begin{matrix}B,C\\ D\end{matrix}\middle| x\right)-\overline{A}(-x)\binom{C\overline{A}}{D\overline{A}} \tag{3.3}$$
$$+(q-1)A(-1)\overline{D}(x)C\overline{D}(1-x)\delta(B).$$

From the definition of $F_2(a;b,b';c,c';x,y)$ we know that

$$F_2(a;b,0;c,c';x,y)={}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\middle| x\right),$$
$$F_2(a;0,b';c,c';x,y)={}_2F_1\left(\begin{matrix}a,b'\\ c'\end{matrix}\middle| y\right).$$

We now give a finite field analogue of the above identities.

Theorem 3.1. *For any $A,B,B',C,C'\in\widehat{\mathbb{F}_q}$ and $x,\ y\in\mathbb{F}_q$, we have*

$$\begin{aligned}(3.4)\qquad & F_2(A;B,\varepsilon;C,C';x,y)\\ &= -\varepsilon(y)C'(-1)\,{}_2F_1\left(\left.\begin{matrix}A,B\\ C\end{matrix}\right| x\right)+\overline{C'}(y)\overline{A}C'(1-y)\binom{A\overline{C'}}{A}{}_2F_1\left(\left.\begin{matrix}A\overline{C'},B\\ C\end{matrix}\right|\frac{x}{1-y}\right)\\ &\quad+(q-1)A(-1)\overline{C}(x)\overline{C'}(y)\overline{AB}C^2C'(1-y)B\overline{C}(1-x-y)\delta(A\overline{C'})\end{aligned}$$

for $y\neq 1$,

$$\begin{aligned}(3.5)\qquad & F_2(A;\varepsilon,B';C,C';x,y)\\ &= -\varepsilon(x)C(-1)\,{}_2F_1\left(\left.\begin{matrix}A,B'\\ C'\end{matrix}\right| y\right)+\overline{C}(x)\overline{A}C(1-x)\binom{A\overline{C}}{A}{}_2F_1\left(\left.\begin{matrix}A\overline{C},B'\\ C'\end{matrix}\right|\frac{y}{1-x}\right)\\ &\quad+(q-1)A(-1)\overline{C'}(y)\overline{C}(x)\overline{AB'}C'^2C(1-x)B'\overline{C'}(1-x-y)\delta(A\overline{C})\end{aligned}$$

for $x\neq 1$.

Proof. We first prove (3.4). It follows from (3.1) that

$$\begin{aligned}\sum_{\lambda}\binom{A\chi\lambda}{\lambda}\binom{\lambda}{C'\lambda}\lambda(y)&=(q-1)\,{}_2F_1\left(\left.\begin{matrix}A\chi,\varepsilon\\ C'\end{matrix}\right| y\right)\\ &=(q-1)\binom{C'}{A\chi}A\chi(-1)\overline{C'}(y)\overline{A}\overline{\chi}C'(1-y)-(q-1)C'(-1)\varepsilon(y).\end{aligned}$$

Then, using the above identity in Theorem 1.3, by (2.1)–(2.3), (1.1) and (3.3), and canceling some terms, we get

$$\begin{aligned}& F_2(A;B,\varepsilon;C,C';x,y)\\ &=\frac{1}{(q-1)^2}\sum_{\chi}\binom{A\chi}{\chi}\binom{B\chi}{C\chi}\chi(x)\sum_{\lambda}\binom{A\chi\lambda}{\lambda}\binom{\lambda}{C'\lambda}\lambda(y)+\overline{A}(-x)\overline{C'}(y)C'(1-y)\binom{\overline{A}B}{B\overline{C}}\\ &=\overline{C'}(y)\overline{A}C'(1-y)\,{}_3F_2\left(\left.\begin{matrix}A,A\overline{C'},B\\ A,C\end{matrix}\right|\frac{x}{1-y}\right)-\varepsilon(y)C'(-1)\,{}_2F_1\left(\left.\begin{matrix}A,B\\ C\end{matrix}\right| x\right)\\ &\quad+\overline{A}(-x)\overline{C'}(y)C'(1-y)\binom{\overline{A}B}{B\overline{C}}\\ &=-\varepsilon(y)C'(-1)\,{}_2F_1\left(\left.\begin{matrix}A,B\\ C\end{matrix}\right| x\right)+\overline{C'}(y)\overline{A}C'(1-y)\binom{A\overline{C'}}{A}{}_2F_1\left(\left.\begin{matrix}A\overline{C'},B\\ C\end{matrix}\right|\frac{x}{1-y}\right)\\ &\quad+(q-1)A(-1)\overline{C}(x)\overline{C'}(y)\overline{AB}C^2C'(1-y)B\overline{C}(1-x-y)\delta(A\overline{C'}).\end{aligned}$$

This proves (3.4).

Identity (3.5) follows from (3.4) and (1.3). This completes the proof of Theorem 3.1. □

In [1, §9.5, (3)], Bailey gave the following reduction formula:

$$F_2(a;b,b';b,c';x,y)=(1-x)^{-a}{}_2F_1\left(\left.\begin{matrix}a,b'\\ c'\end{matrix}\right|\frac{y}{1-x}\right).$$

We also deduce the finite field analogue of the above formula.

Theorem 3.2. *For any $A, B, B', C' \in \widehat{\mathbb{F}_q}$ and $x \in \mathbb{F}_q \backslash \{1\},\ y \in \mathbb{F}_q$, we have*

$$\begin{aligned}
&F_2(A; B, B'; B, C'; x, y) \\
&= -\varepsilon(x)\overline{A}(1-x){}_2F_1\left(\left.\begin{matrix}A, B'\\ C'\end{matrix}\right| \frac{y}{1-x}\right) + \overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y) \\
&\quad + \overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y)\left(-1+(q-1)\delta(A\overline{B})\right) \\
&\quad + \overline{B}(x)\binom{A\overline{B}}{\overline{B}}{}_2F_1\left(\left.\begin{matrix}A\overline{B}, B'\\ C'\end{matrix}\right| y\right).
\end{aligned}$$

Proof. We know from [6, (3.11)] that for any $A,\ B \in \widehat{\mathbb{F}_q}$ and $x \in \mathbb{F}_q$, we have

$$\sum_{\chi}\binom{A\chi}{B\chi}\chi(x) = \overline{B}(x)\sum_{\chi}\binom{A\overline{B}\chi}{\chi}\chi(x) = (q-1)\overline{B}(x)\overline{A}B(1-x). \tag{3.6}$$

It is easily seen from (3.2) that

$$\begin{aligned}
\sum_{\chi}\binom{A\chi}{\chi}\binom{A\lambda\chi}{A\chi}\chi(x) &= (q-1){}_2F_1\left(\left.\begin{matrix}A, A\lambda\\ A\end{matrix}\right| x\right) \\
&= (q-1)\binom{A\lambda}{A}\varepsilon(x)\overline{A\lambda}(1-x) - (q-1)\overline{A}(-x).
\end{aligned}$$

Then, by (1.1) and (3.6),

$$\begin{aligned}
&\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{A\chi\lambda}{A\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y) \\
&= \sum_{\lambda}\binom{B'\lambda}{C'\lambda}\lambda(y)\sum_{\chi}\binom{A\chi}{\chi}\binom{A\lambda\chi}{A\chi}\chi(x) \\
&= (q-1)^2\varepsilon(x)\overline{A}(1-x){}_2F_1\left(\left.\begin{matrix}A, B'\\ C'\end{matrix}\right| \frac{y}{1-x}\right) - (q-1)^2\overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y).
\end{aligned} \tag{3.7}$$

Thus, by (2.1) and (2.4),

$$\begin{aligned}
&F_2(A; B, B'; B, C'; x, y) \\
&= \frac{1}{(q-1)^2}\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{A\chi\lambda}{A\chi}\left(-1+(q-1)\delta(B\chi)\right)\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y) \\
&\quad + \overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y)\left(-1+(q-1)\delta(A\overline{B})\right) \\
&= -\varepsilon(x)\overline{A}(1-x){}_2F_1\left(\left.\begin{matrix}A, B'\\ C'\end{matrix}\right| \frac{y}{1-x}\right) + \overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y) \\
&\quad + \overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y)\left(-1+(q-1)\delta(A\overline{B})\right) \\
&\quad + \overline{B}(x)\binom{A\overline{B}}{\overline{B}}{}_2F_1\left(\left.\begin{matrix}A\overline{B}, B'\\ C'\end{matrix}\right| y\right).
\end{aligned}$$

This concludes the proof of Theorem 3.2. □

From Theorem 3.2 and (1.3) we can easily derive the following identity which is the finite field analogue of the formula

$$F_2(a;b,b';c,b';x,y)=(1-y)^{-a}{}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\middle|\frac{x}{1-y}\right).$$

Theorem 3.3. *For any $A,B,B',C\in\widehat{\mathbb{F}_q}$ and $x\in\mathbb{F}_q$, $y\in\mathbb{F}_q\backslash\{1\}$, we have*

$$\begin{aligned}&F_2(A;B,B';C,B';x,y)\\&=-\varepsilon(y)\overline{A}(1-y)\,{}_2F_1\left(\begin{matrix}A,B\\ C\end{matrix}\middle|\frac{x}{1-y}\right)+\overline{A}(-y)\overline{C}(x)\overline{B}C(1-x)\\&\quad+\overline{A}(-y)\overline{C}(x)\overline{B}C(1-x)\left(-1+(q-1)\delta(A\overline{B'})\right)\\&\quad+\overline{B'}(y)\binom{A\overline{B'}}{\overline{B'}}{}_2F_1\left(\begin{matrix}A\overline{B'},B\\ C\end{matrix}\middle|x\right).\end{aligned}$$

From the definition of $F_2(A;B,B';C,C';x,y)$, we can easily deduce the following transformation formulae for $F_2(A;B,B';C,C';x,y)$.

Theorem 3.4. *For any $A,B,B',C,C'\in\widehat{\mathbb{F}_q}$ and $x,y\in\mathbb{F}_q$, we have*

$$\begin{aligned}&F_2(A;B,B';C,C';x,y)\\&=\overline{A}(1-x)F_2\left(A;\overline{B}C,B';C,C';-\frac{x}{1-x},\frac{y}{1-x}\right)\\&=\overline{A}(1-y)F_2\left(A;B,\overline{B'}C';C,C';\frac{x}{1-y},-\frac{y}{1-y}\right)\\&=\overline{A}(1-x-y)F_2\left(A;\overline{B}C,\overline{B'}C';C,C';-\frac{x}{1-x-y},-\frac{y}{1-x-y}\right).\end{aligned}\tag{3.8}$$

Proof. Using the definition of $F_2(A;B,B';C,C';x,y)$ and then making the substitutions (1) $u=1-u'$, $v=v'$, (2) $u=u'$, $v=1-v'$, (3) $u=1-u'$, $v=1-v'$ at the left side of (3.8) we can easily obtain these transformation formulae. □

It is easily seen that these transformation formulae in (3.8) can be regarded as the finite field analogue of [1, §9.4, (6)–(8)].

4. Generating functions

In this section, we establish some generating functions for $F_2(A;B,B';C,C';x,y)$.

We first state a result of Greene in our notations.

Proposition 4.1. (See [6, (2.15)]) *For any $A,B,C\in\widehat{\mathbb{F}_q}$, we have*

$$\binom{A}{B}\binom{C}{A}=\binom{C}{B}\binom{C\overline{B}}{A\overline{B}}-(q-1)B(-1)\delta(A)+(q-1)AB(-1)\delta(B\overline{C}).$$

The following theorem involves a generating function for $F_2(A;B,B';C,C';x,y)$.

Theorem 4.1. *For any $A, B, B', C, C' \in \widehat{\mathbb{F}_q}$ and $x \in \mathbb{F}_q \backslash \{0\},\ y \in \mathbb{F}_q,\ t \in \mathbb{F}_q \backslash \{0, 1\},$ we have*

$$
\begin{aligned}
&\sum_{\theta} \binom{A\theta}{\theta} F_2(A\theta; B, B'; C, C'; x, y)\theta(t) \\
&= (q-1)\overline{A}(1-t) F_2\left(A; B, B'; C, C'; \frac{x}{1-t}, \frac{y}{1-t}\right) \\
&- (q-1)\overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-t-y)B'(1-t)\binom{\overline{A}B}{B\overline{C}} \\
&- (q-1)\overline{A}(-t)\overline{C'}(y)\overline{B'}C'(1-y)\,{}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \middle| -\frac{x}{t}\right) \\
&- (q-1)\overline{A}(-t)\left(\binom{B}{C}\binom{B'}{C'}\varepsilon(y) - F_2(\varepsilon; B, B'; C, C'; x, y)\right) \\
&+ (q-1)\overline{A}(-x)\overline{C'}((1-t)y)\overline{B'}C'(1-(1-t)y)\binom{\overline{A}B}{\overline{A}C} \\
&+ (q-1)BC(-1)\overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y)\,{}_2F_1\left(\begin{matrix} A, A\overline{C} \\ A\overline{B} \end{matrix} \middle| -\frac{t}{x}\right).
\end{aligned}
$$

Proof. It is easily seen from (3.2) that

$$
{}_2F_1\left(\begin{matrix} A\chi, A\chi\lambda \\ A\chi \end{matrix} \middle| t\right) = \binom{A\chi\lambda}{A\chi}\overline{A}\overline{\chi}\overline{\lambda}(1-t) - \overline{A}\overline{\chi}(-t).
$$

Then from (3.3) we know that

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} A, A\chi, A\chi\lambda \\ A, A\chi \end{matrix} \middle| t\right) \\
&= \binom{A\chi}{A}\,{}_2F_1\left(\begin{matrix} A\chi, A\chi\lambda \\ A\chi \end{matrix} \middle| t\right) - \overline{A}(-t)\binom{\chi\lambda}{\chi} + (q-1)A(-1)\overline{A}\overline{\chi}(t)\lambda(1-t)\delta(A\chi) \\
&= \binom{A\chi}{\chi}\binom{A\chi\lambda}{A\chi}\overline{A}\overline{\chi}\overline{\lambda}(1-t) - \binom{A\chi}{\chi}\overline{A}\overline{\chi}(-t) - \overline{A}(-t)\binom{\chi\lambda}{\chi} \\
&+ (q-1)A(-1)\overline{A}\overline{\chi}(t)\lambda(1-t)\delta(A\chi).
\end{aligned}
$$

Thus

$$\begin{aligned}
(4.1)\qquad &\sum_{\theta,\chi,\lambda}\binom{A\theta}{\theta}\binom{A\chi\theta}{A\theta}\binom{A\chi\lambda\theta}{A\chi\theta}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)\theta(t)\\
&=(q-1)\sum_{\chi,\lambda}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)_3F_2\left(\begin{matrix}A, A\chi, A\chi\lambda\\ A, A\chi\end{matrix}\middle| t\right)\\
&=(q-1)\overline{A}(1-t)\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{A\chi\lambda}{A\chi}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi\left(\frac{x}{1-t}\right)\lambda\left(\frac{y}{1-t}\right)\\
&-(q-1)\overline{A}(-t)\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi\left(-\frac{x}{t}\right)\lambda(y)\\
&-(q-1)\overline{A}(-t)\sum_{\chi,\lambda}\binom{\chi\lambda}{\chi}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)\\
&+(q-1)^2\overline{A}(-x)\binom{\overline{A}B}{\overline{A}C}\sum_{\lambda}\binom{B'\lambda}{C'\lambda}\lambda((1-t)y).
\end{aligned}$$

From Theorem 1.3 we see that

$$\begin{aligned}
(4.2)\qquad &\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{A\chi\lambda}{A\chi}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi\left(\frac{x}{1-t}\right)\lambda\left(\frac{y}{1-t}\right)\\
&=(q-1)^2F_2\left(A;B,B';C,C';\frac{x}{1-t},\frac{y}{1-t}\right)\\
&-(q-1)^2\overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-t-y)AB'(1-t)\binom{\overline{A}B}{B\overline{C}}.
\end{aligned}$$

By (2.2) and (3.6),

$$\begin{aligned}
(4.3)\qquad &\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi\left(-\frac{x}{t}\right)\lambda(y)\\
&=\sum_{\chi}\binom{A\chi}{\chi}\binom{B\chi}{C\chi}\chi\left(-\frac{x}{t}\right)\sum_{\lambda}\binom{B'\lambda}{C'\lambda}\lambda(y)\\
&=(q-1)^2\overline{C'}(y)\overline{B'}C'(1-y)_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\middle| -\frac{x}{t}\right).
\end{aligned}$$

It can be deduced from (2.4) and (3.6) that

$$\begin{aligned}
\sum_{\lambda}\binom{\lambda}{\lambda}\binom{B'\lambda}{C'\lambda}\lambda(y)&=-\sum_{\lambda}\binom{B'\lambda}{C'\lambda}\lambda(y)+(q-1)\binom{B'}{C'}\varepsilon(y)\\
&=-(q-1)\overline{C'}(y)\overline{B'}C'(1-y)+(q-1)\binom{B'}{C'}\varepsilon(y).
\end{aligned}$$

This combines (2.4) to give

$$\begin{aligned}
&\sum_{\chi,\lambda}\binom{\chi}{\chi}\binom{\chi\lambda}{\chi}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)\\
&=(q-1)\binom{B}{C}\sum_{\lambda}\binom{\lambda}{\lambda}\binom{B'\lambda}{C'\lambda}\lambda(y)-\sum_{\chi,\lambda}\binom{\chi\lambda}{\chi}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)\\
&=(q-1)^2\binom{B}{C}\binom{B'}{C'}\varepsilon(y)-(q-1)^2\overline{C'}(y)\overline{B'}C'(1-y)\binom{B}{C}\\
&-\sum_{\chi,\lambda}\binom{\chi\lambda}{\chi}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y).
\end{aligned}$$

From Theorem 1.3 and (2.1) we have

$$\begin{aligned}
\sum_{\chi,\lambda}\binom{\chi}{\chi}\binom{\chi\lambda}{\chi}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)&=(q-1)^2F_2(\varepsilon;B,B';C,C';x,y)\\
&-(q-1)^2\overline{C'}(y)\overline{B'}C'(1-y)\binom{B}{C}.
\end{aligned}$$

So we deduce from the above two identities that

$$\begin{aligned}
\sum_{\chi,\lambda}\binom{\chi\lambda}{\chi}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)&=(q-1)^2\binom{B}{C}\binom{B'}{C'}\varepsilon(y)\\
&-(q-1)^2F_2(\varepsilon;B,B';C,C';x,y).
\end{aligned}\tag{4.4}$$

By Theorem 1.3 and (2.1)–(2.3),

$$\begin{aligned}
&\sum_{\theta}\binom{A\theta}{\theta}F_2(A\theta;B,B';C,C';x,y)\theta(t)\\
&=\frac{1}{(q-1)^2}\sum_{\theta,\chi,\lambda}\binom{A\theta}{\theta}\binom{A\chi\theta}{A\theta}\binom{A\chi\lambda\theta}{A\chi\theta}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)\theta(t)\\
&+BC(-1)\overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y)\sum_{\theta}\binom{A\theta}{\theta}\binom{A\overline{C}\theta}{A\overline{B}\theta}\theta(-t/x).
\end{aligned}$$

Using (4.2)–(4.4) and (3.6) in (4.1) and then substituting (4.1) in the above identity, we obtain the result. This ends the proof of Theorem 4.1. □

Theorem 4.1 is actually the finite field analogue of [3, (2.2)].

We also establish two other generating functions for $F_2(A;B,B';C,C';x,y)$.

Theorem 4.2. *For any $A, B, B', C, C' \in \widehat{\mathbb{F}_q}$ and $x, y \in \mathbb{F}_q,\ t \in \mathbb{F}_q \backslash \{1\}$, we have*

$$\sum_{\theta} \binom{B\overline{C}\theta}{\theta} F_2(A; B\theta, B'; C, C'; x, y)\theta(t) \tag{4.5}$$
$$= (q-1)\varepsilon(t)\overline{B}(1-t)F_2\left(A; B, B'; C, C'; \frac{x}{1-t}, y\right)$$
$$- (q-1)\overline{B}C(-t)\varepsilon(x)\overline{A}(1-x){}_2F_1\left(\begin{matrix}A, B'\\ C'\end{matrix}\middle| \frac{y}{1-x}\right)$$

for $x \neq 1$,

$$\sum_{\theta} \binom{B'\overline{C'}\theta}{\theta} F_2(A; B, B'\theta; C, C'; x, y)\theta(t) \tag{4.6}$$
$$= (q-1)\varepsilon(t)\overline{B'}(1-t)F_2\left(A; B, B'; C, C'; x, \frac{y}{1-t}\right)$$
$$- (q-1)\overline{B'}C'(-t)\varepsilon(y)\overline{A}(1-y){}_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\middle| \frac{x}{1-y}\right)$$

for $y \neq 1$.

Proof. We first prove (4.5). It is easy to know from Theorem 1.3 that

$$\sum_{\theta} \binom{B\overline{C}\theta}{\theta} F_2(A; B\theta, B'; C, C'; x, y)\theta(t) \tag{4.7}$$
$$= \frac{1}{(q-1)^2} \sum_{\theta,\chi,\lambda} \binom{B\overline{C}\theta}{\theta}\binom{B\theta\chi}{B\overline{C}\theta}\binom{A\chi}{\chi}\binom{A\chi\lambda}{\lambda}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)\theta(t)$$
$$+ \overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y)\sum_{\theta}\binom{B\overline{C}\theta}{\theta}\binom{\overline{A}B\theta}{B\overline{C}\theta}\theta(t).$$

It can be seen from Proposition 4.1 that

$$\binom{B\overline{C}\theta}{\theta}\binom{B\theta\chi}{B\overline{C}\theta} = \binom{B\theta\chi}{\theta}\binom{B\chi}{B\overline{C}} - (q-1)\theta(-1)\delta(B\overline{C}\theta) + (q-1)BC(-1)\delta(\overline{B}\overline{\chi}).$$

Then

$$\sum_{\theta,\chi,\lambda} \binom{B\overline{C}\theta}{\theta}\binom{B\theta\chi}{B\overline{C}\theta}\binom{A\chi}{\chi}\binom{A\chi\lambda}{\lambda}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)\theta(t) \tag{4.8}$$
$$= \sum_{\theta,\chi,\lambda} \binom{B\theta\chi}{\theta}\binom{B\chi}{B\overline{C}}\binom{A\chi}{\chi}\binom{A\chi\lambda}{\lambda}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)\theta(t)$$
$$- (q-1)\overline{B}C(-t)\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{A\chi\lambda}{\lambda}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y).$$

It follows from (2.1), (3.6) and Theorem 1.3 that

$$
\begin{aligned}
(4.9)\qquad & \sum_{\theta,\chi,\lambda}\binom{B\theta\chi}{\theta}\binom{B\chi}{B\overline{C}}\binom{A\chi}{\chi}\binom{A\chi\lambda}{\lambda}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)\theta(t)\\
&=\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{A\chi\lambda}{\lambda}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi(x)\lambda(y)\sum_{\theta}\binom{B\chi\theta}{\theta}\theta(t)\\
&=(q-1)\varepsilon(t)\overline{B}(1-t)\sum_{\chi,\lambda}\binom{A\chi}{\chi}\binom{A\chi\lambda}{\lambda}\binom{B\chi}{C\chi}\binom{B'\lambda}{C'\lambda}\chi\left(\frac{x}{1-t}\right)\lambda(y)\\
&=(q-1)^3\varepsilon(t)\overline{B}(1-t)F_2\left(A;B,B';C,C';\frac{x}{1-t},y\right)\\
&\quad-(q-1)^3\varepsilon(t)A\overline{B}(1-t)\overline{A}(-x)\overline{C'}(y)\overline{B'}C'(1-y)\binom{\overline{A}B}{B\overline{C}}.
\end{aligned}
$$

It is easily known from (3.2) that

$$
\begin{aligned}
(4.10)\qquad \sum_{\theta}\binom{B\overline{C}\theta}{\theta}\binom{\overline{A}B\theta}{B\overline{C}\theta}\theta(t)&=(q-1)\,{}_2F_1\left(\left.\begin{matrix}B\overline{C},\overline{A}B\\ B\overline{C}\end{matrix}\right|t\right)\\
&=(q-1)\binom{\overline{A}B}{B\overline{C}}\varepsilon(t)A\overline{B}(1-t)-(q-1)\overline{B}C(-t).
\end{aligned}
$$

Using (4.9) and (3.7) in (4.8), combining (4.7), (4.8) and (4.10) and canceling some terms, we get

$$
\begin{aligned}
&\sum_{\theta}\binom{B\overline{C}\theta}{\theta}F_2(A;B\theta,B';C,C';x,y)\theta(t)\\
&=(q-1)\varepsilon(t)\overline{B}(1-t)F_2\left(A;B,B';C,C';\frac{x}{1-t},y\right)\\
&\quad-(q-1)\overline{B}C(-t)\varepsilon(x)\overline{A}(1-x)\,{}_2F_1\left(\left.\begin{matrix}A,B'\\ C'\end{matrix}\right|\frac{y}{1-x}\right),
\end{aligned}
$$

which proves (4.5).

Identity (4.6) follows easily from (4.5) and (1.3). This finishes the proof of Theorem 4.2. □

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